# Surjective Isometries Between Real JB$^*$-triples

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## 1. Introduction

In [**18**], R. Kadison proved that every surjective linear isometry $\Phi : \mathcal{A} \to \mathcal{B}$ between two unital C$^*$-algebras is of the form

$$\Phi(x) = uT(x),\ \ x \in \mathcal{A},$$

where $u$ is a unitary element in $\mathcal{B}$ and $T$ is a Jordan $*$-isomorphism from $\mathcal{A}$ to $\mathcal{B}$. This result extends to non-abelian unital C$^*$-algebras the classical Banach-Stone Theorem [**3, 31**] obtained in the 1930's. Kadison's result was extended to surjective isometries between C$^*$-algebras by A. L. T. Paterson and A. M. Sinclair [**27**], by replacing the unitary element $u$ by a unitary element in the multiplier C$^*$-algebra of the range algebra. In particular, every surjective isometry between C$^*$-algebras preserves the triple products of the form $\{x, y, z\} = 2^{-1}(xy^*z + zy^*x)$.

In the non-associative case, J. Wright and M. Youngson [**34**, Theorem 6], established that every unital surjective linear isometry between two unital JB$^*$-algebras is a Jordan $*$-isomorphism. In 1995, J. M. Isidro and A. Rodríguez [**17**, Theorem 1.9] show that every surjective linear isometry $\Phi$ between two JB-algebras is of the form

$$\Phi(x) = bT(x),$$

where $b$ is a central symmetry in the algebra of multipliers of the range JB-algebra and $T$ is a surjective algebra isomorphism. It also follows from [**17**, Theorem 1.9] that a bijective linear map $\Phi$ between two JB-algebras is an isometry if, and only if, $\Phi$ is an triple-isomorphism with respect to the triple product $\{x, y, z\} = (x \circ y) \circ z + (z \circ y) \circ x - (x \circ z) \circ y$.

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Another extensions of the Banach-Stone Theorem for non-associative Banach algebras can be found in [**29**] and [**19**].

C$^*$-algebras and JB$^*$-algebras belong to a more general class of algebraic-topological structures known with the name of (complex) JB$^*$-triples (see Section §2 for the definition). The Banach-Stone Theorem for JB$^*$-triples, due to W. Kaup [**21**], establishes that a bijective linear map $\Phi$ between two JB$^*$-triples is an isometry if, and only if, it is a triple isomorphism. An alternative proof of Kaup's version of the Banach-Stone Theorem was obtained by T. Dang, Y. Friedman and B. Russo in [**8**].

Along the history, others extensions of the Banach-Stone Theorem have come out. One of the first is due to T. Dang, [**6**], by studying real linear surjective isometries between (complex) JB$^*$-triples. Motivated by the quantum mechanics, it turned out the study of invertible affine maps on the unit sphere of the dual of a JB$^*$-triples and these maps coincide with the adjoints of real (not necessarily complex) linear surjective isometries (compare [**6**, first paragraph in page 972]). The Banach-Stone Theorem is not, in general, true for real linear isometries between (complex) JB$^*$-triples (see [**6**, Remark 2.7]). However, T. Dang shows in [**6**, Theorem 3.1] that if $\Phi : \mathcal{E} \to \mathcal{F}$ is a surjective real linear isometry between two (complex) JB$^*$-triples such that $\mathcal{E}^{**}$ does not contain nontrivial Cartan factors of rank 1, then $\Phi$ is a triple isomorphism.

The structures of C$^*$-algebras, JB$^*$-algebras and (complex) JB$^*$-triples have been generalized to real C$^*$-algebras, J$^*$B-algebras and real JB$^*$-triples, respectively (see [**13, 5**], [**2**] and [**16**], see also §2 for completeness). The study of the Banach-Stone Theorem in these last structures is another line of generalization. For some of this structures, the corresponding Banach-Stone Theorems have been obtained by several authors as follows.

In 1990, M. Grzesiak proves an extension of the Banach-Stone Theorem for abelian real C$^*$-algebras [**23**, Corollary 5.2.4]. For non-necessarily abelian real C$^*$-algebras the Banach-Stone Theorem was obtained by C.-H. H. Chu, T. Dang, B. Russo and B. Ventura (see [**5**, Theorem 6.4]), showing that a linear bijection between two real C$^*$-algebras is an isometry if, and only if, it is a triple isomorphism.

The study of surjective linear isometries between real JB$^*$-triples begins in [**16**], where the authors prove that every triple isomorphism between real JB$^*$-triples is an isometry, [**16**, Theorem 4.8]. However, as we have seen before, not every surjective isometry is a triple isomorphism (see [**6**, Remark 2.7]). Recently, W. Kaup, [**22**], has studied the Banach-Stone Theorem for real Cartan factors (real forms of complex

Cartan factors, see §2 for completeness). In [**22**, Theorem 5.18], Kaup has proved that every surjective real linear mapping from a non exceptional real or complex Cartan factor of rank $> 1$ to a real JBW*-triple is a triple isomorphism if, and only if, it is an isometry. In the just referred paper, the author left open the cases of the two exceptional real Cartan factors $V^{\mathbb{O}_0}$ and $VI^{\mathbb{O}_0}$ (compare [**22**, page 217]). This problem is the starting point of this paper.

In our first goal, Proposition 2.14, we conclude that every surjective linear isometry between two real reduced Cartan factors is a triple isomorphism. The novelties in our techniques lie in the concept of real reduced JB*-triple, already introduced by O. Loos [**25**, 11.9]. Most of real Cartan factors are real reduced Cartan factors, in particular, the exceptional real Cartan factors $V^{\mathbb{O}_0}$ and $VI^{\mathbb{O}_0}$ are real reduced Cartan factors. This fact and our Proposition 2.14 give a positive answer to the problem left open by Kaup. Moreover, our result for real reduced Cartan factors, jointly with Kaup's result for the non-reduced real or complex Cartan factors, allow us to remove the hypothesis of non-exceptionality in [**22**, Theorem 5.18] (see Corollary 2.16). Finally, in our main result, Theorem 3.2, we get an extension of Dang's Banach-Stone Theorem to the setting of real JB*-triples. As a consequence of our main result we also obtain a Banach-Stone Theorem for J*B-algebras (Corollary 3.4).

Let $X$ be a real or complex Banach space and let $S \subset X$. We denote by $X^*$ and $S^\circ$ the dual space of $X$ and the polar of $S$ in $X^*$, respectively. If $X$ is a dual Banach space, $X_*$ will denote a predual of $X$ and $S_\circ$ will stand for the pre-polar of $S$ in $X_*$. The canonical embedding of $X$ into $X^{**}$ will be denoted by $j : X \to X^{**}$. The Banach space of all bounded linear operator between two Banach spaces $X$ and $Y$ is denoted by $L(X, Y)$ and $L(X)$ will stand for $L(X, X)$.

## 2. Surjective Isometries Between Real Cartan factors

We recall that a *JB*-triple* is a complex Banach space $\mathcal{E}$ together with a triple product $\{.,.,.\} : \mathcal{E} \times \mathcal{E} \times \mathcal{E} \to \mathcal{E}$, which is continuous, symmetric and linear in the outer variables and conjugate linear in the middle one, satisfying

a) *Jordan Identity*: for all $a, b, x, y, z \in \mathcal{E}$

$$L(a,b)\{x,y,z\} = \{L(a,b)x,y,z\} - \{x,L(b,a)y,z\} + \{x,y,L(a,b)z\},$$

where $L(a,b)x := \{a,b,x\}$;

b) For each $a \in \mathcal{E}$ the operator $L(a,a)$ is hermitian with nonnegative spectrum, and $\|L(a,a)\| = \|a\|^2$.

