GPI for nonassociative algebras

M. Cabrera and A. Moreno
Departamento de Análisis Matemático. Facultad de Ciencias.
Universidad de Granada. 18071 Granada. Spain.
e-mail: cabrera@ugr.es, agalindo@ugr.es

Abstract

We will introduce the concept of generalized multiplicative polynomial identity (in short GMPI) for multiplicatively prime algebras. Our main result is a GMPI-theorem, which provides the following characterization of GMPI-algebras: Let A be a multiplicatively prime algebra with extended centroid C and central closure Q. Then the following assertions are equivalent: (i) There is a nonzero GMPI on $\mathcal{P}$ for some nonzero ideal $\mathcal{P}$ of $M(A)$. (ii) $M(Q)$ is a primitive algebra with nonzero socle and there is a nonzero idempotent $E \in M(Q)$ such that $EM(Q)E$ is a finite dimensional division C-algebra. (iii) $M(A)$ is GPI. (iv) A is GMPI.

Introduction

The GPI-theory has a wide scope of applications. One area of application is that of concrete nonassociative structures (such as alternative, Lie and Jordan) related to associative algebras. Although the PI-theory for general nonassociative algebras has been developed (see [16, 18, 19, 20]), as far as we know, the same cannot be said for the GPI-theory. Nonetheless, inspired by the connection between the GPI-theory and the local PI-theory for associative algebras [12], a GPI-theory for Jordan systems has been developed [14, 15].

In order to find a GPI-theory in a general nonassociative context, it seems that the framework for such a theory should be the class of multiplicatively semiprime algebras. This is based on the fact that these algebras

The authors are supported by Junta de Andalucía Grant FQM290 and D.G.I. project BFM2002-01529

behave well in relation to the extended centroid and the central closure [6]. Let us recall that a (not necessarily associative) algebra A is said to be *multiplicatively semiprime* (in short m.s.p.) whenever both A and $M(A)$ (the multiplication algebra of A) are semiprime. On the other hand, the class of multiplicatively semiprime algebras is quite large; for example, it contains: semiprime associative algebras [6], and more generally, nondegenerate alternative algebras [9]; nondegenerate Jordan algebras [9]; semiprime skew Lie algebras associated with a prime associative algebra with an involution [4]; semiprime algebras with a nondegenerate symmetric associative bilinear form [17]; and the free nonassociative algebra generated by a nonempty set [7]. Recently, a structure theory for multiplicatively semiprime algebras has been developed [3], which shows that the atomic multiplicatively semiprime algebras are just the essential subdirect products of families of multiplicatively prime algebras. This is the reason why the development of a GPI-theory must be firstly center on multiplicatively prime algebras.

Let us recall that a (not necessarily associative) algebra A is said to be *multiplicatively prime* (in short m.p.) whenever both A and $M(A)$ are prime. Given an algebra A, for $F \in M(A)$ and $a \in A$ we denote by $W_{F,a}$ the linear operator from $M(A)$ into A given by

$$W_{F,a}(T) = FT(a) \quad \text{for} \quad \text{all} \quad T \in M(A).$$

We recall that an algebra A with nonzero product is m.p. if, and only if, $W_{F,a} = 0 \quad (F \in M(A), a \in A)$ implies either $F = 0$ or $a = 0$ [5, Proposition 1].

The first section is devoted to obtaining some preliminary results on m.p. algebras. Among them, we emphasize Corollary 1.3. This result is the m.p.-version of a well-known lemma by Martindale and asserts that *if A is an m.p. algebra with extended centroid C, and if $W_{F,a} = W_{G,b}$ ($F, G \in M(A)\backslash\{0\}$, $a, b \in A\backslash\{0\}$), then there exists $\lambda \in C$ such that $b = \lambda a$ and $F = \lambda G$.*

The other main result in this section is Theorem 1.5 that reads as follows: *If A is an m.p. algebra with extended centroid C and central closure Q, if $F \in M(Q)\backslash\{0\}$ and $q \in Q\backslash\{0\}$ are such that the rank of $W_{F,q}$ has finite dimension over C, then $M(Q)$ is a primitive algebra with nonzero socle.*

