UNDERSTANDING RANDOMNESS: CHALLENGES FOR RESEARCH AND TEACHING

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The ubiquity of randomness and the consequent need to understand random phenomena in order to make adequate decisions led many countries to include probability throughout the curricula from primary education to University level. This need was also recognized by the mathematicians who developed the probability theory. This is a relatively young field, and is not free of controversies, which are also reflected in the lack of agreement on a common definition of randomness. Psychological and didactical research, suggest widespread misconceptions and misperceptions of randomness; however, these results have not always been taken into account in teaching, where randomness is considered a transparent concept.

Keywords: randomness; epistemology; subjective views; teaching and learning.

> The most decisive conceptual event of twentieth century physics has been the discovery that the world is not deterministic. Causality, long the bastion of metaphysics was toppled, or at least tilted... A space was cleared for chance (Hacking, 1990, p. 1).

When I was kindly asked to contribute with a plenary talk to this conference, I decided to select a topic that reflected a part of European research to stochastic education. Randomness is a good example, since it may be examined from the philosophical, psychological, mathematical and didactic perspectives, each of which has been dealt with by European researchers, and which globally reflect the European perspective for didactics. Furthermore, there is an increasing tendency to teach probability to very young children. However, as we will analyse in my presentation, this concept is far from elementary and we still have to find adequate ways to introduce it to students at different ages.

With this view in mind, I will first describe some of the meanings attributed to the idea of randomness since its emergence; secondly, I will summarize the main research dealing with the personal meanings that people attribute to randomness. I will finish with some personal suggestions for teaching and future stochastic education research that may help to increase our understanding and managing of random situations.

FROM CHANCE TO RANDOMNESS

From childhood, we are surrounded by uncertainty, in our personal lives, our social activities and professional work. The omnipresence of randomness implies our need to understand random phenomena in order to make adequate decisions when confronted with uncertainty. Mathematicians developed the field of probability as a set of models that can be applied to uncertain situations; however, progress in mathematical methods did not solve the philosophical debate around randomness.

Today, mathematics curricula for compulsory teaching levels increase the study of random phenomena. Expressions such as "random experiments", "random digit", "random variable", "random variation", "random even", "random sampling", "randomly", "randomization", "random variable" appear in curricular documents (e.g., CCSSI, 2010; Franklin et al., 2007), as well as in the school textbooks.

However, in these documents, the meaning of randomness is not always clear and unequivocal, because these expressions refer to an abstract entity, not entirely defined; thus, increasing potential difficulties for students or teachers arise. Randomness is a multifaceted object, as shown in the various interpretations received throughout history (Batanero, Green, & Serrano, 1998; Batanero, Henry, & Parzysz, 2005; Bennett, 1999; Saldanha & Liu, 2014). Even today, we find no simple definition that we can use unambiguously to classify a given event or process as being random or not. In the following, the reflections about the nature of randomness by eminent statisticians, philosophers, psychologists and researchers in mathematics education are summarised.

Primitive Ideas

Early notions of chance were found in many ancient cultures. However, for centuries there was no theoretical speculation about the nature of randomness or systematic study of frequencies of results of these games. Possible reasons for the tardy development of probability, such as the connection of chance and divination to predict the future, were discussed by David (1962). Borovcnik and Kapadia (2014a) suggest that in Greek mathematics, data about variability was ignored as contrary to their ideal of scientific argument. Later, many different conceptions of chance arose, in particular (Batanero, Henry, & Parzysz, 2005, p. 27):

> Believing in a destiny predetermined by God or spirits; Assuming a personal chance factor, unequal for different individuals; Accepting natural necessity, ineluctably subjected to laws which still are partially unknown and which govern the world's origin and evolution; Arguing the inextricable complexity of the infinitesimal causes generating macroscopic phenomena, which we consider fortuitous as the only possible reasonable interpretation; Assuming the existence of a fundamental, chaotic and absolute natural randomness.

Bennet (1999) analysed different historical conceptions of chance that was later formalized in the mathematical concept of randomness (Saldanha & Liu, 2014).

Some of these conceptions still appear in students and teachers (Batanero, Arteaga, Serrano, & Ruiz, 2014; Engel & Sedlmeier, 2005). Below I give a brief summary of these developments.

Chance and Causality

We can find a first meaning of randomness in the Spanish dictionary by Moliner (2000) where “random” is defined as “Uncertain. It is said of what depends on luck or chance” (p. 123), and “chance” is defined as “the presumed cause of events that are neither explained by natural necessity nor by a human or divine intervention” (p. 320). In this description, random is something with unknown causes and chance is the assumed cause of random phenomena.

This meaning was prevalent in a first historical phase in the development of randomness according to Bennett (1999). According to David (1962) the astralagus was used in games of chance around 3500 B.C. Cubic dice were abundant in primitive civilizations like the Egyptian or Chinese civilizations, which used games of chance in an attempt to predict or control fate in decision-making. In spite of this wide use, there was no scientific idea of randomness until the Middle Ages. Whether it was attributed to supernatural forces or not, randomness suppressed the possibility that human will, intelligence or knowledge would influence decisions or destiny (Poincaré, 1909/2011).

Throughout this period, some philosophers related chance to causality (Bennet, 1999): Democritus suggested that everything is the combined fruit of chance and need. Leucippus believed that nothing happens at random; everything happens for a reason and out of necessity. Aristotle considered that chance results from the coincidence of several independent events whose interaction results in an unexpected result. Implicit in this meaning is to believe that every phenomenon has a cause. Randomness is only the measure of our ignorance. Random phenomena are, by definition, those whose laws are unknown (Poincaré, 1909/2011).

A deterministic vision of the world was common throughout the Renaissance as is visible in Bernoulli (1713/1987, p. 14):

> All which benefits under the sun from past, present or future, being or becoming, enjoys itself an objective and total certainty… since if all what is future would not arrive with certainty, we cannot see how the supreme Creator could preserve the whole glory of his omniscience and omnipotence."

This conception of chance as opposed to cause and due to our ignorance remained until the 19th century: "*Present events are connected with preceding ones by a link based upon the evident principle that a thing cannot occur without a cause which produces it"* (Laplace, 1814/1995, p. vi).

Modern Concept of Chance

This conception changed at the beginning of the 20th century. For example, Poincaré

(1912/1987) noticed that some processes with unknown laws, such as death, are considered deterministic. Moreover, other phenomena, such as Brownian motion, are described by deterministic laws at a macroscopic level, while the behaviour of particles is random. Other situations are considered to be random because "*A very small cause, which escapes us, determines a considerable effect that we cannot fail to see, and then we say that this effect is due to chance*" (Poincaré, 1912/1987, p. 4).

Among the phenomena with unknown laws, Poincaré distinguished random phenomena that can be studied with probability calculus from other phenomena where probability is not applicable. Furthermore, probability will not lose its validity when we find out the rules governing the random phenomena. Thus, the director of a life insurance company is ignorant of the precise date when each person taking the insurance will die. Moreover, the distribution of the entire population´s lifetime does not change when we add knowledge about the precise death for each particular individual. Today, we accept the existence of fundamental chance around us and, in addition to the theory of probability, other theories, such as those of complexity or chaos, may be used to describe randomness.

