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The theorem of Schur in the Minkowski plane

Rafael López

Departamento de Geometría y Topología, Universidad de Granada, 18071 Granada, Spain

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ABSTRACT

We prove a Lorentzian analogue of the theorem of Schur for spacelike (or timelike) curves in the Minkowski plane.

1. Introduction

A classical theorem in differential geometry of curves in Euclidean space $\mathbb{E}^3$ compares the lengths of the chords of two curves, one of them being a planar convex curve [1–3].

Theorem 1 (*Schur*)**.** *Let $\alpha_1, \alpha_2 : [0, L] \to \mathbb{E}^3$ be two (sufficiently smooth) curves of length L parametrized by an arc length s. Suppose that α_1 is a planar curve such that α_1 together with the chord joining its endpoints bounds a convex set in the plane. Let d_1 and d_2 denote the lengths of the chords joining the endpoints of α_1 and α_2, respectively. Assume that*

$$\kappa_1(s) \geq \kappa_2(s), \quad s \in [0, L].$$

Then

$$d_1 \leq d_2,$$

and the equality holds if and only if α_1 and α_2 are congruent.

In the words of S.S. Chern, "if an arc is 'stretched', the distance between its endpoints becomes longer" [2, p. 36]. The curvatures κ_i are assumed in the space sense, that is, non-negative by definition, even for plane curves. This result is known in the literature under the name *Schur's theorem*. Exactly, Schur [4] proved the result when both curvatures agree pointwise (based on ideas of an unpublished work of H.A. Schwarz in 1884). The proof in the general case is due to Schmidt [5]. Some generalizations and extensions can be seen in [6,7] and a historical account of this theorem appears in [7, p. 151]. When the curves are polygons, Schur's theorem is related with the so-called Cauchy's arm lemma [8,9].

In this work we present the Lorentzian version of Schur's theorem for planar curves in the Minkowski plane $\mathbb{E}^2_1$.

E-mail address: rcamino@ugr.es.

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Theorem 2. *Let $\alpha_1, \alpha_2 : [0, L] \to \mathbb{E}_1^2$ be two (sufficiently smooth) spacelike curves of length L parametrized by the arc length s. Suppose that α_1 together with the chord joining its endpoints bounds a convex set. Let d_1 and d_2 denote the lengths of the chords joining the endpoints of α_1 and α_2, respectively. Assume that*

$$\kappa_1(s) \geq |\kappa_2(s)|, \quad s \in [0, L].$$

Then

$$d_1 \geq d_2,$$

and the equality holds if and only if α_1 and α_2 are congruent. The same result holds if both curves are timelike.

Here the length d_i of each chord is measured with the Lorentzian metric, that is, $d_i = |\alpha_i(L) - \alpha_i(0)|$, $i = 1, 2$.

We point out that Schur's theorem cannot be extended for space curves in the Minkowski three-dimensional space $\mathbb{E}_1^3$, even if both curvatures κ_i agree pointwise. We now present a variety of counterexamples. Assume that $\mathbb{E}_1^3$ is the real vector space $\mathbb{R}^3$ with the metric $(+, +, -)$. We consider the (planar) spacelike curves in $\mathbb{E}_1^3$ given by

$$\alpha_1(r; s) = (r\cos(s/r), r\sin(s/r), 0), \qquad \alpha_2(r; s) = (0, r\sinh(s/r), r\cosh(s/r)),$$

with $r > 0$. Both curves are parametrized by the arc length s and the curvatures are $\kappa_i(r; s) = 1/r$, for any s, $i = 1, 2$. Fix L. Then $d_1 = d_1(r) = |\alpha_1(r; L) - \alpha_1(r; 0)| = r\sqrt{2(1 - \cos(L/r))}$ and $d_2 = d_2(r) = |\alpha_2(r; L) - \alpha_2(r; 0)| = r\sqrt{2(\cosh(L/r) - 1)}$.