In the second section we introduce a concept of generalized multiplicative polynomial identity (in short GMPI) for m.p. algebras. Roughly speaking, for a given m.p. algebra A, a GMPI is a finite sum of monomials of the form $F_0X_{i_1}F_1X_{i_2}...F_{n-1}X_{i_n}(q)$ for $n \in \mathbb{N}$, $q \in Q$ (the central closure of A),

$F_0, F_1, ..., F_{n-1} \in M(Q)$, and $X_{i_1}, X_{i_2}, ..., X_{i_n}$ noncommutative formal variables. Our main result is the following GMPI-theorem (Theorem 2.1): *Let A be an m.p. algebra with extended centroid C and central closure Q. Then there is a nonzero GMPI Φ on $\mathcal{P}$ for some nonzero ideal $\mathcal{P}$ of $M(A)$ if and only if $M(A)$ has a nonzero idempotent E such that $EM(A)$ is a minimal right ideal of $M(A)$ (hence $M(A)$ is a primitive algebra with nonzero socle) and $EM(A)E$ is a finite dimensional division algebra over C.* From this theorem, we obtain the following characterization of GMPI-algebras (Corollary 2.2): *Let A be an m.p. algebra with extended centroid C and central closure Q. Then the following assertions are equivalent: (i) There is a nonzero GMPI on $\mathcal{P}$ for some nonzero ideal $\mathcal{P}$ of $M(A)$. (ii) $M(Q)$ is a primitive algebra with nonzero socle and there is a nonzero idempotent $E \in M(Q)$ such that $EM(Q)E$ is a finite dimensional division C-algebra. (iii) $M(A)$ is GPI. (iv) A is GMPI.*

1 Preliminary results on m.p. algebras

In this talk, we will deal with algebras which are not necessarily associative over a fixed field $\mathbf{K}$ of zero characteristic. For an algebra A and for a in A, let L_a and R_a stand for the operators of left and right multiplication by a on A. Let $L(A)$ denote the algebra of all linear operators from A into A and let $M(A)$ denote the *multiplication algebra* of A, namely the subalgebra of $L(A)$ generated by the identity operator Id_A and the set $\{L_a, R_a : a \in A\}$.

An algebra A is said to be *multiplicatively prime* (in short *m.p.*) whenever both A and $M(A)$ are prime algebras. For $F \in M(A)$ and $a \in A$ we denote by $W_{F,a}$ the linear operator from $M(A)$ into A given by

$$W_{F,a}(T) = FT(a) \quad \text{for} \quad \text{all} \quad T \in M(A).$$

There is a useful characterization of m.p. algebras in terms of these operators. Namely, if A has nonzero product we have: A is m.p. if, and only if, $W_{F,a} = 0 \quad (F \in M(A), a \in A)$ implies either $F = 0$ or $a = 0$ [5, Proposition 1]. Moreover, if A is m.p., then for each nonzero ideal $\mathcal{P}$ of $M(A)$ we see that $W^{\mathcal{P}}_{F,a} = 0 \quad (F \in M(A), a \in A)$ implies either $F = 0$ or $a = 0$, where $W^{\mathcal{P}}_{F,a}$ denotes the restriction of $W_{F,a}$ to $\mathcal{P}$.

The theory of extended centroid and central closure of a prime algebra will become a key tool in the proof of our results. We refer the reader to [11] and [1] for a detailed account in a nonassociative context, and to [2] in an associative context (see also [8]). We recall that the extended centroid

$C(A)$ of a prime algebra A is a field extension of the base field, and the central closure $Q(A)$ of A is an algebra extension of A. Moreover, $Q(A)$ is a prime C-algebra which is generated by A, and the extended centroid of $Q(A)$ is isomorphic to $C(A)$. Also, we remember the following result of [5, Theorems 1 and 2], in which two isomorphisms appear, which that will be used frequently without explicit mention.

Theorem 1.1. *Let A be an m.p. algebra. Then the extended centroids of A and of $M(A)$ are isomorphic. More precisely, there exists an isomorphism φ from $C(A)$ onto $C(M(A))$ which is uniquely determined by the following condition: For each $\lambda \in C(A)$, we have*

$$\varphi(\lambda)(F)(x) = F(\lambda(x))$$

for all $F \in dom(\varphi(\lambda))$ and $x \in dom(\lambda)$. Moreover, if (via φ) both extended centroids are identified and denoted by C, then we have a C-algebra isomorphism Φ from $Q(M(A))$ onto $M(Q(A))$ given by

$$\Phi(\sum_{i=1}^{n} \lambda_i F_i)(\sum_{j=1}^{m} \mu_j a_j) = \sum_{\substack{1\leq i\leq n \\ 1\leq j\leq m}} \lambda_i \mu_j F_i(a_j).$$

As a consequence, $Q(A)$ is also an m.p. algebra.