The different philosophical conceptions of chance are compatible with the axiomatic mathematical theory of probability, which provides a system of concepts and procedures that serve to analyse uncertain situations (Batanero, Henry, & Parzysz, 2005). Mathematical probability does not enter philosophical debates and uses the ideas of random experiment and randomness as primitive (with no consideration of the nature of chance in each particular application). However, even today, the interpretations of randomness and probability continue to be subject of philosophical debates and the teacher of probability needs to be aware of these interpretations, because they influence students' reasoning when confronted with chance situations.

CONCEPTUALIZING RANDOMNESS

According to Hacking (1975), probability was conceptualized from two complementary perspectives since its emergence: as a personal degree of belief in the likelihood of random events (epistemic view), and as method to find objective mathematical rules through data and experiments (statistical view). These two views unfolded in multiple perspectives that described what random events are, and how can we assign probabilities to them.

Randomness as Equiprobability

In the earlier applications of probability, randomness was related to equiprobability, which was a reasonable assumption in games such as flipping coins or drawing balls from an urn. Consequently, it was assumed that a member of a class was random (or was selected at random), when there was exactly the same probability to obtain this object or any other member of the same class. Thus, there is exactly the same probability to get the number 1 or any other number from 1 to 6, when throwing a dice. We can find, for example, this interpretation of randomness in the Liber de

Ludo Aleae by Cardano (1663/1961, p. 189).

> The most fundamental principle of all in gambling is simply equal conditions...of money, of situation...and of the dice itself. To the extent to which you depart from that equality, if it is in your opponent's favour, you are a fool, and if in your own, you are unjust.

Accordingly, in the classical definition of probability given by de Moivre (1718/ 1967) and refined by Laplace (1814/1995), probability is simply the number of favourable cases to a particular event divided by the number of all cases possible in that experiment, provided all the possible cases are equiprobable.

Kyburg (1974) criticised this definition of randomness since it imposes unnatural restrictions to its applications. We can only consider that an object is a random member of a class if the class is finite. If the class is infinite, then the probability for selecting each member is zero, and so (apparently) identical, even when the selection method is biased. Applying this definition in order to discriminate a random from non-random member in a given class is difficult, even in games of chance. How could we know, for example, that a given coin is not slightly biased?

Randomness as Stability of Frequencies

By the end of the 18th century, the study of random phenomena was extended beyond the world of games of chance to natural and social sciences. In these applications, for example, to the study of the blood type of a newborn, we cannot apply equiprobability. The concept of *independence* was essential to assure randomness in successive trials (Bennet, 1999). In this new view, we consider an object as a random member of a class if we could select it through a method providing a given a priori relative frequency in the long run to each member of this class.

In his attempt to extend the scope of probability to insurance and life-table problems, Jacques Bernoulli (1713/1987) gave the first proof of the Law of Large Numbers and justified the use of relative frequencies to estimate the value of probabilities. In the frequentist approach sustained later by von Mises (1928/1952) or Renyi (1966/1992), probability is defined as the hypothetical number towards which the relative frequency tends. Such a convergence had been observed in many natural phenomena so that the frequentist approach extended the range of applications enormously. A practical drawback of this conception is that we never get the exact value of probability; its estimation varies from one repetition of the experiment (called sample) to another. Moreover, this approach is not appropriate when it is impossible to repeat the experiment under exactly the same conditions. Another theoretical problem is that the number of experiments that provides sufficient evidence to prove the random nature of the object is undefined.

Subjective View of Randomness

In the classical and in the frequentist approaches, randomness is an objective property of an event or an element of a class. Kyburg (1974) criticized this view and proposed a subjective interpretation of randomness composed of the following four elements:

- The object that is supposed to be a random member of a class;
- The set of which the object is a random member (population or collective);
- The property with respect to which the object is a random number of the given class;
- The knowledge of the person giving the judgement of randomness.

In this interpretation, randomness depends on the person's knowledge. Consequently, what is random for one person might be non-random for another; randomness is no longer a physical objective property, but a subjective judgement. We recognize here the parallelism with the subjective conception of probability, in which all probabilities are conditioned by information, and this is adequate in situations where some information may affect our judgement of randomness.

This view was reinforced by the Bayes's theorem, published in 1763, that proved that the probability for a hypothetical event could be revised in light of new available data. Following this new interpretation, some mathematicians like Keynes (1921), Ramsey (1931), or de Finetti (1937/1974) considered probability as a personal degree of belief that depends on a person's knowledge or experience. Via the Bayes' theorem, an initial (prior) distribution about an unknown probability changes by relative frequencies into a posterior distribution. However, the subjective character of the prior distribution in this approach was criticized; even if the impact of the prior diminishes by objective data and de Finetti (1934/1974) proposed a system of axioms to justify this view.

Axiomatization and Formal Mathematical Views

Despite the fierce discussion on the foundations, the application of probability in all sciences and sectors of life was enormous. Throughout the 20th century, different mathematicians tried to formalize the mathematical theory of probability. Following Borel's work on set and measure theory, Kolmogorov (1933/1950) proposed an axiomatic theory that was accepted by the different probability schools because the different view of probability (no matter the classical, frequentist or subjectivist view) may be encoded by Kolmogorov's axioms. However, the particular interpretation of probability and the method used to assign probabilities to events differ according to the school one adheres to.

The development of statistical inference and the importance of assuring random sampling to apply inferential methods led to the practical interest to find procedures to produce sequences of "pseudo-random" digits. This need induced new discussion about theoretical models of randomness (Zabell, 1992). The need to distinguish two components in randomness was clear: the generation process (*random experiment*) and the pattern of the *random sequences* produced. We can generate random sequences with two different methods: one is using physical devices, such as coins or dice. Another is using deterministic algorithms; therefore, we can separate the generating process from the result (random sequence). More correctly, these results are called pseudo-random, because the generating process is a deterministic algorithm, although the sequence can pass some statistical tests for randomness. Most computer packages and calculators incorporate these algorithms, and thus we can easily obtain pseudo-random sequences of a given length with particular characteristics.

Different approaches served to describe the properties of a random sequence (Fine, 1971). Von Mises (1928/1952) defined a *collective* (population) as a mass phenomenon, a repetitive event or a long series of observations, for which we could accept the hypothesis of stabilization or the relative frequency towards a fixed limit. Starting from this idea, he defined a sequence of events to be random if, in any infinitely long series of outcomes, the relative frequencies of the various events have limiting values, and these values do not change in an infinite subsequence arbitrarily selected. Thus, contrary to the belief held by many players, there is no algorithm (at least theoretically) that serves to predict the behaviour of random sequences. However, since no statistical test can consider all potential pattern generators—because there are infinitely many—the possibility that a given sequence, in spite of having passed all our tests must always remain, and it should have some unnoticed pattern and so not really be random. Another problem is that we only produce finite sequences, so inevitably some tests will fail. In this sense, randomness is a theoretical concept and can only be applied to a process producing infinite sequences.

Kolmogorov (1965) defined the randomness of a sequence using the idea of *computational complexity*, taken from automata and computability theory. The complexity of a sequence is the difficulty in describing the sequence with a code that we can use later to reconstruct the sequence (or to store in a computer). The minimal number of signs needed to codify a sequence provides a scale of complexity. For example, 0101010101 can easily be coded with just a few symbols: 5{01}; while 0100110001 defies finding a shorter code, and then the first sequence is more complex than the second. In this approach, a sequence would be random if any coded description of the same is as long as the sequence itself. Therefore, a sequence would be random if the simplest way in which we could describe it is by listing all its components. Chaitin (1975) suggested that this definition establishes a hierarchy in the degree of randomness for different sequences and that perfect randomness is only theoretical.