Every C$^*$-algebra (respectively, every JB$^*$-algebra) is a complex JB$^*$-triple with respect to the triple product $\{x,y,z\} = \frac{1}{2}(xy^*z + zy^*x)$ (respectively, $\{a,b,c\} = (a \circ b^*) \circ c + (c \circ b^*) \circ a - (a \circ c) \circ b^*$).

A real Banach space $E$ together with a trilinear map $\{.,.,.\} : E \times E \times E \to E$ is called a *real JB*-triple* if there is a complex JB$^*$-triple $\mathcal{E}$ and an $\mathbb{R}$-linear isometry $\lambda$ from $E$ to $\mathcal{E}$ such that $\lambda\{x,y,z\} = \{\lambda x, \lambda y, \lambda z\}$ for all $x, y, z$ in $E$ (see [**16**]).

Real JB$^*$-triples are essentially the closed real subtriples of complex JB$^*$-triples and, by [**16**, Proposition 2.2], given a real JB$^*$-triple $E$ there exists a unique complex JB$^*$-triple $\mathcal{E}$ and a unique *conjugation* (conjugate linear and isometric mapping of period 2) $\tau$ on $\mathcal{E}$ such that $E = \mathcal{E}^\tau := \{x \in \mathcal{E} : \tau(x) = x\}$. In fact, $\mathcal{E} = E + iE$ is the complexification of the vector space $E$, with triple product extending in a natural way the triple product of $E$ and a suitable norm. Along the paper, the complexification of a real JB$^*$-triple $E$, equipped with the structure of complex JB$^*$-triple, will be denoted by $\widehat{E}$. Given a complex Banach space $X$ and a conjugation $\tau$ on $X$, the real Banach space $X^\tau = \{x \in X : \tau(x) = x\}$ is called a real form of $X$. Given a conjugation $\tau$ on a complex JB$^*$-triple, then by Kaup's Banach-Stone Theorem for complex JB$^*$-triples [**21**] we can assure that $\tau$ is a conjugate-linear triple isomorphism and hence, the real JB$^*$-triples coincide with the real forms of complex JB$^*$-triples.

Every complex JB$^*$-triple is a real JB$^*$-triple regarded as a real Banach space. The class of real JB*-triples also includes all JB-algebras [**14**], all real C*-algebras [**13**], and all J*B-algebras [**2**]. Another examples of real JB$^*$-triples are the so-called real and complex Cartan factors, that are introduced below.

*Cartan Factors*

Classical Cartan factors can be classified in six different types (see [**20**]. The Cartan factor of type 1, denoted by $I_{n,m}$, is the complex Banach space, $L(H,K)$, of bounded linear operators between two complex Hilbert spaces $H$ and $K$ of dimensions $n$, $m$ respectively, where the triple is defined by $\{x,y,z\} = 2^{-1}(xy^*z + zy^*x)$.

We recall that given a conjugation, $q$, on a complex Hilbert space $H$, we can define the following linear involution $x \mapsto x^t := qx^*q$ on $L(H)$. The Cartan factor of type 2, denoted by $II_n$, (respectively, type 3, $III_n$) is the subtriple of $L(H)$ formed by the skew-symmetric operators (respectively, symmetric) for the involution $t$. Moreover, $II_n$ and $III_n$ are, up to isomorphism, independent of the conjugation $q$ on $H$.

The Cartan factor of type 4, $IV_n$ (also called complex Spin factor), is a complex Hilbert space provided with an conjugation $x \mapsto \overline{x}$, triple product

$$\{x, y, z\} = \langle x/y \rangle\, z + \langle z/y \rangle\, x - \langle x/\bar{z} \rangle\, \bar{y},$$

and norm given by $\|x\|^2 = \langle x/x \rangle + \sqrt{\langle x/x \rangle^2 - |\,\langle x/\overline{x} \rangle\,|^2}$.

The Cartan factor of type 6 is the 27-dimensional exceptional JB*-algebra
$VI = H_3(\mathbb{O}^{\mathbb{C}})$ of all symmetric 3 by 3 matrices with entries in the complex Octonions $\mathbb{O}^{\mathbb{C}}$ [**36**]. The Cartan factor of type 5 is the subtriple, $V = M_{1,2}(\mathbb{O}^{\mathbb{C}})$ , of the Cartan factor of type 6. We also refer to [**14**], [**1**], [**30**], [**32**], and [**25**] as a basic bibliographic about the exceptional Cartan factors.

Following [**22**], real Cartan factors are real forms of complex Cartan factors. They are completely described in [**22**, Corollary 4.4] and [**25**, pages 11.5-11.7]. Real Cartan factors can be described, up to isomorphism, as follows:

Let $X$ and $Y$ be two real Hilbert spaces of dimensions $n$, and $m$ respectively. Let $P$ and $Q$ be two Hilbert spaces of dimensions $p$, and $q$ respectively, over the quaternion field $\mathbb{H}$. Finally let $H$ be a complex Hilbert space of dimension $n$.

(1) $I^{\mathbb{R}}_{n,m} := L(X, Y)$
(2) $I^{\mathbb{H}}_{2p,2q} := L(P, Q)$
(3) $I^{\mathbb{C}}_{n} := \{z \in L(H) : z^* = z\}$
(4) $II^{\mathbb{R}}_{n} := \{x \in L(X) : x^* = -x\}$
(5) $II^{\mathbb{H}}_{2p} := \{w \in L(P) : w^* = w\}$
(6) $III^{\mathbb{R}}_{n} := \{x \in L(X) : x^* = x\}$
(7) $III^{\mathbb{H}}_{2p} := \{w \in L(P) : w^* = -w\}$

(8) $IV^{r,s}_{n} := E$ where $E = X_1 \oplus^{\ell_1} X_2$ and $X_1$,$X_2$ are closed linear subspaces of a real Hilbert space, $X$, of dimension greater or equal three, such that $X_2 = {X_1}^{\perp}$, with triple product

$$\{x, y, z\} = \langle x/y \rangle\, z + \langle z/y \rangle\, x - \langle x/\bar{z} \rangle\, \bar{y},$$

where $\langle ./. \rangle$ is the inner product in $X$ and the involution $x \to \bar{x}$ on $E$ is defined by $\bar{x} = (x_1, -x_2)$ for every $x = (x_1, x_2)$. This factor is known as *real spin factor*.

(9) $V^{\mathbb{O}_0} := M_{1,2}(\mathbb{O}_0)$
(10) $V^{\mathbb{O}} := M_{1,2}(\mathbb{O})$
(11) $VI^{\mathbb{O}_0} := H_3(\mathbb{O}_0)$
(12) $VI^{\mathbb{O}} := H_3(\mathbb{O})$

Where $\mathbb{O}_0$ is the real split Cayley algebra over the field of the real numbers and $\mathbb{O}$ is the real division Cayley algebra (also called the algebra of real division octonions). The real Cartan factors $9 - 12$ are called *exceptional real Cartan factors.*

Given a real or complex JB*-triple $U$ and a tripotent $e \in U$ (i. e. $\{e,e,e\} = e$), then $e$ induces the following two decompositions of $U$

$$U = U_0(e) \oplus U_1(e) \oplus U_2(e) = U^1(e) \oplus U^{-1}(e) \oplus U^0(e)$$

where $U_k(e) := \{x \in U : L(e,e)x = \frac{k}{2}x\}$ is a subtriple of $U$ and $U^k(e) = \{x \in U : Q(e)(x) := \{e,x,e\} = kx\}$ is a real Banach subspace of $U$ (compare [**25**, Theorem 3.13]). The natural projection of $U$ onto $U_k(e)$ (respectively, $U^k(e)$) will be denoted by $P_k(e)$ (respectively, $P^k(e)$). The first decomposition is called the Peirce decomposition with respect to the tripotent $e$. The following *Peirce rules* are satisfied for the Peirce decomposition

$$\{U_i(e), U_j(e), U_k(e)\} \subseteq U_{i-j+k}(e), \text{ where } i,j,k = 0,1,2 \text{ and}$$

$$U_l(e) = 0 \text{ for } l \neq 0,1,2.$$

$$\{U_0(e), U_2(e), U\} = \{U_2(e), U_0(e), U\} = 0.$$

The following identities and rules are also satisfied

$$U_2(e) = U^1(e) \oplus U^{-1}(e), \qquad U_1(e) \oplus U_0(e) = U^0(e)$$

$$\left\{U^i(e), U^j(e), U^k(e)\right\} \subseteq U^{ijk}(e), \text{ whenever } ijk \neq 0.$$

It is known that for every tripotent $e$ in a real or complex JB*-triple $U$, $Q(e)$ preserves the triple product in $U_2(e)$.