- By taking the same value r in both curves, and for any $0 < L < 2\pi r$, we have $\kappa_1(r; s) = \kappa_2(r; s)$ for any s and $d_1(r) < d_2(r)$.
- Let $0 < L < \pi$. Then $\kappa_1(1; s) > \kappa_2(2; s)$ and $d_1(r) < d_2(r)$.
- For any $L > 0$, $\kappa_2(1; s) > \kappa_2(2; s)$ and $d_2(1) > d_2(2)$.

This note is organized as follows. In Section 2 we present some preliminaries about the geometry of the Minkowski plane and we show Theorem 2. In Section 3 we present some applications of Schur's theorem.

2. Preliminaries and proof of the result

Let $\mathbb{E}_1^2$ denote the Minkowski plane, that is, the real planar vector $\mathbb{R}^2$ endowed with the metric $\langle , \rangle = dx^2 - dy^2$, where (x, y) are the usual coordinates of $\mathbb{R}^2$. A vector $v \in \mathbb{E}_1^2$ is said to be spacelike if $\langle v, v\rangle > 0$ or $v = 0$, timelike if $\langle v, v\rangle < 0$, and lightlike if $\langle v, v\rangle = 0$ and $v \neq 0$. The length of a vector v is given by $|v| = \sqrt{|\langle v, v\rangle|}$. If $p, q \in \mathbb{E}_1^2$, we define the distance between p and q as $|p - q|$.

It is important to point out that, in contrast to what happens in Euclidean space, in the Minkowski ambient space we cannot define the angle between two vectors, except that both vectors are of timelike type. If $u, v \in \mathbb{E}_1^2$ are two timelike vectors, then $\langle u, v\rangle \neq 0$. In such a case, there is a Cauchy–Schwarz inequality given by

$$|\langle u, v\rangle| \geq |u|\,|v|$$

and the equality holds if and only if u and v are two proportional vectors. In the case where $\langle u, v\rangle < 0$, there exists a unique number $\theta \geq 0$ such that

$$\langle u, v\rangle = -|u|\,|v|\cosh(\theta).$$

This number θ is called the *hyperbolic angle* between u and v.

Given a regular smooth curve $\alpha : I \subset \mathbb{R} \to \mathbb{E}_1^2$, we say that α is spacelike (resp. timelike) if $\alpha'(t)$ is a spacelike (resp. timelike) vector for all $t \in I$, where $\alpha'(t) = d\alpha/dt$. If α is a spacelike curve parametrized by $\alpha(t) = (x(t), y(t))$, then $x'(t)^2 - y'(t)^2 > 0$, and so, $x'(t) \neq 0$ for any t. This means that $x := x(t) : I \to \mathbb{R}$ is a local diffeomorphism, and so, $x : I \to J = x(I)$ is a diffeomorphism between interval of the real line $\mathbb{R}$. Then we can reparametrize α as the graph of a function $y = f(x)$, $x \in J \subset \mathbb{R}$ and $1 - f'(x)^2 > 0$, $x \in J$. In a similar way, any timelike curve in $\mathbb{E}_1^2$ is the graph of a function $x = g(y)$, with $1 - g'(y)^2 > 0$, $y \in J$. In conclusion, there are no *closed* spacelike (or timelike) curves in $\mathbb{E}_1^2$.

The following result will be useful in the proof of Theorem 2.

Lemma 3. *Let $\alpha : I \to \mathbb{E}_1^2$ be a spacelike (resp. timelike) curve. Then for any $s_1, s_2 \in I$, the vector $\alpha(s_2) - \alpha(s_1)$ is spacelike (resp. timelike).*

Proof. We write α as the graph of a function f. By using the mean value theorem, the straight line joining $\alpha(s_1)$ and $\alpha(s_2)$ is parallel to other one that it is tangent at some point $s_3 \in I$. Since $\alpha'(s_3)$ is a spacelike (resp. timelike) vector, then the vector $\alpha(s_2) - \alpha(s_1)$ is also spacelike (resp. timelike). □