Epistemic meanings of randomness

To sum up, we can use some ideas from the onto-semiotic approach to mathematics education. In this framework, mathematical knowledge has a socio-epistemic dimension, since it is linked to the activity in which the subject is involved and depends on the institutional and social context in which it is embedded. Mathematical activity is described in terms of practices or sequences of actions, regulated by institutionally established rules, oriented towards solving a problem (Drijvers, Godino, Font, & Trouche, 2013).

Table 1. Example of mathematical objects linked to different epistemic meanings of randomness

Meaning of randomness	Problem	Concepts/Properties	Procedures
Intuitive	− Divination − Attempt to control chance	− Luck, fate − Opinion, belief	− Physical devices (dice, coins...)
Classical	− Establishing the fair betting in a game of chance	− Equiprobability − Proportionality − Favourable/possible cases − Expectation	− Enumeration − Combinatorial analysis − Laplace's rule − A priori analysis of the experiment
Frequentist	− Estimating the tendency in the long run	− Repeatable experiment − Frequency − Convergence	− Collecting data − Estimation − Limit in the long run
Subjective	− Updating a degree of belief	− Subjective character − Depends on information − Non repeatable − Conditional probability − Prior distribution − Posterior distribution − Likelihood, risk	− Bayes' theorem − Decision theory and methods
Formal	− Describing mathematical properties of randomness	− Random experiment − Sample space − Events algebra − Measure − Complexity − Random sequence	− Abstract mathematics (e.g., set theory) − Randomness tests − Simulation − Algorithms that produce pseudo-random sequences

In this framework, the meaning of mathematical objects is linked to the mathematical practices carried out by somebody (a person or an institution) to solve specific mathematical problems. Around the mathematical practices linked to these specific problems, different rules (concepts, propositions, procedures) emerge (Godino, Batanero, & Font, 2007); these rules are supported by mathematics language (terms and expressions, symbols, graphs, etc.), which, in turn is regulated by the rules. All these objects are linked to arguments that serve to communicate the problem solution properties and procedures, and to validate and generalize them to other contexts and problems. An epistemic configuration (either institutional or personal) is the system of objects involved in the mathematical practices carried out to solve a specific problem (Figure 1). Each different epistemic meaning of randomness is linked to a specific type of problem whose solution involves particular mathematical objects, part of which are summarised in Table 1. Consequently, there is a specific onto-semiotic configuration linked to each of these meanings, which differ from each other not only in the philosophical aspects debated in the previous sections, but in the mathematical objects that characterize them. As a result, reducing the teaching of randomness to only one or a few of these views implies a reduction of the overall meaning of the concept.

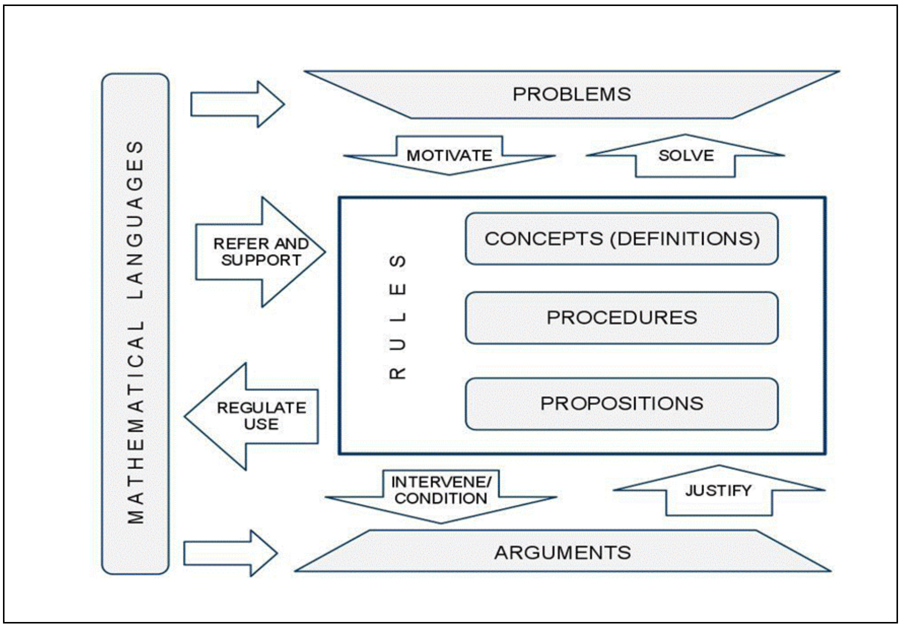

Figure 1. Onto-semiotics configurations involved in mathematical practices (Drijvers, Godino, Font, & Trouche, 2013, p. 28)

SUBJECTIVE CONCEPTUALIZATION OF RANDOMNESS

Research dealing with people's conceptualization of randomness has a long history, and different research paradigms are visible in this research.

Levels of Acquisition of the Concept of Randomness

The pioneer work in probabilistic reasoning was due to Piaget and Inhelder (1951) who described levels (or stages) in children's understanding of chance and probability. These authors assumed that randomness is produced by the interference of independent causes, and then, children first have to understand deterministic cause-and-effect phenomena before they can grasp the nature of random events. Another prerequisite for understanding randomness, according to these authors is combinatorial reasoning, which is needed to describe the set of possibilities in random phenomena, and to accept that each isolated outcome is unpredictable.

Piaget and Inhelder (1951) investigated the understanding of patterns in random distributions by children. They designed an experiment simulating the fall of raindrops on the tiles of a pavement. After situating a few counters (raindrops) on the pavement, they asked the children where the next raindrop would fall. In a first stage (6-9 year-old), the children assumed that the raindrops would approximately fall in equal numbers on each square of the pavement. When there was one drop in every square of the pavement, except for one empty square, the children invariably located the next drop in the empty square, so that a uniform distribution was achieved. With increasing age, Piaget and Inhelder assumed that the irregularity of the distribution would be accepted and that adolescents would understand randomness.

Later research, however, contradicted this assumption: Green (1989) investigated the probabilistic reasoning of 2930 children in the United Kingdom and used some tasks related to perception of randomness at age 11-16; his findings suggest that the percentage of students recognizing random distributions does not improve with age (stagnation of children's perception of randomness during this period). Similar results were found by Green (1991) and Engel and Sedlmeier (2005) with different tasks (to students at age 10-15).

Intuitions and Personal Beliefs

While Inhelder and Piaget focused on the formal understanding of randomness, other authors tried to describe personal beliefs and intuitive understanding of this concept. This research suggests that the paradoxes and controversies about the meaning of randomness are reproduced in the intuitions people build when they face random situations; these intuitions often contradict the mathematical rules of probability (Borovcnik & Kapadia, 2014a).

Children use qualitative expressions (probable, unlikely, feasible, etc.) to express their degrees of belief in the occurrence of random events; however, their ideas are too imprecise and have difficulty in differentiating random and deterministic phenomena (Fischbein & Gazit, 1984). Young children may not see stable properties in random generators such as dice or marbles in urns and believe that such generators have a mind of their own or can be controlled by them (Fischbein, Nello, & Marino, 1991; Truran, 1994). Although older children may accept the need to assign numbers (probabilities) to events to compare their likelihood, a correct probabilistic reasoning rarely develops spontaneously without a specific instruction (Fischbein, 1975); for this reason, adults often have wrong intuitions about probability.