Two non zero elements $x$, $y$ in a real or complex JB*-triple $U$ are said to be *orthogonal* and write $x \perp y$ if $L(x,y) = 0$ (equivalently $L(y,x) = 0$). In particular if $e$ and $f$ are tripotents in $U$, we have $e \perp f$ if and only if $e \in U_0(f)$; $e$ and $f$ are said to be *colinear*, $e \top f$, if $e \in U_1(f)$ and $f \in U_1(e)$). We say that $e$ *governs* $f$, $e \vdash f$, whenever $f \in U_2(e)$ and $e \in U_1(f)$. A non-zero tripotent $e$ is called *minimal* if $U^1(e) = \mathbb{R}e$ (since in the complex case $U^{-1}(e) = i\, U^1(e)$, this definition is equivalent to $U_2(e) = \mathbb{C}e$).

A real or complex JBW*-triple is a JB*-triple which is a dual Banach space. Every real or complex JBW*-triple has a unique predual and its triple product is separately weak*-continuous (compare [**4**] and [**26**]).

Following [**25**, 11.9], we say that a real JB*-triple $E$ is *reduced* whenever $E_2(e) = \mathbb{R}e$ (equivalently, $E^{-1}(e) = 0$) for every minimal tripotent $e \in E$. The reduced real Cartan factors have been studied and classified in [**25**, 11.9] in the finite dimensional case and in [**22**, Table 1] (in the last case they correspond to those factors with the parameter $z = 1$). The non reduced real Cartan factors are the following $IV_n^{n,0}$, $V^{\mathbb{O}}$, $I_{2p,2q}^{\mathbb{H}}$ and $III_{2n}^{\mathbb{H}}$ the remaining real Cartan factors are all reduced.

REMARK 2.1. Let $E$ be a real Cartan factor of type $IV_n^{n,0}$. It is easy to check that every norm-one element, $e$, in $E$ is a minimal tripotent which is also unitary (i. e. $E_2(e) = E$), in particular, $E_1(e) = 0$.

Let now $E$ denote the real Cartan factor $V^{\mathbb{O}}$. Let $e = (1, 0)$ in $E$. In this case, we can easily see that

$$E_1(e) = \{(0, z) : z \in \mathbb{O}\}, \quad E^1(e) = \mathbb{R}e, \text{ and}$$

$$E^{-1}(e) = \{(y, 0) : y \in \mathrm{Span}_{\mathbb{R}}\{e_1, \ldots, e_7\}\},$$

where $\{1, e_1, \ldots, e_7\}$ denotes the canonical basis of $\mathbb{O}$. Every tripotent element in $E_1(e)$ is of the form $f = (0, z)$ with $zz^* = 1$, and for such a tripotent $e$ lies in $E_1(f)$.

In the two remaining non reduced real Cartan factors ($I^{\mathbb{H}}_{2p,2q}$, $III^{\mathbb{H}}_{2n}$) it is easy to see that given a minimal tripotent $e$ and a tripotent $f \in E_1(e)$, we have $e \in E_1(f) \cup E_2(f)$.

The next lemma shows that the situation studied in the above Remark for non reduced real Cartan factors remains true for every real JB*-triple.

LEMMA 2.2. *Let $E$ be a real or a complex JB*-triple, $v$ a minimal tripotent in $E$ and $e$ a tripotent in $E_1(v)$. Then $v \in E_2(e) \cup E_1(e)$.*

PROOF. If $E$ is a complex JB*-triple, the proof follows from [**9**, Lemma 2.1].

Suppose now, that $E$ is a real JB*-triple. By [**16**, Lemma 4.2 and Theorem 4.4], the bidual, $E^{**}$, of $E$ is a real JBW*-triple with separate weak*-continuous triple product extending the product of $E$. Therefore, given a tripotent $e \in E$ we can assure, by Banach-Alaouglu's Theorem, that

$$(E^{**})^j(e) = \overline{E^j(e)}^{w^*} \quad \text{and} \quad (E^{**})_k(e) = \overline{E_k(e)}^{w^*},$$

for every $j \in \{0, 1, -1\}$, $k \in \{0, 1, 2\}$. As a consequence, every minimal tripotent in $E$ is also a minimal tripotent in $E^{**}$. Thus, we can assume from now on, that $E$ is a real JBW*-triple.

By [**28**, Theorem 3.6] there are two weak*-closed ideal $A$ and $N$ of $E$ such that

$$E = A \oplus^\infty N,$$

where $A$ is the weak*-closed real linear span of all minimal tripotents of $E$, $N$ contains no minimal tripotents and $A \perp N$. Moreover it follows from the proof of [**28**, Theorem 3.6] that $A$ can be decomposed in the following $\ell_\infty$-sum

$$A = \oplus^\infty C_\alpha,$$

where each $C_\alpha$ is real Cartan factors or a complex Cartan factors regarded as a real JB*-triple and every minimal tripotent of $E$ belongs to a unique $C_\alpha$. Thus we can suppose that $v \in C_\gamma$ for a unique $\gamma$. Since for $\beta \neq \alpha$ we have $C_\alpha \perp C_\beta$, then $E_1(v) = (C_\gamma)_1(v)$. Therefore we can assume that $E = C_\gamma$ is a real Cartan factor or a complex Cartan factor regarded as real.

If $E$ is a complex Cartan factor regarded as real the statement follows from [**9**, Lemma 2.1].

Suppose now that $E$ is a reduced real Cartan factor. Then $v$ is a minimal tripotent in the complexification $\widehat{E}$ of $E$. Therefore $v \in \widehat{E}_2(e) \cup \widehat{E}_1(e)$. But, since $\tau(v) = v$ and $\tau(e) = e$, we have $v \in \left(\widehat{E}_2(e) \cup \widehat{E}_1(e)\right)^\tau = E_2(e) \cup E_1(e)$.