If $\alpha = \alpha(t)$ is a spacelike or timelike curve, we can reparametrize α by the arc length $s = s(t)$, that is, $|\alpha'(s)| = 1$ for all $s \in I$. We say then that α is parametrized by the arc length. In such a case, it is possible to define a Frenet frame $\{\mathbf{T}(s), \mathbf{N}(s)\}$ associated with each point s [10]. Here $\mathbf{T}$ and $\mathbf{N}$ are the tangent and normal vector field, respectively and they are defined by $\mathbf{T}(s) = \alpha'(s)$ and $\mathbf{N}(s)$ is a unit smooth vector field orthogonal to $\mathbf{T}(s)$ for all s. The geometry of the curve α can be described by the differentiation of the Frenet frame, which leads to the corresponding Frenet equations. If $\langle \mathbf{T}(s), \mathbf{T}(s)\rangle = \epsilon$, $\epsilon = 1$ or -1 if α is a spacelike or timelike curve, respectively, we have

$$\begin{pmatrix} \mathbf{T}'(s) \\ \mathbf{N}'(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) \\ \kappa(s) & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T}(s) \\ \mathbf{N}(s) \end{pmatrix}, \tag{1}$$

where $\kappa(s) = -\epsilon\langle \mathbf{T}'(s), \mathbf{N}(s)\rangle$ is the curvature of α. If α is a spacelike curve and it writes as the graph of a function $y = f(x)$, the curvature of α at s, $\alpha(s) = (x, f(x))$, is

$$\kappa(x) = \frac{f''(x)}{(1 - f'(x)^2)^{3/2}}.$$

Here $1 - f'(x)^2 > 0$ for all x by the spacelike condition. If α is a timelike curve being the graph of a function $x = g(y)$, the curvature at $\alpha(s) = (g(y), y)$ is given by

$$\kappa(y) = -\frac{g''(y)}{(1 - g'(y)^2)^{3/2}}.$$

Anyway, if $\kappa(s) > 0$ for all $s \in I$, then the curve α is convex.

As in Euclidean plane, there exist (spacelike or timelike) curves in $\mathbb{E}^2_1$ with constant curvature. If the curvature vanishes for all s, then α parameterizes a straight line of $\mathbb{E}^2_1$. Assume now that $\kappa(s) = c$ is a non-zero constant. After a motion of $\mathbb{E}^2_1$, the only spacelike (resp. timelike) curves in $\mathbb{E}^2_1$ with curvature c are $\alpha(s) = r(\sinh(s/r), \cosh(s/r))$ (resp. $\alpha(s) = r(\cosh(s/r), \sinh(s/r))$), with $r = 1/|c|$. Both curves describe hyperbolas of $\mathbb{R}^2$ of equations $x^2 - y^2 = -r^2$ and $x^2 - y^2 = r^2$. We will call them as the (spacelike or timelike) *circles* of $\mathbb{E}^2_1$ of radius $r > 0$.

We proceed to show Theorem 2 by distinguishing two cases:

I. *Case that α_i are spacelike curves*. We put $\alpha_i(s) = (x_i(s), y_i(s))$, $s \in I$. Since $\alpha_i'(s) = (x_i'(s), y_i'(s))$ and α_i is parametrized by the arc length, $x_i'(s)^2 - y_i'(s)^2 = 1$. Thus there exist functions $\theta_i : I \to \mathbb{R}$ such that $\alpha_i'(s) = (\cosh(\theta_i(s)), \sinh(\theta_i(s)))$. Then $\mathbf{N}_i(s) = (\sinh(\theta_i(s)), \cosh(\theta_i(s)))$ and the curvature $\kappa_i(s)$ is

$$\kappa_i(s) = -\langle \mathbf{T}_i'(s), \mathbf{N}_i(s)\rangle = \theta_i'(s).$$