Fischbein's assumption has been confirmed by research in the field of decision making under uncertainty, where erroneous judgements in out-of-school settings are pervasive. The widely known studies by Kahneman and his collaborators (e.g., Kahneman, Slovic, & Tversky, 1982) support the idea that people violate probabilistic rules and use specific *heuristics*[2] to simplify uncertain decisions. According to these authors, heuristics such as *representativeness* or *availaibility* reduce the complexity of probability tasks and may be useful in many situations; however, under specific circumstances these heuristics cause systematic biases with serious consequences. Furthermore, some people do not understand the purpose of probabilistic methods, which allow us to predict the behaviour of a distribution, but are invalid to predict each specific outcome (Konold, 1989). A detailed survey of students' intuitions, strategies and learning at different ages may be found in Chernoff and Sriraman (2014), Jones (2005), Jones, Langrall, and Mooney (2007), and Shaughnessy (1992).

Generating and Recognizing Randomness

There is a wide research into adults' subjective perception of randomness (e.g., Bar-Hillel & Wagenaar, 1991; Batanero & Serrano, 1999; Chernoff, 2009; 2011; Engel & Sedlmeier, 2005; Falk, 1981; Kahneman & Tversky, 1972; Wagenaar, 1972), Two types of tasks have commonly been used: (a) In generation tasks subjects follow standard instructions to invent a series of outcomes from a typical random process, such as tossing a coin; (b) In comparative likelihood tasks (Chernoff, 2011), people are asked to select the most or least likely of several sequences of results that have been produced by a random device or to decide whether some given sequences were produced by a random mechanism. Related tasks have also been proposed using two-dimensional random distributions of points on a squared grill (e.g., Batanero & Serrano, 1999; Green, 1991; Engel & Sedlmeier, 2005; Toohey, 1995).

Generation tasks: Producing random distributions

In a longitudinal study on randomness with 7 to 11 year-old children, Green (1991) asked them to invent random sequences of heads and tails representing the results of flipping 50 times a fair coin. He first analysed whether the children produced approximately the same number of heads and tails in their sequences and found that they were very exact in reproducing equiprobability (the average number of heads was close to 25); furthermore, the children produced sequences with very consistent first and second parts (about 12 heads in each part). Green concluded that children were too consistent to reflect the random variability. Moreover, these children did not perceive the independence, as they produced sequences with too short runs (of heads or of tails), as compared to the length we expect in a random sequence. As suggested by Bryant and Nunes (2012), the independence of random events is hard to grasp and many adults believe that a head is more likely to appear on the sixth flipping of a coin after a run of five tails.

Comparative likelihood tasks: Properties attributed to randomness

Results from research asking people to distinguish random from non-random sequences of events suggest that our judgements about what random sequences are, is subjected to biases (e.g., Batanero & Serrano, 1999; Chernoff, 2009, 2011; Green, 1983, 1991; Kahneman & Tversky, 1972; Konold, Pollatsek, Well, Lohmeier, & Lipson, 1993; Shaughnessy, 1977).

An example of this research is Green's (1983) study with 11-16 year-old children. In his questionnaire, he asked the children to discriminate between a random and a non-random sequence (both sequences consisted of results of flipping a coin 150 times). Most participants in the study chose the non-random sequence, regardless of their age. Some of them provided incorrect reasons to justify their choice; for example, they argued that the pattern of the sequence was too irregular (and then they did not accept the variability of the random sequence). Other participants expected exactly 50% of heads and tails in a random sequence or rejected the possibility of long runs. Very similar results were found in another study (Green, 1991) where the author asked the children to discriminate random and non-random sequences of heads and tails, as well as random and non-random bi-dimensional distribution of points.

Toohey (1995) used part of Green's tasks in a study with 75 12-16 year olds in Melbourne. He suggested that the understanding of randomness involves accepting the ideas of equal/unequal likelihood, multiple possibilities, model, causality and unpredictability. He also identified two different possibilities (local and global) in attributions of randomness. The local perspective of randomness is based on isolated results, while global perspective is reliant on the frequency distribution of the different outcomes.

Batanero and Serrano (1999) proposed some items taken from Green (1991) to 277 students aged 14 and 18 and analysed the reasons they gave to decide that a sequence or a distribution was random. The students' arguments were related to the observed frequencies of events (close or different from the expected value), the overall pattern of the distribution (uniform distribution or variability), the length of the runs (too short or too long runs), the existence of multiple possibilities and the unpredictability of results. Even when the authors found some widespread misconceptions, they also noticed that the students were able to perceive the characteristics of the random sequences presented to them and that this recognition improved by age. They also identified some partly correct conceptions that reproduced the conceptions of randomness described in the first sections of this paper, which were considered correct in different historical periods. Consistent results were reported by Engel and Sedlmeier (2005) in a cross sectional study that examined German students' understanding of variability in empirical data and by Batanero, Gómez, Gea, and Contreras (2014) in a study with Spanish prospective primary school teachers.

Facing the subjects with their own misconceptions: statistical analysis of their own data

As analysed in the previous sections, the perception of randomness is deduced with either generation or recognition tasks. Batanero and colleagues (2014) combined both tasks in a study with 208 Spanish prospective primary school teachers, using a formative activity with two parts. In the first part (a classroom session), the prospective teachers carried out an experiment to decide whether the group had good intuitions on randomness or not. The experiment consisted of trying to write down apparent random results of flipping a fair coin 20 times (without really throwing the coin, just inventing the results) in such a way that other people would think the coin was flipped at random (invented sequence). Participants recorded the invented sequences on a recording sheet (this is a typical generation tasks). These sequences were analysed by the researchers and were consistent with previous research reported on generation tasks.

Batanero and colleagues also asked the prospective teachers to analyse some variables deduced from their invented sequences (number of heads, number of runs, and length of the longer run) and compare them with the same variables in real coin-flipping sequences as a result of flipping a fair coin 20 times. The prospective teachers were asked to carry out the statistical analysis of differences between the same variables in the real coin-flipping and invented sequences for the whole group. They were also asked to prepare a report with their conclusions about how good their perception of randomness based on the statistical analysis, was. Each prospective teacher analysed the results in his or her group (30-40 students per group) using elementary graphs and statistics; they had freedom to use any method they wished. This second part of the activity is a sophisticated version of a comparative likelihood task, because the participants were not only asked to discriminate the random (real flipping) from non-random (invented) sequences. They were only asked to perform (intuitively) the type of analysis that researchers use to study people's perception of randomness. This activity was highly motivating for the prospective teachers and served to simultaneously increase their statistical and didactical knowledge.

Results from this second part firstly showed that these prospective teachers were able to use their statistical knowledge to solve a real world problem (deciding if the perception of randomness in the group was good). They secondly completed a modelling cycle: they started from a real problem (studying the intuitions on randomness), simplified the problem, and decided which aspects were relevant. They thirdly built some mathematical models to study the problem, worked with the models, and finally interpreted the results to answer the real world question. As regards their perception of randomness, many of the primitive conceptions described in Batanero and Serrano (1999) appeared and part of them were identified by the prospective teachers' themselves. Participants also recognised that the classroom showed a good perception of the expected value and a poor conception of both independence and variation. Some new results emerged; for example, some prospective teachers believed that it is not possible to apply mathematical methods (statistics) to study random phenomena, because of their unpredictability. A few participants also believed they could predict or control the outcomes in a random process (illusion of control described by Langer, 1975).