Let A be an m.p. algebra with extended centroid C and central closure Q. For $F \in M(Q)$, $q \in Q$, and $\mathcal{P}$ nonzero ideal of $M(A)$ we denote by $W_{F,q}^{\mathcal{P}} : \mathcal{P} \to Q$ the linear mapping defined by

$$W_{F,q}^{\mathcal{P}}(T) = FT(q) \quad \text{for} \quad \text{all} \quad T \in \mathcal{P}.$$

It is clear that $W_{F,q}^{\mathcal{P}} = 0$ implies $W_{F,q}^{C\mathcal{P}} = 0$, and therefore either $F = 0$ or $q = 0$.

The following result is analogous to [2, Lemma 6.1.2], and, using a terminology that will be introduced in the second section, the first statement asserts that "for m.p. algebras there are no nonzero linear GMPI's in one variable".

Lemma 1.2. *Let A be an m.p. algebra with extended centroid C and central closure Q. Assume that $F_1, ..., F_n \in M(Q)$ are C-independent, $q_1, ..., q_n \in Q$ are C-independent, and $\mathcal{P}$ is a nonzero ideal of $M(A)$. Then*

(1) $\sum_{i=1}^{n} W^{\mathcal{P}}_{F_i,q_i} \neq 0$.

(2) If $dim_C(CRank(\sum_{i=1}^{n} W^{\mathcal{P}}_{F_i,q_i})) < \infty$*, then there exist* $F \in M(A)\backslash\{0\}$ *and* $a \in A\backslash\{0\}$ *such that* $dim_C(CRank(W_{F,a})) < \infty$*.*

Proof. (1) Suppose that $\sum_{i=1}^{n} W^{\mathcal{P}}_{F_i,q_i} = 0$. By [11, Theorem 3.1] applied to Q, there exists $G \in M(Q)$ such that $G(q_1) \neq 0$ and $G(q_i) = 0$ for all $i \in \{2, ..., n\}$. Now, we consider the mapping $\phi : M(Q) \to Q$ given by

$$\phi(T) = \left(\sum_{i=1}^{n} W_{F_i,q_i}\right)(TG).$$

Note that

$$\phi(T) = \sum_{i=1}^{n} F_i TG(q_i) = F_1 TG(q_1) = W_{F_1,G(q_1)}(T).$$

Therefore, taking into account that $\sum_{i=1}^{n} W^{\mathcal{P}}_{F_i,q_i} = 0$ implies $\phi(\mathcal{P}) = 0$, we deduce that $W^{\mathcal{P}}_{F_1,G(q_1)} = 0$, and hence either $F_1 = 0$ or $G(q_1) = 0$, which is a contradiction.

(2) It is clear that $dim_C(C\phi(\mathcal{P})) < \infty$. Put $q_1' = G(q_1)$. Since $\phi = W_{F_1,q_1'}$, it follows that $dim_C(CF_1\mathcal{P}(q_1)) < \infty$. Take $S \in \mathcal{P}$ such that $0 \neq F_1 S \in M(A)$, and $x \in A$ such that $0 \neq q_1' x \in A$. Setting $F = F_1 S$ and $a = q_1' x$, we have as above that $CFM(A)(a) \subseteq CF_1\mathcal{P}(q_1')$, and therefore $dim_C(CRank(W_{F,a})) < \infty$. □

As a consequence we obtain the m.p.-version of one of Martindale's well-known lemma [13, Theorem 1] (see also [2, Theorem 2.3.4]).