We use a translation followed by a linear isometry of $\mathbb{E}^2_1$ of type

$$\begin{pmatrix} \cosh\varphi & \sinh\varphi \\ \sinh\varphi & \cosh\varphi \end{pmatrix}. \tag{2}$$

We suppose that $\alpha_1(s_1)$ and $\alpha_1(s_2)$ belong to the x-axis, that is, $y_1(s_1) = y_2(s_2) = 0$. Let $s_0 \in I$ such that $\alpha_1'(s_0)$ is parallel to the x-axis. For the curve α_2, we perform a motion of type (2) in such a way that $\alpha_2'(s_0)$ is parallel to $\alpha_1'(s_0)$. As $\kappa_1(s) \geq |\kappa_2(s)|$, then $\theta_1'(s) \geq |\theta_2'(s)|$. As $\theta_i(s_0) = 0$, then $\theta_1(s) \geq |\theta_2(s)|$ in $[s_0, s_2]$ and $\theta_1(s) \leq -|\theta_2(s)| \leq 0$ in $[s_1, s_0]$. As the function $\cosh(t)$ is increasing on t, for $t \geq 0$ and decreasing for $t \leq 0$, we have $\cosh(\theta_1(s)) \geq \cosh(|\theta_2(s)|)$ for all $s \in [s_1, s_2]$. From Lemma 3, the segment joining the endpoints $\alpha_i(s_1)$ and $\alpha_i(s_2)$ is spacelike, and so, the length of the chord between the endpoints of α_1 is $d_1 = |x_1(s_2) - x_1(s_1)|$. Then

$$\begin{aligned} d_1 &= |x_1(s_2) - x_1(s_1)| = \left|\int_{s_1}^{s_2} x_1'(s)\,\mathrm{d}s\right| = \int_{s_1}^{s_2} \cosh(\theta_1(s))\,\mathrm{d}s \\ &\geq \int_{s_1}^{s_2} \cosh(|\theta_2(s)|)\,\mathrm{d}s = \int_{s_1}^{s_2} |x_2'(s)|\,\mathrm{d}s \geq \left|\int_{s_1}^{s_2} x_2'(s)\,\mathrm{d}s\right| \\ &= |x_2(s_2) - x_2(s_1)| \geq |\alpha_2(s_2) - \alpha_2(s_1)| = d_2, \end{aligned}$$

where in the last inequality, we have use the fact that if (a, b) is a spacelike vector, then $|a| \geq \sqrt{a^2 - b^2}$. If we have the equality $d_1 = d_2$, then $\theta_1(s) = \theta_2(s)$ for all $s \in I$. This means that $\kappa_1(s) = \kappa_2(s)$ and the fundamental theorem for spacelike curves in the Minkowski plane ensures that they differ in a motion of the ambient space $\mathbb{E}^2_1$.

II. *Case that α_i are timelike curves*. The reasoning is similar and we only write the differences. Since $\langle \alpha_i'(s), \alpha_i'(s)\rangle = -1$, there exist functions θ_i such that $\alpha_i'(s) = (\sinh(\theta_i(s)), \cosh(\theta_i(s)))$. Moreover, $\mathbf{N}_i(s) = (\cosh(\theta_i(s)), \sinh(\theta_i(s)))$ and

$$\kappa_i(s) = \langle \mathbf{T}_i'(s), \mathbf{N}_i(s)\rangle = \theta_i'(s).$$

Again, after a motion of type (2), we suppose that $\alpha_1(s_1)$ and $\alpha_1(s_2)$ belong to the y-axis, that is, $x_1(s_1) = x_2(s_2) = 0$. Again, let $s_0 \in I$ denote the value so $\alpha_1'(s_0)$ is parallel to the y-axis. For α_2, we assume that $\alpha_2'(s_0)$ is parallel to $\alpha_1'(s_0)$. As $\kappa_1(s) \geq |\kappa_2(s)|$, then $\theta_1'(s) \geq |\theta_2'(s)|$. As the above case, we have $\cosh(\theta_1(s)) \geq \cosh(|\theta_2(s)|)$ for all $s \in [s_1, s_2]$, and so,

$$\begin{aligned} d_1 &= |y_1(s_2) - y_1(s_1)| = \int_{s_1}^{s_2} \cosh(\theta_1(s))\,\mathrm{d}s \geq \int_{s_1}^{s_2} \cosh(|\theta_2(s)|)\,\mathrm{d}s \\ &\geq |y_2(s_2) - y_2(s_1)| \geq |\alpha_2(s_2) - \alpha_2(s_1)| = d_2. \end{aligned}$$