Other Research Paradigms

A different approach to evaluate people's perception was taken by Konold, Lohmeier, Pollatsek, and Well (1991) who concentrated on the random process (instead of concentrating on the random sequence). They asked the subjects in their study to decide whether different types of situations (processes) were or were not random and justify their responses. They used processes with equiprobable and non-equiprobable outcomes. While they found no differences in the subjects' categorization of the situations as random, novices tended to feel that the non-equiprobable situations were not random. The analysis of students' arguments served to describe the following conceptions of randomness:

- *Randomness as equiprobability:* Subjects that only consider randomness where all the possible results are equally probable.
- *Randomness as opposed to causality*, or as a special type of cause.
- *Randomness as uncertainty;* existence of multiple possibilities in the same conditions.
- *Randomness as a model* to represent some phenomenon, depending on our information about it.

Randomization is an important statistical procedure that assures the proper application of statistical methods, such as statistical tests. Pratt (2000) and Pratt and Noss (2002) investigated children's understanding of randomization when playing chance games and found 10-year olds that understood the connection between randomness and fairness, and the role of randomization in ensuring fairness (see also Johnston-Wilder & Pratt, 2007; Paparistodemou, Noss, & Pratt, 2008). Pratt (2000) suggests that children reason with two different meanings for randomness (very close to the description by Toohey, 1995): a local perception is related to the impossibility to predict the process behaviour in each trial, while a global perception involves the children´s understanding of patterns in the long run and in the distributions.
As it is apparent in our survey, research into people's perception of randomness has been faced with different paradigms that provided complementary results. Yet new questions remain open; in particular, it is not clear what model of randomness is better suited for children at different ages, or how we can help students acquire progressively more complete models of randomness as they become adult. We now analyse the way the topic has been taken into account in the curricula.

TEACHING AND LEARNING

Randomness in School Curricula

The different views of probability have been reflected on the teaching of probability in schools, and on the way, randomness has been conceptualized in the curricula in Spain and other European countries; the concept itself is often only introduced via examples of random and non-random situations, or with indirect reference to isolated properties (e.g., unpredictability) (Azcárate, Cardeñoso, & Serradó, 2005), but is not formally defined.

According to Henry (2010), the classical view of probability based on combinatorial calculus dominated the French school curricula until the 80s, and this was also the case in Spain and other European countries. Since combinatorial reasoning is difficult, the teaching of probability was postponed until grades 8 or 9 (14 year-olds), an age where wrong intuitions difficult to eradicate are already acquired. Throughout the "modern mathematics" era, probability was used to illustrate set theory; there was little interest in modelling random phenomena from the real world. In these two approaches, the applications were restricted to games of chance; consequently, many school teachers considered probability as a part of recreational mathematics, with not much value for the education of children and tended to reduce its teaching.

Today, due to the technology available, we use the frequentist view to introduce probability as the limit of relative frequencies in a long series of trials. This change also involves a shift from a formula-based approach to an emphasis on providing probabilistic experience. Even very young children are encouraged to perform random experiments or simulations, formulate questions or predictions about the tendency of outcomes in a series of these experiments, collect and analyse data to test their conjectures, and justify their conclusions based on these data. This view also connects to the current interest for modelling in school mathematics (Henry, 2010), since simulation can also help students distinguish between model (the theoretical probability) and reality (frequencies of experimental results) (Girard, 1997; Engel & Vogel, 2004).

Randomness receives prominence today at high school level in relation to the introduction of inference (or "informal inference"). For example, in the CCSSI (2010) for grade 7 we find "use random sampling to draw inferences about a population" and "understand and evaluate random processes underlying statistical experiments". For high school (grades 9-12), this curriculum specifies "define a random variable for a quantity of interest by assigning a numerical value", "use random number generators", and "collect data from a random sample of a population". Many other curricula in Europe, as well as in the Australia, New Zealand and the United States approach probability and inference in a frequentist way, using simulation and resampling to estimate the probabilities of interest (e.g., Frischemeier & Biehler, 2013). The subjective view, that takes into account one-off decisions, which are frequent in everyday life, and where we cannot apply the frequentist view, is hardly considered in the curriculum. Moreover, the experiments we often simulate are atypical examples of random situations, in the sense that in few real-life applications of probability can we repeat a process many times in exactly the same conditions (Borovcnik & Kapadia, 2014a).

A Didactic Approach to Randomness

The many perspectives and properties of randomness described in the previous sections suggest that a complete understanding of randomness is only achieved gradually. Moreover, probability models do not exactly fit reality and therefore should be viewed more as scenarios to explore reality than as images of this reality

(Borovcnik, 2006). Since feedback in probability is only indirect (after a long series of trials), understanding of probability is not easy.

Throughout primary school, we can encourage children to discriminate certain, possible and impossible events in different context, and use the language of chance. Starting with specific materials with symmetrical properties, such as dice or coins, the children can compare their predictions from the a-priori analysis of the structure with frequency from data collected from repeated experiments to estimate probability.

In a second stage, we can progressively move to the study of materials lacking symmetry properties—spinners with unequal areas, thumbtacks, etc.—, where we only can estimate probability from frequencies. Once this phase is successful, we can turn to real life (e.g., sports, demographic, or social phenomena), using data available from the daily press, Internet, or other sources. Subjective situations (e.g., should the teacher ask me next time?) where only personal probabilities can be applied, can complete the field of application of probability.

By the end of primary school or in early middle school (10-11 year olds) children can start simulating simple situations using devices such as the box model simulator in the National Library of Virtual Manipulatives (http://nlvm.usu.edu/). Today, there are plenty of technological resources, including software specially designed to explore probability (see also Lee & Lee, 2009). With simulation, we introduce a modelling approach where the essential features of the situation are modelled by the simulator and irrelevant properties are disregarded. As shown by Pratt (2000), simulation of familiar objects like dice, can help 10-11 year-old children express their previous beliefs and articulate a more complete meaning for randomness in the light of their experiences with the simulator.

Towards the end of secondary school (15-16 year olds) a deeper analysis of the properties of the random numbers generated through a calculator or computer may be introduced. The experiments, recording and analysis of the sequences produced in these simulation activities will help to integrate study of probability and statistics. Eichler and Vogel (2014) propose a modelling approach for each of the main views of probability (classical, frequentist and subjective) and discuss the role of simulation in supporting students' understanding in each of these perspectives. The context of decision making, such as for example, taking insurance, is useful to introduce subjective views. When facing the uncertainty of a single decision, this decision could be made more transparent if we ask the students to weigh up the different possibilities, and compute the expected values of costs or prizes (Borovcnik, 2006).

The gradual introduction of concepts and notation will serve to mathematically explain the regularities observed in the data. Exploration of microworlds (e.g, Cerulli, Chioccariello, & Lemut, 2006) may serve to confront children´s intuitions to mathematical ideas. Johnston-Wilder and Pratt (2007) suggest that these tools help children see randomness as a dynamic process, since a printout of a random sequence loses the essence of what random is to be.