Finally, we assume that $E$ is a non reduced real Cartan factor. By [**22**, Table 1, page 210] (see also [**25**, 11.9]), $E$ is one of the following $I^{\mathbb{H}}_{2p,2q}$, $III^{\mathbb{H}}_{2n}$, $IV^{n,0}_n$, $V^{\mathbb{O}}$. By [**22**, Proposition 5.8], given two minimal tripotents $u, v \in E$, there is an automorphism of $E$ interchanging $u$ and $v$. This implies that it is enough to check the statement of the lemma for a particular minimal tripotent in each one of the previous four factors to finish the proof. Therefore, the statement follows from Remark 2.1 above. $\square$

Let $U$ be a real or complex JB*-triple. We recall (see [**7**]) that an ordered triplet $(v, u, \tilde{v})$ of tripotents in $U$, is called a *trangle* if $v \perp \tilde{v}$, $u \vdash v$, $u \vdash \tilde{v}$ and $v = Q(u)\tilde{v}$. If $u \vdash v$, we say $(v, u)$ form a *pre-trangle*. It is easy to see that $(u, v, \tilde{v})$ form a trangle with $\tilde{v} = Q(u)v$. An ordered quadruple $(u_1, u_2, u_3, u_4)$ of tripotents is called a *quadrangle* if $u_1 \perp u_3$, $u_2 \perp u_4$, $u_1 \top u_2 \top u_3 \top u_4 \top u_1$ and $u_4 = 2\{u_1, u_2, u_3\}$ (The Jordan identity assures that the above equality is still true if the indices are permutated cylcically, e.g. $u_2 = 2\{u_3, u_4, u_1\}$). If $u_1$, $u_2$, $u_3$ are tripotents such that $u_1 \perp u_3$, $u_1 \top u_2 \top u_3$, we say $(u_1, u_2, u_3)$ form a *pre-quadrangle*). In this case we have that $u_4 = 2\{u_1, u_2, u_3\}$ is a tripotent and $(u_1, u_2, u_3, u_4)$ form a quadrangle. The following lemma can be obtained by applying Peirce rules and the definition of quadrangle.

LEMMA 2.3. *Let $(u_1, u_2, u_3, u_4)$ be a quadrangle in a real or complex JB*-triple $U$. Then $\varepsilon((u_1 + u_2 + u_3 + u_4)$ (respectively, $\varepsilon((u_1 + u_2 + u_3 - u_4))$ is a tripotent if, and only if, $|\varepsilon| = \pm 2^{-1}$ (respectively, $|\varepsilon| = \pm 2^{-\frac{1}{2}}$).*

Lemma 2.2 allows us to translate the result known as "Triple System Analyzer" (see [**7**, proposition 2.1]), to the setting of real JB*-triples by replacing [**9**, Lemma 2.1] in the proof of [**7**, Proposition 2.1] with Lemma 2.2.

Proposition 2.4. *Let $U$ be a real or complex JBW*-triple containing a minimal tripotent $v$. Let $u$ be a tripotent in $U_1(v)$. Then exactly one of the following 3 cases will occur:*

(1) *$u$ is minimal in $U$. This ocurrs if and only if $u$ and $v$ are colinear.*
(2) *$u$ is not minimal in $U$ but is minimal in $U_1(v)$. In this case $(v,u)$ form a pre-trangle and $\tilde{v} = \{u,v,u\}$ is a minimal tripotent in $U$.*
(3) *Finally if $u$ is not minimal in $U_1(v)$, then there exists two orthogonal minimal tripotents of $U$, $u_1$, $\tilde{u}_1$, contained in $U_1(v)$, such that $u = u_1 + \tilde{u}_1$. Moreover $\tilde{v} = \{u,v,u\}$ is a minimal tripotent of $U$ and $(u_1, v, \tilde{u}_1, \tilde{v})$ form a quadrangle.*

Let $U$ be a real or complex JB*-triple. We recall that the rank of a $U$, $r(U)$, is the minimal cardinal number satisfying $card(S) \leq r$ whenever $S$ is an orthogonal subset of $U$, that is $0 \notin S$ and $x \perp y$ for every $x \neq y$ in $S$. The rank of a JBW*-triple is preserved by surjective isometries (see [**16**, Proposition 3.8 and the Proof of Theorem 4.8]).

Corollary 2.5. *Let $v$ be a minimal tripotent in a real or complex JBW*-triple $U$. Then rank $(U_1(v)) \leq 2$.*

Remark 2.6. Following [**22**] we recall that a real spin factor is a Banach space $E$ such that $E = X_1 \oplus^{\ell_1} X_2$, where $X_1$,$X_2$ are closed linear subspaces of a real Hilbert space, $X$, of dimension greater or equal three, such that $X_2 = {X_1}^{\perp}$, with triple product

$$\{x,y,z\} = \langle x/y \rangle\, z + \langle z/y \rangle\, x - \langle x/\bar{z} \rangle\, \bar{y},$$

where $\langle ./. \rangle$ is the inner product in $X$ and the involution $x \to \bar{x}$ on $E$ is defined by $\bar{x} = (x_1, -x_2)$ for every $x = (x_1, x_2)$. When $X_1$ and $X_2$ are both non zero, then it is easy to check that the set of minimal tripotents of $E$ is

$$\mathrm{MinTrip}(E) = \left\{ \frac{1}{2}(x_1 + x_2) : x_1 \in X_1, x_2 \in X_2 \text{ and } \|x_1\| = \|x_2\| = 1 \right\}.$$

Let $u = \frac{1}{2}(x_1 + x_2)$ be a minimal tripotent in $E$. It is easy to see that $E_0(u) = \mathbb{R}\bar{u}$, $E_2(u) = \mathbb{R}u$, and $E_1(u) = \big(\{x_1\}^{\perp} \cap X_1\big) \oplus \big(\{x_2\}^{\perp} \cap X_2\big)$. When $X_i = 0$ for some $i = 1, 2$, then it is easy to see that

$$\mathrm{MinTrip}(E) = \{x : x \in E, \|x\| = 1\}.$$

In the latter case, given a tripotent $e \in E$ we have $E_0(e) = E_1(e) = 0$, $E_2(e) = E$, $E^1(e) = \mathbb{R}e$, and $E^{-1}(e) = \{e\}^{\perp}$.

When $X_1$ and $X_2$ are non zero, then $\mathrm{rank}(E) = 2$, while $\mathrm{rank}(E) = 1$ whenever $X_1$ or $X_2$ is zero.

Let $(u, v, \tilde{u})$ be a trangle in $E$ with $u, \tilde{u}$ minimal. We note that in this case rank $(E) = 2$ and hence $X_1, X_2 \neq 0$. One can check that $u, v, \tilde{u}$ must have the following form $u = \frac{1}{2}(x_1 + x_2)$, $\tilde{u} = \pm\frac{1}{2}(x_1 - x_2)$, and $v = y$, where $\|x_1\| = \|x_2\| = 1$, $y$ is in the disjoint union of $\big(\{x_1\}^{\perp} \cap X_1\big)$ and $\big(\{x_2\}^{\perp} \cap X_2\big)$, and $\|y\| = 1$. Moreover, since $\{u, v, \tilde{u}\} = \frac{1}{2}v$, it can be concluded that $\varepsilon(u + v + \tilde{u})$ is a minimal tripotent if and only if $\varepsilon = \pm\frac{1}{2}$ while $\varepsilon(u + v - \tilde{u})$ is a tripotent if and only if $\varepsilon = \pm\frac{1}{\sqrt{2}}$.

LEMMA 2.7. *Let $\mathcal{U}$ be a JBW*-triple and let $u$, $v$ two orthogonal minimal tripotents in $\mathcal{U}$. Then $\mathcal{U}_2(u+v)$ is $\mathbb{C} \oplus^{\infty} \mathbb{C}$ or a spin factor.*

PROOF. Since $\mathcal{U}_2(u + v)$ is a JBW*-triple and $u$, $v$ are orthogonal minimal tripotents in $\mathcal{U}_2(u+v)$, we can assume without loss of generality that $\mathcal{U} = \mathcal{U}_2(u + v)$. Under this assumptions, applying [**7**, Lemma 2.4], we have

$$\mathcal{U} = \mathbb{C}u \oplus \mathbb{C}v \oplus \mathcal{W} ,$$

where $\mathcal{W} = \mathcal{U}_1(u) \cap \mathcal{U}_1(v)$ is a JBW*-triple.