In this case, we use that if (a, b) is a timelike vector, then $|b| \geq |(a, b)| = \sqrt{-a^2 + b^2}$. If $d_1 = d_2$, we conclude that $\kappa_1 = \kappa_2$ and both curves are congruent. This completes the proof of Theorem 2.

3. Applications of Schur's theorem

The theorem of Schur has some applications in problems of search for shortest curves connecting two points under certain hypotheses on the curvature bounded from above [2,3]. The first one is the following.

Theorem 4. *Let p and q be two points of $\mathbb{E}_1^2$ such that the vector $q - p$ is spacelike (resp. timelike). Denote by d, the distance between p and q. Let α be a spacelike (resp. timelike) curve joining p and q with curvature $\kappa(s) \leq 1/r$. Then the length of α is greater than or equal to the length of a circle of radius r joining p and q.*

We point out that in contrast to what happens in Euclidean plane (it is necessary that the radius r satisfies $r \geq d/2$), in the Minkowski plane there are no restrictions between the distance d and the radius r of circle through two points.

Proof. We derive the proof in the case where $q - p$ is a spacelike vector. We point out that given two points p and q, where the vector $q - p$ is spacelike, and given $r > 0$, there exists a spacelike circle joining p and q. The points p and q determine one bounded arc and two unbounded arcs in the circle through p and q. Let $L > 0$ be the length of α. After a motion, we can assume that the straight line joining p and q is parallel to the x-axis. Consider the circle $\alpha_1(s) = r(\sinh(s/r), \cosh(s/r))$ of radius r and an arc of α_1 of length L whose endpoints determine a horizontal chord C of length d_1. Letting $\alpha_2 = \alpha$, Schur's theorem insures $d_1 \geq d$. Therefore, there exists a chord S' parallel to S of length d whose endpoints lie in the trace of α_1. As $d_1 \geq d$, S' is nearer to the vertex of the hyperbola α_1 than S. In particular, the length of the (bounded) arc of α_1 determined by S' is less than or equal to that of S, that is, L. □

Before the next result, recall that given two points p and q in the Minkowski plane $\mathbb{E}_1^2$ such that the vector $q - p$ is spacelike, the segment of line connecting p and q is the *largest* spacelike curve joining p and q. A simple proof of this fact is the following. After a rigid motion of $\mathbb{E}_1^2$, we assume that p and q lies in the x-axis. Let α be a spacelike curve connecting both points and parametrized as $\alpha(t) = (x(t), y(t))$, with $\alpha(a) = p$ and $\alpha(b) = q$. We know that $x'(t) \neq 0$ and let us assume that $x'(t) > 0$. Then

$$\begin{aligned} \mathrm{Length}(\alpha) &= \int_a^b \sqrt{x'(t)^2 - y'(t)^2}\,\mathrm{d}t \leq \int_a^b x'(t)\,\mathrm{d}t = x(b) - x(a) \\ &= |\alpha(b) - \alpha(a)|. \end{aligned}$$

Theorem 5. *Let p and q be two distinct points of $\mathbb{E}_1^2$ such that the vector $q - p$ is spacelike (resp. timelike) and $c > 0$. Among all spacelike (resp. timelike) curves whose endpoints are p and q and whose curvature $\kappa(s)$ satisfies $\kappa(s) \leq c$, the shortest spacelike (resp. timelike) curve is a circle of radius $r = 1/c$.*

There is no Euclidean version of this result since the shortest arc between two different points is the arc of the straight line through p and q (whose curvature is 0). In Euclidean plane, the problem attains its interest when $p = q$, that is, the curve joining p and q is closed. The answer is, obviously, a circle. In contrast, and as pointed out in Section 2, there do not exist closed spacelike (or timelike) curves.