Corollary 1.3. *Let* A *be an m.p. algebra with extended centroid* C *and central closure* Q*. Assume that* $F_1, F_2 \in M(Q)\backslash\{0\}$ *and* $q_1, q_2 \in Q\backslash\{0\}$ *are such that* $W_{F_1,q_2} = W_{F_2,q_1}$*. Then there exists* $\lambda \in C$ *such that* $F_2 = \lambda F_1$ *and* $q_2 = \lambda q_1$*.*

Proof. According to the part (1) in the lemma above, either $\{F_1, F_2\}$ are C-dependent or $\{q_1, q_2\}$ are C-dependent. If there exists $\lambda \in C$ such that $F_2 = \lambda F_1$, then we have

$$0 = W_{F_1,q_2} - W_{F_2,q_1} = W_{F_1,q_2} - W_{\lambda F_1,q_1} = W_{F_1,q_2-\lambda q_1},$$

hence $q_2 - \lambda q_1 = 0$, and so $q_2 = \lambda q_1$. Analogously, if there exists $\lambda \in C$ such that $q_2 = \lambda q_1$, then we have

$$0 = W_{F_1,q_2} - W_{F_2,q_1} = W_{F_1,\lambda F_1} - W_{F_2,q_1} = W_{\lambda F_1 - F_2,q_1},$$

hence $\lambda F_1 - F_2 = 0$, and so $F_2 = \lambda F_1$. □

Now we are in a position to prove the following generalization of the above corollary. This generalization resembles [2, Theorem 2.3.7].

Theorem 1.4. *Let A be an m.p. algebra with extended centroid C and central closure Q. Assume that $n \in \mathbb{N}$, $n \geq 2$, $F_1, ..., F_n \in M(Q)\backslash\{0\}$ and $q_1, ..., q_n \in Q\backslash\{0\}$. Set $M = \sum_{i=1}^n CF_i$. Then the following conditions are equivalent:*

(i) For all $T_1, ..., T_{n-1} \in M(Q)$

$$\sum_{\sigma \in S_n} \epsilon(\sigma) F_{\sigma(1)} T_1 F_{\sigma(2)} T_2 ... F_{\sigma(n-1)} T_{n-1}(q_{\sigma(n)}) = 0,$$

where S_n is the permutation group;

(ii) Either $dim_C(M) = n - 1$ and $\sum_{i=1}^n \lambda_i q_i = 0$ when $\sum_{i=1}^n \lambda_i F_i = 0$, or $dim_C(M) \leq n - 2$.

Proof. Note that the case $n = 2$ follows from Corollary 1.3. To prove that (i) implies (ii), we proceed by induction on n. Let us assume that the implication is true for a natural number $n \geq 2$, and prove its veracity for $n+1$. Firstly, we will assume that $dim_C(M) = n+1$, and we will encounter a contradiction. By [2, Theorem 2.3.3], there exist $m \in \mathbb{N}$, $A_i, B_i \in M(Q)$ $(1 \leq i \leq m)$, such that

$$\sum_{i=1}^m A_i F_1 B_i \neq 0 \quad \text{and} \quad \sum_{i=1}^m A_i F_j B_i = 0 \quad (2 \leq j \leq n+1).$$

Next, substituting $B_j T_1$ for T_1 in the formula

$$\sum_{\sigma \in S_{n+1}} \epsilon(\sigma) F_{\sigma(1)} T_1 F_{\sigma(2)} T_2 ... F_{\sigma(n)} T_n(q_{\sigma(n+1)}) = 0,$$

then multiplying by A_j on the left, and adding we have

$$0 = \sum_{j=1}^n \sum_{\sigma \in S_{n+1}} \epsilon(\sigma) A_j F_{\sigma(1)} B_j T_1 F_{\sigma(2)} T_2 ... F_{\sigma(n)} T_n(q_{\sigma(n+1)}) =$$

$$\sum_{\sigma\in S_{n+1}} \epsilon(\sigma)\left(\sum_{j=1}^{n} A_j F_{\sigma(1)} B_j\right) T_1 F_{\sigma(2)} T_2 ... F_{\sigma(n)} T_n(q_{\sigma(n+1)}) =$$

$$\left(\sum_{j=1}^{n} A_j F_1 B_j\right) T_1 \left(\sum_{\substack{\sigma\in S_{n+1}\\ \sigma(1)=1}} \epsilon(\sigma) F_{\sigma(2)} T_2 ... F_{\sigma(n)} T_n(q_{\sigma(n+1)})\right) =$$

$$W_{\sum_{j=1}^{n} A_j F_1 B_j\ ,\ \sum_{\substack{\sigma\in S_{n+1}\\ \sigma(1)=1}} \epsilon(\sigma) F_{\sigma(2)} T_2 ... F_{\sigma(n)} T_n(q_{\sigma(n+1)})}(T_1),$$