If $\mathcal{W} = \{0\}$, then it is easy to see that $\mathcal{U} = \mathbb{C} \oplus^{\infty} \mathbb{C}$. Suppose dim $(W) = 1$. In this case $\mathcal{W}$ is generated by a minimal tripotent $w$. Therefore, when we apply Proposition 2.4, it follows that only case (1) or (2) can occur. Since, $(u, w, v)$ form a pre-quadrangle if $w$ is minimal in $\mathcal{U}$, we could extend it to a quadrangle (see [**7**, Proposition 1.7]), which contradicts that dim $(\mathcal{U}) = 3$. Thus, only case (2) occur, and hence $w$ governs $u$ and $v$. Now we can define a scalar product $\langle ./. \rangle : \mathcal{U} \times \mathcal{U} \to \mathbb{C}$, by

$$\langle \alpha u + \beta v + \gamma w / \delta u + \lambda v + \mu w \rangle = \frac{1}{2}(\alpha\bar{\delta} + \beta\bar{\lambda} + 2\gamma\bar{\mu})$$

and a conjugation $^{-}: \mathcal{U} \to \mathcal{U}$ given by $\bar{u} = v$, $\bar{v} = u$ and $\bar{w} = w$, which gives $\mathcal{U}$ a spin factor structure.

Suppose now dim $(\mathcal{W}) \geq 2$. By [**7**, Case 3. Proposition, page 312], it is enough to show that rank $\mathcal{U}_1(u) = 2$.

We claim that $\mathcal{W}$ contains a minimal tripotent $w_1$. Otherwise, taking a tripotent $e$ in $\mathcal{W}$ we have $e$ is not minimal in $\mathcal{W}$. By Proposition 2.4(3), we can assure that there are two minimal tripotents in $\mathcal{U}$ contained in $\mathcal{W}$ (and hence minimal in $\mathcal{W}$) such that $e$ is the sum of them, which contradicts that $\mathcal{W}$ contains no minimal tripotents.

Suppose first that $w_1$ is also minimal in $\mathcal{U}$. Then $Q(u+v)(w_1) \in \mathcal{W}$ is a minimal tripotent in $\mathcal{U}$, which is orthogonal to $w_1$. Therefore, rank $\mathcal{U}_1(u) \geq 2$, and it follows from Corollary 2.5 that rank $\mathcal{U}_1(u) = 2$.

Suppose now that $w_1$ is not minimal in $\mathcal{U}$. If $\mathcal{W}_0(w_1) \neq \{0\}$, there exits a tripotent in $\mathcal{W}_0(w_1) \subseteq \mathcal{W}$ that is orthogonal to $w_1$, therefore rank $\mathcal{U}_1(u) = 2$. Suppose, on the contrary, that $\mathcal{W}_0(w_1) = \{0\}$. Since dim $(\mathcal{W}) \geq 2$ and

$$\mathcal{W} = \mathcal{W}_2(w_1) \oplus \mathcal{W}_1(w_1) = \mathbb{C}w_1 \oplus \mathcal{W}_1(w_1)$$

then, following the same reasoning given to obtain $w_1 \in \mathcal{W}$, it follows that there exist another minimal tripotent $w_2$ in $\mathcal{W}_1(w_1)$. If $w_2$ is not minimal in $\mathcal{W}$, by Proposition 2.4(2), then $Q(w_2)(w_1)$ is a minimal tripotent in $\mathcal{W}$ which is orthogonal to $w_1$. This implies rank $\mathcal{W} \geq 2$ and hence rank $\mathcal{U}_1(u) = 2$, as we wanted. We can finally suppose that $w_2$ is minimal in $\mathcal{W}$ and hence, by Proposition 2.4(1), $w_1 \top w_2$. In this case we have two minimal tripotents $w_1$,$w_2$ in $\mathcal{W}$ that are not minimal in $\mathcal{U}$. By Proposition 2.4(2), applied to $u$, $w_1$ and $v$, $w_2$, we obtain $\{w_1, u, w_1\}$ is a minimal tripotent in $\mathcal{U}_0(u) = \mathbb{C}v$. On the other hand, by the Peirce rules $\{w_1, u, w_1\} \in \mathcal{U}_0(w_2)$. This contradicts $v \in \mathcal{U}_2(w_2)$. □

In [**22**, page 215], Kaup affirms that $E_2(u + v)$ is a spin factor whenever $u$, $v$ are minimal tripotents in the real JBW*-triple $E = I^{\mathbb{R}}_{n,m}$. Since the latter is a real reduced JBW*-triple, our next result include the above affirmation.

Corollary 2.8. *Let $E$ be a real reduced JBW*-triple and let $u$, $v$ be orthogonal minimal tripotent in $E$. Then $E_2(u + v)$ is $\mathbb{R} \oplus^\infty \mathbb{R}$ or a real spin factor.*

Proof. Clearly $E_2(u+v) = \mathbb{R}u \oplus \mathbb{R}v \oplus [E_1(u) \cap E_1(v)]$. If $E_1(u) \cap E_1(v)] = \{0\}$ , we have that $E_2(u + v)$ can be identified as real JBW*-triple with $\mathbb{R} \oplus^\infty \mathbb{R}$. Otherwise, since $u$, $v$ are two orthogonal minimal tripotents in $\widehat{E}$, we have $\widehat{E}_2(u + v)$ is a JBW*-triple with $dim \widehat{E}_2(u + v) \geq 3$. Now by the above lemma we have that $\widehat{E}_2(u + v)$ is a spin factor, so $E_2(u + v)$ is a real spin factor. □

The next proposition summarizes some known facts about surjective isometries between real JB*-triples from [**16**].

Proposition 2.9. *Let $\Phi : E \to F$ be a surjective linear isometry between two real JB*-triples. The following assertions holds*

(1) $\Phi(x) \perp \Phi(y)$ *iff* $x \perp y$*;*
(2) $\Phi(\{x, x, x\}) = \{\Phi(x), \Phi(x), \Phi(x)\}$*;*
(3) *For every tripotent $e \in E$, $\Phi$ maps the spaces $E^1(e)$, $E_0(e)$, and $E^{-1}(e) \oplus E_1(e)$ into the corresponding spaces with respect to $\Phi(e)$;*

(4) $\Phi$ *preserves the symmetrized triple product*

$$< x, y, z >= \frac{1}{3}(\{x, y, z\} + \{z, x, y\} + \{y, z, x\}).$$

PROOF. By passing to the bi-transpose of $\Phi$ we can suppose that we have a surjective weak$^*$-continuous linear isometry between two real JBW$^*$-triples. Since in a real JBW$^*$-triple the algebraic elements are dense (compare $(i) \Rightarrow (ii)$ in the proof of [**16**, Theorem 4.8]), the statement follows from [**16**, Proposition 3.8, Theorem 4.8]. □

When we have a surjective real linear isometry from a complex Cartan factor with a unitary element to another complex Cartan factor, we can deduce, from the last statement of the above proposition, that the isometry is in fact a triple isomorphism.

COROLLARY 2.10. *Let $\Phi : \mathcal{E} \to \mathcal{F}$ be a surjective real linear isometry between two complex Cartan factors. Suppose that $\mathcal{E}$ contains a unitary element $u$. Then $\Phi$ is a triple isomorphism.*

PROOF. We may assume rank of $\mathcal{E} > 1$. By [**22**, Proposition 5.7], we have $\mathcal{F} = \Phi(\mathcal{E}) = \Phi(\mathcal{E}_2(u)) = \mathcal{F}_2(\Phi(u))$, which assures that $v = \Phi(u)$ is a unitary element in $\mathcal{F}$. It is well known that $\mathcal{E}$ (respectively, $\mathcal{F}$) is a JBW$^*$-algebra with product $x \circ_1 y = \{x, u, y\}$ and involution $x^{\sharp_1} = \{u, x, u\}$ (respectively, $x \circ_2 y = \{x, v, y\}$, $x^{\sharp_2} = \{v, x, v\}$) [**33**, Proposition 19.13]. Moreover, in this case, the the triple product is determined by the algebraic structure through the identity

$$\{x, y, z\} = (x \circ y^*) \circ z + (z \circ y^*) \circ x - (x \circ z) \circ y^*.$$

Therefore, $\Phi$ is a unital surjective real linear isometry between two JBW$^*$-algebras. By Proposition 2.9 (4), $\Phi$ preserves the symmetrized triple product and is unital. Then it can be easily seen that $\Phi\sharp_1 = \sharp_2\Phi$, $\Phi(x \circ_1 y) = \Phi(x) \circ_2 \Phi(y)$, $(x, y \in \mathcal{E})$. Thus $\Phi$ is a Jordan $^*$-isomorphism and hence a triple isomorphism. □

The following result is the main tool for the study of surjective isometries between real reduced JB$^*$-triples.