Through these activities, students will progressively acquire understanding of the following essential characteristics of random phenomena:

- In a random situation there is uncertainty; more than one result is possible.
- The actual result, which will occur, is unpredictable (local variability of random processes).
- We can analyse either the process (random generator) or the sequence of random results: these two aspects can be separated.
- In a few situations (e.g., games of chance) we can analyse the process before the experiment; this analysis will inform us of the likelihood of possible results
- Commonly, there is the possibility—at least in the imagination—of repeating the experiment (or observation) many times in (almost) similar conditions.
- In this case, the sequence of results obtained through repetition lacks a pattern; we cannot control or predict each result (local variability).
- In this apparent disorder, a multitude *of global regularities* can be discovered, the most obvious being the stabilization of the relative frequencies of each possible result. This global regularity is the basis that allows us to study random phenomena using the theory of probability.
- In one-off uncertain situations we still can apply probability if our initial degrees of beliefs are consistent (have reasonable properties).
- To conclude, randomness is a model we apply to some situations, because this model is useful to predict or control the situations.

As argued by Konold et al. (1991) it is preferable to consider randomness as a label with which we associate many concepts, such as experiment, event, sample space, probability, etc. In this sense, the word randomness refers to a collection of mathematical concepts and procedures, which we can apply to uncertain situations. We need to think about the orientation we take towards the phenomenon that we qualify as "random" rather than think of randomness as an objective quality of the phenomenon itself. We apply a mathematical model to the situation, because it is useful to describe it and to understand it; but we do not believe that the situation is identical to the model. Deciding when a probability model is more appropriate for the situation than other mathematical models is a part of the competence we want the students to develop.

FINAL REFLECTIONS

The complexity of the idea of randomness explains the counterintuitive results that abound even in basic probabilistic concepts (Székely, 1986; Borovcnik & Peard, 1996). This complexity is also reflected at higher levels in probability theorems (e.g., the Central Limit theorem) that are expressed in terms of probability. According to Borovcnik and Kapadia (2014b) our poor intuitions in this field may be explained by our desire for deterministic explanations, but they might also be attributed to an inadequate education.

In spite of this complexity, "*Probability is the only reliable means we have to predict and plan for the future; it plays a huge role in our lives, so we cannot ignore it, and we must teach it to all future citizens*" (Devlin, 2014, p. ix). It is then important to reinforce probability in the school curricula and to find appropriate conceptualizations of randomness for different ages.

One goal of probability education is to take advantage of childrens' intuitions from elementary school as a basis for the acquisition of probability reasoning. One important insight into this line of research is the power of representation formats, such as natural frequencies (Gigerenzer & Hoffrage, 1995) or "tinker cubes" and other manipulatives (Martignon, Laskey, & Kurz-Milcke, 2007). Experimental interaction with mathematical modelling in a co-operative setting can likewise help children develop secondary intuitions (Nilson, 2003). Besides, as suggested by Andrà and Stanja (2013) it is important to pay attention to the interpretation and use of signs, which is not self-evident in probability, and may be interfered with experience with the same signs in other mathematical domains.

It is also important to confront the students with their own misconceptions and erroneous beliefs. As discussed by Borovcnik and Kapadia (2014b), progress in the development of mathematical concepts is usually accompanied by ruptures and conflicts, but there is an opportunity for learning when one tries to solve the conflict and understand paradoxical results.

Eichler and Vogel (2014) analyse the role of simulation to explore a model that already exists, develop an unknown model approximately, and represent data generation. However, though simulation is vital to improve students' probabilistic intuitions and to materialize probabilistic problems, a genuine knowledge of probability can only be achieved through the study of some formal probability theory. Of course, the acquisition of such formal knowledge by students should be gradual and supported by experience with random experiments.

We should also complement the objective and subjective views of probability. Even when many people believe that events have a unique probability rather than considering probability as a measure of our knowledge (Devlin, 2014), the idea of updating previous information in the light of new data is very intuitive as it reflects the way how people think.

It is also important to empower teachers with a specific preparation to teach probability because teachers' beliefs influence their instructional planning, their classroom practices, and have an impact on their students' learning (Eichler, 2011).

Even if prospective teachers have a major in mathematics, they may be unfamiliar with different meanings of randomness and probability, or with their students' most common misconceptions. Teachers should also be conscious that teaching principles valid for other areas of mathematics, are not always appropriate in the field of probability (Batanero & Díaz, 2012). As described in a teaching experiment reported by Brousseau, Brousseau, and Warfield (2002), the teacher may fail to produce a specific random result when needed (even if he/she manages to assure a good probability of happening for the given result). Thus, even a reasonable knowledge of probability would not suffice for the teacher to be able to reproduce the didactic situation exactly as he/she prefers, and this could be a source of challenge for the teacher.

The preparation of teachers requires the design of activities where teachers are first confronted with their previous ideas and then perform and discuss experiments (e.g., Batanero, Biehler, Engel, Maxara, & Vogel, 2005; Batanero et al., 2014) in order to simultaneously increase teachers' probabilistic and didactic knowledge.

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NOTES

1. Piaget and Inhelder based their research on the classical view of probability.

2. The specific meaning of word *heuristics* in this research is a cognitive process that helps to solve a problem by reducing part of the data.

3. Prospective primary school teachers do not follow a specific course of probability. They study elementary probability in their first year of studies and along secondary school (as a part of mathematics).

REFERENCES

Andrà, C., & Stanja, J. (2013). What does it mean to do probability? Ideas, symbols and procedures. In B. Ubuz, Ç. Haser, & A. Mariotti (Eds.), *Proceedings of the Eight Congress of European Research in Mathematics Education* (pp. 736-745). Ankara: Middle East Technical University.

Azcárate, P., Cardeñoso, J. M., & Serradó, S. (2006). Randomness in textbooks: the influence of deterministic thinking. In M. Bosch (Ed.), *Proceedings of the Fourth Congress of European Research in Mathematics Education* (559-568). Sant Feliu de Guixols: ERME.

Bar-Hillel, M., & Wagenaar, W. A. (1991). The perception of randomness. *Advances in Applied Mathematics 12,* 428-454.

Batanero, C., Arteaga, P., Serrano, L., & Ruiz. B. (2014). Prospective primary school teachers' perception of randomness. In E. J. Chernoff, & B. Sriraman, (Eds.), *Probabilistic thinking: presenting plural perspectives* (pp. 345-366). Dordrecht:

Springer.

Batanero, C., Biehler, R., Engel, J., Maxara, C., & Vogel, M. (2005). Simulation as a tool to bridge teachers' probabilistic and pedagogical knowledge. Paper presented at the ICMI Study 15. *Professional development of mathematics teachers.* Aguas de Lindoia, Brazil, 2005.

Batanero, C., & Díaz, C. (2012). Training teachers to teach probability: Reflections and challenges. *Chilean Journal of Statistics, 3*(1), 3-13.

Batanero, C., Gómez, E., Gea, M. M., & Contreras, J. M. (2014). Assessing and developing prospective teachers' understanding of random sequences. In U. Sproesser, S. Wessolowski, & C. Wörn (Eds.), *Daten, Zufall und der Rest der Welt – Didaktische Perspektiven zur anwendungsbezogenen Mathemati*k. Heidelberg: Springer Spektrum Verlag.