Proof. As in Theorem 4, we assume that the vector $q - p$ is spacelike and that both points lie in a straight line parallel to the x-axis. Let α_1 be the circle of radius $r = 1/c$ through p and q and let L_1 denote the length of the arc of α_1 between p and q. After a translation of $\mathbb{E}_1^2$ followed a symmetry with respect to the x-axis, the curve α_1 parameterizes as $\alpha_1(s) = r(\sinh(s/r), \cosh(s/r))$ again. Let α_2 be a spacelike curve with curvature $\kappa_2(s)$, $\kappa_2(s) \leq c = \kappa_1$ connecting p and q and with length L_2. By contradiction, we assume that $L_2 < L_1$.

Consider the arc β of α_1 of length L_2 such that the chord S' of its endpoints is parallel to the x-axis. As $L_2 < L_1$, this means that S' lies below the chord S connecting p and q, that is, between S and the vertex of the hyperbola α_1. In particular, their lengths d' and d respectively, satisfy $d' < d$. However, if we use Schur's theorem for β and α_2, we obtain $d' \geq d$, which leads to a contradiction. This shows the result. □

Finally, a theorem due to Vogt [11] compares the angles at the endpoints of a convex planar curve in Euclidean space whose curvature decreases with the arc length parameter. Hopf realizes a proof of this result using the theorem of Schur [12, p. 26]. This result has its Lorentzian version, although we have not been able to obtain the proof using Theorem 2. We present the result.

Theorem 6. *Let $\alpha : [s_1, s_2] \to \mathbb{E}_1^2$ be a spacelike (or timelike) curve parametrized by the arc length s and such that α together with the chord C between its endpoints $\alpha(s_1)$ and $\alpha(s_2)$ bounds, a convex set. Denote by θ_i the angle between α and the chord C at the points s_i, $i = 1, 2$. If the curvature $\kappa(s)$ is a decreasing function, then $\theta(s_1) \geq \theta(s_2)$. The equality holds if and only if κ is a constant function.*

When we consider the angle between the curve and the chord, we mean in its usual sense, that is, the angle between their unit normal vectors. If α is spacelike, then both α_1 and the chord have normal vectors of timelike type. If α_1 and the chord are timelike curves, we refer the angles that make their (timelike) tangent vectors.

Proof. We derive the proof when α is a spacelike curve, since the case that α is timelike is analogous. After a motion of $\mathbb{E}_1^2$, we assume that $\alpha(s_i)$ belongs to the x-axis and by convexity, that $\alpha(s) = (x(s), y(s))$ lies below that axis. We know that $\mathbf{T}(s) = (\cosh(\theta(s)), \sinh(\theta(s)))$ and $\mathbf{N}(s) = (\sinh(\theta(s)), \cosh(\theta(s)))$. Moreover, $\theta'(s) = \kappa(s)$ and $\theta(s_1) \geq \theta(s_2)$. An integration by parts leads to

$$
\begin{aligned}
0 &\leq \int_{s_1}^{s_2} \kappa'(s)y(s)\,\mathrm{d}s = -\int_{s_1}^{s_2} \kappa(s)y'(s)\,\mathrm{d}s = -\int_{s_1}^{s_2} \theta'(s)\sinh(\theta(s)\,\mathrm{d}s) \\
&= \cosh(\theta(s_1)) - \cosh(\theta(s_2)).
\end{aligned}
$$

This proves that $\theta(s_1) \geq \theta(s_2)$. If the equality holds, then the first inequality in the above expression yields $\kappa'(s) = 0$ for all s, that is, κ is constant. □

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