THEOREM 2.11. *Let $\Phi : E \to F$ be a surjective linear isometry between two real reduced JBW$^*$-triples. Then $\Phi$ preserves quadrangles consisting of minimal tripotents. Moreover, if $(u, v, \widetilde{u})$ is a trangle in $E$ with $u, \tilde{u}$ minimal, then $(\Phi(u), \Phi(v), \Phi(\widetilde{u}))$ is a trangle in $F$.*

PROOF. By Proposition 2.9, $\Phi$ preserves tripotents and the relations of minimality and orthogonality between them. $\Phi$ also preserves colinearity since $E$ and $F$ are reduced. Hence, if $(u_1, u_2, u_3, u_4)$ is a quadrangle of minimal tripotents in $E$, we have that

$(\Phi(u_1), \Phi(u_2), \Phi(u_3), \Phi(u_4))$ form a quadrangle unless maybe for the property $\Phi(u_4) = 2\{\Phi(u_1), \Phi(u_2), \Phi(u_3)\}$. The rest of the proof is devoted to prove the last equality.

Denote by $v_i = \Phi(u_i)$ for $i = 1, 2, 3, 4$. By Corollary 2.8, $F_2(v_1 + v_3)$ is a real spin factor of dimension $\geq 4$ and rank 2, since $v_2$ and $v_4$ belongs to $F_2(v_1 + v_3)$. In the other hand, we have that $2\{v_1, v_2, v_3\} = Q(v_1 + v_3)(v_2)$ is a minimal tripotent in $F$ orthogonal to $v_2$, since $Q(e)$ is an automorphism. Therefore, since in a real Spin factor of rank 2, the orthogonal space relative to a minimal tripotent has dimension 1 (see Remark 2.6), we have $2\{v_1, v_2, v_3\} = \pm v_4$. Suppose that $2\{v_1, v_2, v_3\} = -v_4$. By Lemma 2.3, $\frac{1}{\sqrt{2}}(u_1 + u_2 + u_3 - u_4)$ is a tripotent while its image by $\Phi$, $\frac{1}{\sqrt{2}}(v_1 + v_2 + v_3 - v_4)$, is not a tripotent, which contradicts that $\Phi$ preserves tripotents.

To see the last statement let $(u, v, \widetilde{u})$ be a trangle in $E$ with $u, \tilde{u}$ minimal. As we have seen in the first part of the proof,

$$\Phi|_{E_2(u+\tilde{u})} : E_2(u + \tilde{u}) \to F_2(\Phi(u) + \Phi(\tilde{u}))$$

is a surjective isometry between two real spin factors of rank 2 (compare Remark 2.6). Since $v$ is a tripotent in $E_1(u) \cap E_1(\tilde{u})$, and $E$ is reduced, then $\Phi(v) \in F_1(\Phi(u)) \cap F_1(\Phi(\tilde{u}))$. By Lemma 2.2, Proposition 2.4 and the fact that $\Phi(v)$ is not minimal, it follows that $\Phi(v) \vdash \Phi(u)$ and $\Phi(v) \vdash \Phi(\tilde{u})$. Therefore we only have to show that $Q(\Phi(v))(\Phi(u)) = \Phi(\tilde{u})$ to get the statement. By Peirce rules and since $Q(\Phi(v))$ is an automorphism on $F_2(\Phi(v)$, we have $Q(\Phi(v))(\Phi(u)) \in F_0(\Phi(u)) = \mathbb{R}\Phi(\tilde{u})$, which implies that $Q(\Phi(v))(\Phi(u)) = \pm\Phi(\tilde{u})$. Suppose that $Q(\Phi(v))(\Phi(u)) = -\Phi(\tilde{u})$, in which case $(\Phi(u), \Phi(v), -\Phi(\tilde{u}))$ is a trangle. By Remark 2.6, $\frac{1}{2}(u + v + \tilde{u})$ is a tripotent in $E$, while $\Phi(\frac{1}{2}(u + v + \tilde{u})) = \frac{1}{2}(\Phi(u) + \Phi(v) + \Phi(\tilde{u}))$ is not a tripotent. $\square$

Remark 2.12. Let $C$ be a complex Cartan factor of rank $> 1$. By [**7**], there exist a rectangular grid, a symplectic grid, an hermitian grid, a spin grid, or an exceptional grid of type one or two built up from trangles and quadrangles in $C$. Moreover, if $C$ is not a type $III$ Cartan factor, then it is the weak*-closed linear span of the elements of the corresponding grid and all the non vanishing triples product among the elements of the grid are those associated to quadrangles of minimal tripotents or to trangles $(u, v, \tilde{u})$ with $u, \tilde{u}$ minimal.

If $C$ is a type $I^{\mathbb{R}}_{n,m}$, $I^{\mathbb{H}}_{2p,2q}$, $II^{\mathbb{R}}_{n}$ or $III^{\mathbb{H}}_{2p}$ real Cartan factor, then we can define, as in the complex case, a grid built up from quadrangles of minimal tripotents (compare [**22**, Proofs of Propositions 5.14, 5.16 and 5.17]).

Let $C = IV_n^{r,s} = X_1 \oplus^{\ell_1} X_2$ be a real spin factor. We can always assume that $r \geq s$. Let $\{e_i\}_{i\in I}$ and $\{f_j\}_{j\in J}$ be orthonormal basis of $X_1$ and $X_2$, respectively. By hypothesis, there is a set $J_1$ such that $I = J \cup J_1$. We define $u_i = 2^{-1}(e_i + f_i)$, $\tilde{u}_i = 2^{-1}(e_i - f_i) = \overline{u}_i$, whenever $i \in J$ while $u_i = e_i$ for all $i \in I \backslash J = J_1$. It is easy to see that

(1) $u_i$ is a minimal tripotent for all $i \in J$ and $u_i$ is a tripotent for all $i \in J_1$;
(2) $u_j \vdash u_i$, $u_j \vdash \tilde{u}_i$, $Q(u_j)(u_k) = -\overline{u}_k$, for all $j \in J_1$, $i \in J$ and $k \in I$;
(3) $(u_i, u_j, \tilde{u}_i, \tilde{u}_j)$ are odd quadrangles for $i \neq j$, $i, j \in J$;
(4) $C = C_2(u_i + \tilde{u}_i) = C_2(u_j)$ for all $i \in J$ and $j \in J_1$;
(5) The non vanishing triple products among elements of the set correspond to those described in (2) and (3).

The family $\{u_i, \tilde{u}_i, u_j : i \in J, j \in J_1\}$ is called a *real spin grid.*

If $C$ is a type $I_{n,n}^{\mathbb{C}} = \{z \in L(H) : z^* = z\}$ (respectively, $III_n^{\mathbb{R}}$ or $II_{2p}^{\mathbb{H}}$) real Cartan factor (see [**22**, Theorem 4.1]). The real "hermitian grid" in this case is as follows $\{v_{\alpha,\beta}^l\}$, where $v_{\alpha,\beta}^l = (le_\alpha \otimes e_\beta + e_\beta \otimes le_\alpha)$ if $\alpha \neq \beta$, $v_{\alpha,\alpha}^l = e_\alpha \otimes e_\alpha$, such that $\{e_\alpha\}$ is an orthonormal basis of the complex (respectively, real or Hilbertian) Hilbert space $H$ (respectively, $X$ or $P$), $l = 1, i$ (respectively, $l = 1$, $l = 1, i, j, k$) and $(h \otimes k)(x) = (x|k)h$.