Batanero, C., Green, D. R., & Serrano, L. R. (1998). Randomness, its meanings and educational implications. *International Journal of Mathematical Education in Science and Technology, 29*(1), 113-123.

Batanero, C., Henry, M., & Parzysz, B. (2005). The nature of chance and probability. In G. Jones (Ed.), *Exploring probability in school: challenges for teaching and learning* (pp. 15-37). New York: Springer.

Batanero, C., & Serrano, L. (1999).The meaning of randomness for secondary school students. *Journal for Research in Mathematics Education, 30*(5), 558-567.

Bennett, D. J. (1999). *Randomness.* Cambridge, MA: Harvard University Press.

Bernoulli, J. (1987). *Ars conjectandi- 4ème partie.* Rouen: IREM. (Original work published in 1713).

Borovcnik, M. (2006). Probabilistic and statistical thinking. In M. Bosch (Ed.), *Proceedings of the Fourth Congress of European Research in Mathematics Education* (pp. 485-406). Sant Feliu de Guixols: ERME.

Borovcnik, M., & Kapadia, R. (2014a). A historical and philosophical perspective on probability. In E. J. Chernoff & B. Sriraman, (Eds.), *Probabilistic thinking: presenting plural perspectives* (pp. 7-34). Dordrecht: Springer.

Borovcnik, M., & Kapadia, R. (2014b). From puzzles and paradoxes to concepts in probability. In E. J. Chernoff & B. Sriraman, (Eds.), *Probabilistic thinking: presenting plural perspectives* (pp. 35-73). Dordrecht: Springer.

Borovcnik, M., & Peard, R. (1996). Probability. In A. Bishop, K. Clements, C. Keitel, J. Kilpatrick, & C. Laborde (Eds.), *International handbook of mathematics education* (pp. 239-288). Dordrecht: Kluwer.

Brousseau, G., Brousseau, N., & Warfield, N. (2002). An experiment on the teaching of statistics and probability. *Journal of Mathematical Behavior, 20,* 363-411.

Bryant, P., & Nunes, T. (2012). *Children's understanding of probability: a literature review.* London: Nuffield Foundation.

Cardano, G. (1961). *The book on games of chances.* New York: Holt, Rinehart & Winston. (Original work Liber de Ludo Aleae; included in Opera Omnia, Vol. 1, published in 1663).

Cerulli, M., Chioccariello, E., & Lemut, E. (2006). Randomness and LEGO robots. In M. Bosch (Eds.), *Proceedings of the Fourth Congress of European Research in Mathematics Education* (591-600). Sant Feliu de Guixols: ERME.

Chaitin, G. J. (1975). Randomness and mathematical proof. *Scientific American,* 232, 47-52.

Chernoff, E. J. (2009). Sample space partitions: An investigative lens. *Journal of Mathematical Behavior, 28*(1), 19-29.

Chernoff, E. J. (2011). Investigating relative likelihood comparisons of multinomial, contextual sequences. In M. Pytlak, T. Rowland, & E. Swoboda (Eds.), *Proceedings of the Seventh Congress of European Research in Mathematics Education* (591-600). Reszlow: ERME.

Chernoff, E. J., & Sriraman, B. (2014). *Probabilistic thinking: presenting plural perspectives.* Dordrecht: Springer.

Common Core State Standards Initiative (CCSSI) (2010). *Common Core State Standards for Mathematics.* Washington, DC: National Governors Association for Best Practices and the Council of Chief State School Officers. Retrieved from: www.corestandards.org/assets/CCSSI_Math Standards.pdf.

David, F. N. (1962). *Games, gods and gambling.* London: Griffin.

Devlin, K. (2014). The most common misconception about probability? In E. J. Chernoff, & B. Sriraman (Eds.), *Probabilistic thinking: presenting plural perspectives* (pp. ix-xiii). Dordrecht: Springer.

Drijvers, P., Godino, J. D., Font, V., & Trouche, L. (2013). One episode, two lenses. A reflective analysis of student learning with computer algebra from instrumental and onto-semiotic perspectives. *Educational Studies in Mathematics, 82*, 23-49.

Eichler, A. (2011). Statistics teachers and classroom practices. In C. Batanero, G. Burrill, & C. Reading (Eds.), T*eaching statistics in school mathematics-challenges for teaching and teacher education. A joint ICMI/IASE study* (pp. 175-186). New York: Springer.

Eichler, A., & Vogel, M. (2014). Three approaches for modelling situations with randomness. In Chernoff, E. J., & Sriraman, B. (Eds.), *Probabilistic thinking: presenting plural perspectives.* (pp. 75-99). Dordrecht: Springer.

Engel, J., & Sedlmeier, P. (2005). On middle-school students' comprehension of randomness and chance variability in data. *Zentralblatt für Didaktik der*

Mathematik, 37(3), 168-177.

Engel, J., & Vogel, M. (2004). Mathematical problem solving as modeling process. In W. Blum, P. L. Galbraith, H.-W. Henn, & M. Niss (Eds.), *Modelling and applications in mathematics education. ICMI-study 14* (pp. 83–88). New York: Springer.

Falk, R. (1981). The perception of randomness. In C. Laborde (Ed.), *Proceedings of the Fifth International Conference for the Psychology of Mathematics Education* (pp. 222-229). Grenoble: International Group for the Psychology of Mathematics Education.

Fine, T. L. (1971). *Theories of probability. An examination of foundations*. London: Academic Press.

Finetti, B., de (1974). *Theory of probability*. London: John Wiley (Original work published in 1937).

Fischbein, E. (1975). *The intuitive sources of probability thinking in children*. Dordrecht: Reidel.

Fischbein, E., & Gazit, A. (1984). Does the teaching of probability improve probabilistic intuitions? *Educational Studies in Mathematics, 15*(1), 1-24.

Fischbein, E., Nello, M. S., & Marino, M. S. (1991). Factors affecting probabilistic judgements in children and adolescents. *Educational Studies in Mathematics, 22*(6), 523-549.

Franklin, C., Kader, G., Mewborn, D., Moreno, J., Peck, R., Perry, M., & Scheaffer, R. (2007). Guidelines for assessment and instruction in statistics education (GAISE) report: a preK-12 curriculum framework. Alexandria, VA: American Statistical Association. Retrieved from: www.amstat.org/Education/gaise/.

Frischemeier, D., & Biehler, R. (2013). Design and exploratory evaluation of a learning trajectory leading to do randomization tests facilitated by Tinkerplots. In B. Ubuz, Ç. Haser, & A. Mariotti (Eds.), *Proceedings of the Eight Congress of European Research in Mathematics Education* (798-808). Ankara: ERME.

Gigerenzer, G., & Hoffrage, U. (1995). How to improve Bayesian reasoning without instruction: frequency formats. *Psychological Review, 102*, 684-704.

Girard, J. C. (1997). Modélisation, simulation et experience aléatoire [Modeling, simulation and random experience]. In M. Henry (Ed.), *Enseigner les probabilités au lycée* (pp. 73–76). Reims: Commission Inter-IREM Statistique et Probabilités.

Godino, J. D., Batanero, C., & Font, V. (2007). The onto-semiotic approach to research in mathematics education. *ZDM, The International Journal on Mathematics Education, 39*(1-2), 127-135.