and so

$$\sum_{\substack{\sigma\in S_{n+1}\\ \sigma(1)=1}} \epsilon(\sigma) F_{\sigma(2)} T_2 ... F_{\sigma(n)} T_n(q_{\sigma(n+1)}) = 0$$

for all $T_2, ..., T_n \in M(Q)$, which is contrary to the induction hypothesis. Therefore, $dim_C(M) \leq n$. Assume that $dim_C(M) = n$ and, suppose (simplifying the notation) that $F_{n+1} = \sum_{i=1}^{n} \lambda_i F_i$ for suitable $\lambda_1, ..., \lambda_n \in C$. Then, for all $T_1, ..., T_n \in M(Q)$ we have

$$0 = \sum_{\sigma\in S_{n+1}} \epsilon(\sigma) F_{\sigma(1)} T_1 F_{\sigma(2)} T_2 ... F_{\sigma(n)} T_n(q_{\sigma(n+1)}) =$$

$$\sum_{\substack{\sigma\in S_{n+1}\\ \sigma(n+1)=n+1}} \epsilon(\sigma) F_{\sigma(1)} T_1 F_{\sigma(2)} T_2 ... F_{\sigma(n)} T_n(q_{\sigma(n+1)}) +$$

$$\sum_{\substack{\sigma\in S_{n+1}\\ \sigma(n+1)\neq n+1}} \epsilon(\sigma) F_{\sigma(1)} T_1 F_{\sigma(2)} T_2 ... F_{\sigma(n)} T_n(q_{\sigma(n+1)}) =$$

$$\sum_{\sigma\in S_n} \epsilon(\sigma) F_{\sigma(1)} T_1 F_{\sigma(2)} T_2 ... F_{\sigma(n)} T_n(q_{n+1}) +$$

$$\sum_{\sigma\in S_n} \epsilon(\sigma) F_{\sigma(1)} T_1 F_{\sigma(2)} T_2 ... F_{\sigma(n)} T_n(-\sum_{i=1}^{n} \lambda_i q_i) =$$

$$\sum_{\sigma\in S_n} \epsilon(\sigma) F_{\sigma(1)} T_1 F_{\sigma(2)} T_2 ... F_{\sigma(n)} T_n \left(q_{n+1} - \sum_{i=1}^{n} \lambda_i q_i\right) =$$

$$\left[\sum_{\sigma\in S_n}\epsilon(\sigma)F_{\sigma(1)}T_1F_{\sigma(2)}T_2...F_{\sigma(n)}\right]T_n\left(q_{n+1}-\sum_{i=1}^{n}\lambda_iq_i\right)=$$

$$W_{\sum_{\sigma\in S_n}\epsilon(\sigma)F_{\sigma(1)}T_1F_{\sigma(2)}T_2...F_{\sigma(n)}\ ,\ q_{n+1}-\sum_{i=1}^{n}\lambda_iq_i}\ (T_n).$$

Since $\sum_{\sigma\in S_n}\epsilon(\sigma)F_{\sigma(1)}T_1F_{\sigma(2)}T_2...F_{\sigma(n)}\neq 0$ by [2, Theorem 2.3.7], and A is m.p., we conclude that $q_{\sigma(n+1)}-\sum_{i=1}^{n}\lambda_iq_i=0$.

To prove that (ii) implies (i), let us first assume that $n>2$, $dim_C(M)=n-1$ and, suppose (simplifying the notation) that $F_n=\sum_{i=1}^{n-1}\lambda_iF_i$ and $q_n=\sum_{i=1}^{n-1}\lambda_iq_i$ for suitable $\lambda_1,...,\lambda_{n-1}\in C$. Then, taking into account the well-known argument that the standard polynomial vanishes on dependent vectors, we have

$$\sum_{\sigma\in S_n}\epsilon(\sigma)F_{\sigma(1)}T_1F_{\sigma(2)}T_2...F_{\sigma(n-1)}T_{n-1}(q_{\sigma(n)})=$$

$$\sum_{\substack{\sigma\in S_n\\ \sigma(n)=n}}\epsilon(\sigma)F_{\sigma(1)}T_1F_{\sigma(2)}T_2...F_{\sigma(n-1)}T_{n-1}(q_n)+$$

$$\sum_{\sigma\in S_{n-1}}\epsilon(\sigma)F_{\sigma(1)}T_1F_{\sigma(2)}T_2...F_{\sigma(n-1)}T_{n-1}\left(-\sum_{i=1}^{n-1}\lambda_iq_i\right)=$$

$$\sum_{\sigma\in S_{n-1}}\epsilon(\sigma)F_{\sigma(1)}T_1F_{\sigma(2)}T_2...F_{\sigma(n-1)}T_{n-1}\left(q_n-\sum_{i=1}^{n-1}\lambda_iq_i\right)=0.$$