Finally, we study the grids in the exceptional real Cartan factors. Let $C$ be a type $V^{\mathbb{O}_0}$ real Cartan factor. It is easy to check that $C$ contains two minimal orthogonal tripotents $v, \tilde{v}$, such that dim $(C_2(v+\tilde{v})) = 8$ and $C_1(v+\tilde{v}) \neq 0$. Therefore, having in mind that $C$ is reduced and Proposition 2.4, the proof of [**7**, Proposition in page 322] can be literally adapted to get an *exceptional grid of the first type* in $C$. By adapting the proof of [**7**, Proposition in page 323], the above arguments can be applied to get an *exceptional grid of the second type* in $VI^{\mathbb{O}_0}$.

In [**6**, Lemma 2.5] it is proved that every surjective real linear isometry between two complex Cartan factors of rank greater than one is $w^*$-continuous. Our next result shows that the same conclusion still being true for surjective linear isometries between real JBW*-triples.

Let $X$ be a real or complex Banach space. Following [**11**], we define $\mathcal{B}(X)$ as the set of all functionals $\varphi \in X^{***}$ such that for every non-empty closed convex subset $C \subset X$, the mapping

$$\varphi : (\overline{C}^{\sigma(X^{**},X^*)}, \sigma(X^{**}, X^*)) \to \mathbb{F}$$

has at least one point of continuity, where $\mathbb{F}$ denotes the base field. The *universal frame* of $X$, $\gamma(X)$, is defined as

$$\gamma(X) = (\mathcal{B}(X) \cap j(X)^\circ)_\circ.$$

The space $X$ is called *well-framed* iff $\gamma(X) = j(X)$. In [**11**, Théorème 16] it is shown that the well-framed property is inherited by subspaces. The duals (and hence the preduals) of von Neumann algebras and real or complex JBW*-triples are examples of well-framed Banach spaces (compare [**11**, Théorème 18], [**15**], [**26**, Lemma 2.2]).

LEMMA 2.13. *Every linear surjective isometry between real JBW*-triples is $w^*$-continuous.*

PROOF. By [**26**, Lemma 2.2], the predual of every real JBW*-triple is well-framed. This fact assures that the predual of every JBW*-triple satisfies the condition (*) of [**12**, Theorem V.1]. Finally, the statement follows by [**12**, Theorem VII.8]. □

Our next goal will consist in proving that the surjective real linear isometries between two real reduced Cartan factors are triple isomorphisms.

PROPOSITION 2.14. *Let $\Phi : E \to F$ be a surjective linear isometry between two real reduced Cartan factors. Then $\Phi$ is a triple isomorphism.*

PROOF. Let us assume first that both factors are of rank greater than one. Since, as we have seen in Remark 2.12 above, each reduced real Cartan factor of rank greater than one, except the types $III_n^{\mathbb{R}}$, $I_{n,n}^{\mathbb{C}}$, $II_{2p}^{\mathbb{H}}$, and $VI^{\mathbb{O}}$, is the $w^*$-closed real linear span of a grid built up from quadrangles of minimal tripotents and trangles $(u, v, \widetilde{u})$ with $u, \tilde{u}$ minimal, the result follows from Theorem 2.11 and Lemma 2.13. When both factors are reduced of rank one, then they coincide with a type $I_{1,n}^{\mathbb{R}}$ real Cartan factor (compare [**22**, Proposition 5.4] and [**25**, 11.9]). By [**22**, Lemma 5.12] every surjective isometry between type $I_{1,n}^{\mathbb{R}}$ real Cartan factors is a triple isomorphism.

The factors $VI^{\mathbb{O}}$, $I_{n,n}^{\mathbb{C}}$, $II_{2p}^{\mathbb{H}}$, and $III_n^{\mathbb{R}}$ are JB-algebras and hence every surjective isometry between them is a triple isomorphism by [**17**]. □

Following, [**22**, page 214], we denote by $\mathcal{IS}$ the class of all real JB*-triples $E$ where the surjective (real)-linear isometries $\Phi : E \to E$ coincides with the triple automorphisms. Our Proposition 2.14 assures that every real reduced Cartan factor is in the class $\mathcal{IS}$. The exceptional real Cartan factors $V^{\mathbb{O}_0}$ and $VI^{\mathbb{O}_0}$ are real reduced Cartan factors (compare [**22**, Table 1, page 210]), and hence, they are in the class $\mathcal{IS}$. This gives a positive answer to the question posed by Kaup in [**22**, page 217].

Our techniques (Proposition 2.14) are not valid for non-reduced real or complex Cartan factors. The remaining (i. e. the non-reduced real or complex Cartan factor of rank greater than one) are all the complex Cartan factors of rank $> 1$, $I^{\mathbb{H}}_{2p,2q}$ and $III^{\mathbb{H}}_{2n}$ (compare [**22**, Table 1]). In [**6**, Proposition 2.6], Dang shows that every complex Cartan factor of rank $> 1$ is in the class $\mathcal{IS}$. Finally, Kaup shows in [**22**, Theorem 5.18] that every non-exceptional real or complex Cartan factor of rank greater than one is in the class $\mathcal{IS}$. In fact, our result and Kaup's one overlap showing that every non-exceptional real Cartan factor of rank $> 1$ is in the class $\mathcal{IS}$. The following corollary follows now from these results.

Corollary 2.15. *Every real or complex Cartan factor of rank greater than one is in the class $\mathcal{IS}$.*

The previous corollary allows us to extend [**22**, Theorem 5.18] to exceptional real Cartan factors of rank $> 1$, by the same arguments given in [**22**].

Corollary 2.16. *Let $C$ be a real or complex Cartan factor of rank greater than one and $F$ a real JBW*-triple. Then a bijective $\mathbb{R}$-linear map $\Phi : C \to F$ is an isometry if, and only if, it is a triple isomorphism.*

## 3. Real JB*-triples

We begin with the following Gelfand-Naimark type theorem for real JB*-triples whose proof, as in the complex case (see [**10**]), is based in the atomic decomposition of a real JBW*-triple.

Proposition 3.1. *Let $E$ be a real JB*-triple. Then $E$ can be isometrically embedded as a real subtriple of an $\ell_\infty$-sum of real Cartan factors and complex Cartan factors regarded as real. More concretely, if $A$ denotes the atomic part of $E^{**}$ and $\pi : E^{**} \to A$ is the canonical projection, then $A$ is an $\ell_\infty$-sum of real or complex Cartan factors and the mapping $\pi \circ j : E \to A$ is an isometric triple embedding.*

Proof. It is known that $E^{**}$ is a real JBW*-triple whose triple product extends the product of $E$. In particular, $j : E \to E^{**}$ is a triple homomorphism. We recall [**28**, Theorem 3.6] that $E^{**}$ decomposes in the form

$$E^{**} = A \oplus^\infty N,$$

where $A$ and $N$ are weak*-closed ideals, $A$ being the weak*-closed real linear span of all minimal tripotents of $E^{**}$, $N$ contains no minimal tripotents and $A \perp N$. It follows by the proof of [**28**, Theorem 3.6] that $A$ is an $\ell_\infty$-sum of real and complex Cartan factors. It is clear that $\pi : E^{**} \to A$ is a triple homomorphism, and hence, $\pi \circ j$ is a triple homomorphism with norm less or equal to one. Therefore, we only have to show that $\pi \circ j$ is an isometry to get the statement.