Green, D. R. (1983). A survey of probabilistic concepts in 3000 students aged 11-16 years. In D. R. Grey et al. (Eds.), *Proceedings of the First International*

Conference on Teaching Statistics (v. 2, pp. 766-783). Sheffield: International Statistical Institute.

Green, D. R. (1989). Schools students' understanding of randomness. In R. Morris (Ed.), *Studies in mathematics education. Vol 7: The teaching of statistics* (pp. 27-39). Paris: UNESCO.

Green, D. R. (1991). *A longitudinal study of pupil's probability concepts.* Technical Report ME90/91. Loughborough: University of Technology.

Hacking, I. (1975). *The emergence of probability*. Cambridge, MA: Cambridge University Press.

Hacking, I. (1990). *The taming of chance.* Cambridge, MA: Cambridge University Press.

Henry, M. (2010). Evolution de l'enseignement secondaire français en statistique et probabilities [Evolution of French secondary teaching in statistics and probability]. *Statistique et Enseignement, 1*(1), 35-45.

Johnston-Wilder, P., & Pratt, D. (2007). The relationship between local and global perspectives on randomness. In D. Pitta & G. Philippou (Eds.), Proceedings of the Fifth Congress of European Research in Mathematics Education (pp. 742-751). Larnaca: ERME.

Jones, G. A. (Ed.) (2005). *Exploring probability in school: Challenges for teaching and learning*, New York: Springer.

Jones, G. A., Langrall, C., & Mooney, E. (2007). Research in probability: responding to classroom realities In F. Lester (Ed.), *Second handbook of research on mathematics teaching and learning*. Greenwich, CT: Information Age & NCTM.

Kahneman, D., Slovic, P., & Tversky, A. (1982). *Judgment under uncertainty: Heuristics and biases New York:* Cambridge University Press

Kahneman, D., & Tversky, A. (1972). Subjective probability: A judgment of representativeness. *Cognitive Psychology, 3,* 430-454.

Keynes, J. M. (1921). *A treatise on probability.* London: Macmillan.

Kolmogorov, A. N. (1950). *Foundations of probability's calculation.* New York: Chelsea Publishing Company (Original work published in 1933).

Kolmogorov, A. N. (1965). Three approaches for defining the concept of information quantity, *Problems of Information Translation 1*(1), 1-7.

Konold, C. (1989). Informal conceptions of probability. *Cognition and Instruction, 6,* 59-98.

Konold, C., Lohmeier, J., Pollatsek, A., & Well, A. (1991). Novices views on randomness. In R. G. Underhill (Ed.), *Proceedings of the Thirteenth Annual Meeting of the North American Chapter of the International Group for the*

Psychology of Mathematics Education, (Vol. 1, pp. 167-173). Blacksburg VA: Virginia Polytechnic Institute and State University.

Konold, C., Pollatsek, A., Well, A., Lohmeier, J., & Lipson, A. (1993). Inconsistencies in students‘ reasoning about probability. *Journal for Research in Mathematics Education, 24* (5), 392-414.

Kyburg, H. E. (1974). *The logical foundations of statistical inference.* Boston: Reidel.

Langer, E. J. (1975). The illusion of control. *Journal of Personality and Social Psychology, 32,* 311-328.

Laplace, P. S. (1995). *Théorie analytique des probabilités* [Analytical theory of probabilities]. Paris: Jacques Gabay (Original work published in 1814).

Lee, H. S., & Lee, J. T. (2009). Reasoning about probabilistic phenomena: Lessons learned and applied in software design. *Technology Innovations in Statistics Education, 3*(2). Retrieved from: https://escholarship.org/uc/item/1b54h9s9.

Martignon, L., Laskey, K., & Kurz-Milcke, E. (2007). Transparent urns and colored tinker – cubes for natural stochastics in primary school. In D. Pitta, & G. Philippou (Eds.). *Proceedings of the Fifth Congress of European Research in Mathematics Education* (pp. 752-761). Larnaca: ERME.

Mises, R. von (1952). *Probability, statistics and truth*. London: William Hodge (Original work published in 1928).

Moivre, A. de (1967). *The doctrine of chances*. New York: Chelsea (Original work published in 1718).

Moliner, M. (2000). *Diccionario de uso del español* [Dictionary of Spanish uses]. Madrid: Gredos.

Nilson, P. (2003). Experimentation as a tool for discovering mathematical concepts of probability. In M. A. Mariotti (Ed.), *Proceedings of the Third Congress of European Research in Mathematics Education.* Bellaria: ERME. Retrieved from: www.mathematik.uni-dortmund.de/~erme/CERME3/.

Paparistodemou, E., Noss, R., & Pratt, D. (2008). The interplay between fairness and randomness in a spatial computer game. *International Journal for Computer Mathematics Learning, 13,* 89-110.

Piaget J., & Inhelder, B. (1951). *La genése de l'idée de hasard chez l'enfant* [The origin of the idea of chance in children’s thinking]. Paris: Presses Universitaires de France.

Poincaré, H. (1987). *Calcul des probabilités* [Probability calculus]. Paris: Jacques Gabay (Original work published in 1912).

Poincaré, H. (2011). Chance. In H. Poincaré, *Science and method* (pp. 64-91). Dover: Courier (Original work published in 1909).

Pratt, D. (2000). Making sense of the total of two dice. *Journal for Research in Mathematics Education, 31*(5), 602-625.

Pratt, D., & Noss, R. (2002). The microevolution of mathematical knowledge: The case of randomness. *Journal of the Learning Sciences, 11*(4), 453-488.

Ramsey, F. (1931). Truth and probability. In R. B. Braithwaite (Ed.), *The foundations of mathematics and other logical essays* (pp. 156-198). London: Kegan Paul.

Renyi, A. (1992). *Calcul des probabilités* [Probability calculus]. Paris: Jacques Gabay (Original work published in 1966).

Saldanha, L., & Liu, Y. (2014). Challenges of developing coherent probabilistic reasoning: rethinking randomness and probability from a stochastic perspective. In E. J Chernoff, & B. Sriraman, (Eds.), *Probabilistic thinking: presenting plural perspectives* (pp. 367-396). Dordrecht: Springer.

Shaughnessy, J. M. (1977). Misconceptions of probability: An experiment with a small-group, activity-based, model building approach to introductory probability at the college level. *Educational Studies in Mathematics, 8,* 285-316.

Shaughnessy, J. M. (1992). Research in probability and statistics: Reflections and directions. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 465-494). New York: Macmillan.

Székely, G. J. (1986). *Paradoxes in probability and mathematical statistics.* Dordrecht: Reidel.

Toohey, P. G. (1995). *Adolescent perceptions of the concept of randomness.* Unpublished Master Thesis. University of Adelaide.

Truran, K. (1994). Children's understanding of random generators. In J. P. Ponte & J. F. Matos (Eds.), *Proceeding of the 18th International Conference for the Psychology of Mathematics Education* (pp. 328-336). Lisbon: International Group for the Psychology of Mathematics Education.

Wagenaar, W. A. (1972). Generation of random sequences by human subjects: A critical survey of literature. *Psychological Bulletin, 77,* 65-72.

Zabell, S. L. (1992). The quest for randomness and its statistical applications. In F. Gordon & S. Gordon (Eds.), *Statistics for the XXI Century* (pp. 139-166). Washington, DC: The Mathematical Association of America.