Let $x \in E$ with $\|x\| = 1$. By the Kreim-Milman, Hahn-Banach and Banach-Alaoglu's Theorems there exists a extreme point of the unit ball of $E^*$, $\varphi$, such that $\varphi(x) = 1$. By [**28**, Corollary 2.1 and Lemma 2.7], there is a minimal tripotent $u \in E^{**}$ such that $\varphi = \varphi \circ P^1(u)$. In particular $\varphi(N) = 0$ and hence $\varphi = \varphi \circ \pi$. Finally

$$1 = \|x\| = \|j(x)\| \geq \|\pi(j(x))\| \geq \varphi(\pi(j(x))) = \varphi(x) = 1,$$

which assures that $\pi \circ j$ is an isometry. □

The following Theorem extends [**6**, Theorem 3.1] to the real setting.

Theorem 3.2. *Let $\Phi : E \to F$ be a surjective isometry between two real JB*-triples. Suppose that $E^{**}$ does not contain (real or complex) Cartan factors of rank one. Then $\Phi$ is a triple isomorphism.*

Proof. The mapping $\Phi^{**} : E^{**} \to F^{**}$ is a surjective weak$^*$ continuous real-linear isometry between JBW$^*$-triples. By Proposition 2.9, $\Phi^{**}$ preserves tripotents and the relations of minimality and orthogonality between them. Therefore, $\Phi^{**}$ maps the atomic part of $E^{**}$, $A_{E^{**}} = \oplus^{\ell_\infty} C_\alpha$, into the atomic part of $F^{**}$, $A_{F^{**}} = \oplus^{\ell_\infty} C_\beta$. Thus

$$\Psi = \Phi^{**}|_{A_{E^{**}}} : \oplus^{\ell_\infty} C_\alpha \to \oplus^{\ell_\infty} C_\beta$$

is a surjective real-linear isometry from an $\ell_\infty$-sum of a family of real or complex Cartan factors of rank $> 1$ to another $\ell_\infty$-sum of the same type.

We claim that for every $C_\alpha$, there is a unique $C_\beta$ such that $\Psi(C_\alpha) \subseteq C_\beta$. Indeed, since every real or complex Cartan factor of rank greater than one is spanned by a grid built up from quadrangles of minimal tripotents and trangles, $(u, v, \tilde{u})$, with $u, \tilde{u}$ minimal (compare Remark 2.12 and [**7**]), we only have to show that every quadrangle or trangle as above is mapped by $\Psi$ into a unique $C_\beta$. Let $(u_1, u_2, u_3, u_4)$ be a quadrangle of minimal tripotents in a fixed $C_\alpha$. Since $\Psi$ maps minimal tripotents into minimal tripotents and every minimal tripotent belongs to a unique $C_\beta$, it follows that each $\Psi(u_i)$ belongs to a unique $C_\beta$. Now, if $\Psi(u_1)$ and $\Psi(u_2)$ (respectively, $\Psi(u_2)$ and $\Psi(u_3)$ or $\Psi(u_3)$ and $\Psi(u_4)$) lie in different factors then they, an hence $u_1$ and $u_2$ (respectively, $u_2$ and $u_3$ or $u_3$ and $u_4$), must be orthogonal, which is impossible. Let now $(u, v, \tilde{u})$ be a trangle in $C_\alpha$. Since the grids formed by trangles appear only in the case of a spin factor, we can assume that $C_\alpha$ is a spin factor. By Remark 2.6, $m = 2^{-1}(u + v + \tilde{u})$ is a minimal tripotent in $C_\alpha$ which is not orthogonal to $u$ nor $\tilde{u}$. As in the case of quadrangle, this implies that the trangle is contained in a unique $C_\beta$.

Therefore, $\Psi|_{C_\alpha} : C_\alpha \to C_\beta$ is a (real) linear surjective isometry between two real or complex Cartan factors of rank greater than one and then a triple isomorphism by Corollary 2.15. This implies that $\Psi$ is a triple isomorphism. Finally, let $\pi_E : E^{**} \to A_{E^{**}}$, $\pi_F : F^{**} \to A_{F^{**}}$, $j_E$, and $j_F$ be the canonical projections of $E^{**}$ and $F^{**}$ onto their atomic parts and the canonical embeddings of $E$ and $F$ into their biduals, respectively. Since $(\pi_F \circ j_F)\circ\Phi = \Psi\circ(\pi_E\circ j_E)$, it follows, by Proposition 3.1 and the fact that $\Psi$ is a triple isomorphism, that $\Phi$ is a triple isomorphism. $\square$

REMARK 3.3. The conclusion of Theorem 3.2 is not true when $E^{**}$ contains a rank one Cartan factor. Indeed, let $E$ and $F$ be a type $I^{\mathbb{R}}_{1,n}$ and a type $IV^{n,0}_n$ real Cartan factor respectively. Then the identity map from $E$ to $F$, both regarded as n-dimensional real Hilbert spaces, is a surjective isometry which is not a triple isomorphism. Another example of this fact can be found in [**6**, Remark 2.7].

The following result is an application of our main theorem to the case of a (real) J*B-algebra extending [**6**, Corollary 3.2 ]. Following [**2**], we recall that a (real) J*B-algebra is a real Jordan Algebra with unit 1 and an involution $*$ equipped with a complete algebra norm such that $\|U_x(x^*)\| = \|x\|^3$ and $\|x^*\circ x\| \le \|x^*\circ x + y^*\circ y\|$, where $U_x(y) := 2x\circ(x\circ y) - x^2\circ y$. In [**2**, Theorem 4.4] it is shown that the complexification of every J*B-algebra is a complex JB*-algebra, and hence a complex JB*-triple, with a norm extending the original. Therefore, every J*B-algebra is a real JB*-triple with triple product $\{a,b,c\} = a\circ(b^*\circ c) + c\circ(b^*\circ a) - (a\circ c)\circ b^*$.

COROLLARY 3.4. *Let $\Phi : A \to B$ be a surjective isometry between two (real) J*B-algebras. Then $\Phi$ is a triple isomorphism. If $\Phi$ is also unital then it is a $*$-isomorphism.*

PROOF. The unit of $A$ is a unitary element of $A^{**}$ regarded as a real JB*-triple. This implies that every factor in the atomic part of $A^{**}$ contains a unitary element. If $A^{**}$ contains no real or complex rank one Cartan factors we conclude by Theorem 3.2. Otherwise, let $C$ be a real or complex Cartan factor of rank one contained in the atomic part of $A^{**}$. If $C$ is a complex Cartan factor or a reduced real Cartan factor, it follows that $C$ coincides with $\mathbb{C}$ or $\mathbb{R}$, since $C$ contains a unitary element and every tripotent is minimal. If $C$ is a non reduced rank one real Cartan factor with a unitary element, it can be seen that $C$

coincides with a type $IV_n^{n,0}$ real Cartan factor with $n \geq 3$ (compare, [**25**, 11.9] or [**22**, Proposition 5.4]).

Suppose that $A^{**}$ contains a non-trivial rank one real Cartan factor $C_\alpha \equiv IV_n^{n,0}$ $(n \geq 3)$. Now adapting the proof of Theorem 3.2 to this particular case, we can show that $F^{**}$ contains another non-trivial rank one real Cartan factor $C_\beta \equiv IV_n^{n,0}$ $(n \geq 3)$, such that $\Phi^{**}(C_\alpha) = C_\beta$. By [**22**, Lemma 5.13] we get that $\Phi^{**}|_{C_\alpha} : C_\alpha \to C_\beta$ is a triple isomorphism. The proof can be now concluded as in Theorem 3.2. □

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