Finally, assume that $n>2$, $dim_C(M)\leq n-2$ and, suppose (simplifying the notation) that $F_n=\sum_{i=1}^{n-1}\lambda_iF_i$ and $F_{n-1}=\sum_{j=1}^{n-2}\mu_jF_j$. Then, reasoning as above, we have

$$\sum_{\sigma\in S_n}\epsilon(\sigma)F_{\sigma(1)}T_1F_{\sigma(2)}T_2...F_{\sigma(n-1)}T_{n-1}(q_{\sigma(n)})=$$

$$\sum_{\sigma\in S_{n-1}}\epsilon(\sigma)F_{\sigma(1)}T_1F_{\sigma(2)}T_2...F_{\sigma(n-2)}T_{n-2}\left(F_{\sigma(n-1)}T_{n-1}(q_n-\sum_{i=1}^{n-1}\lambda_iq_i)\right).$$

Now, denoting by $q'_j=F_jT_{n-1}(q_n-\sum_{i=1}^{n-1}\lambda_iq_i)$ $(1\leq j\leq n-1)$, and repeating the argumentation we obtain

$$\sum_{\sigma\in S_n}\epsilon(\sigma)F_{\sigma(1)}T_1F_{\sigma(2)}T_2...F_{\sigma(n-1)}T_{n-1}(q_{\sigma(n)})=$$

$$\sum_{\sigma \in S_{n-2}} \epsilon(\sigma) F_{\sigma(1)} T_1 F_{\sigma(2)} T_2 ... F_{\sigma(n-2)} T_{n-2} \left(q'_{n-1} - \sum_{j=1}^{n-2} q'_j \right) = 0,$$

because clearly $q'_{n-1} - \sum_{j=1}^{n-2} q'_j = 0$. □

The following result is analogous to [2, Lemma 6.1.4]. Our proof will be closer to the original proof in [13, Theorem 2].

Theorem 1.5. *Let A be an m.p. algebra with extended centroid C and central closure Q. Assume that $F \in M(Q)\backslash\{0\}$ and $q \in Q\backslash\{0\}$ are such that $dim_C(FM(Q)(q)) < \infty$. Then there exists a nonzero idempotent $E \in M(Q)$ such that $EM(Q)$ is a minimal right ideal of $M(Q)$ and $dim_C(EM(Q)E) < \infty$. As a consequence, $M(Q)$ is a primitive algebra with nonzero socle.*

Proof. Set $U = M(Q)(q)$ and $V = F(U) = FM(Q)(q)$. It is clear that U is an ideal of Q and V is a finite dimensional subspace of Q. Both are nonzero because Q is m.p. Consider the nonzero ideal of $M(Q)$ given by $[U : Q] = \{T \in M(Q) : T(Q) \subseteq U\}$. It is clear that $F[U : Q](Q) \subseteq V$. Therefore, we can consider the mapping $\varphi : F[U : Q] \to L(V)$ given by $\varphi(FT) = FT_{|V}$. It is clear that $F[U : Q]$ is a subalgebra of $M(Q)$ and φ is an algebra homomorphism with kernel

$$Ker(\varphi) = \{FT \in F[U : Q] \ : \ FTF = 0\}.$$

It is easy to see that $\varphi(F[U : Q])$ is a nonzero prime algebra. By the Wedderburn theorem, there exist a natural number n and a finite dimensional division algebra Δ over C such that $\varphi(F[U : Q])$ is isomorphic to $M_n(\Delta)$. In particular, $\varphi(F[U : Q])$ has a unit. Thus, there exists $T_0 \in [U : Q]$ such that

$$\varphi(FT_0)\varphi(FT) = \varphi(FT) \ \text{ and } \ \varphi(FT)\varphi(FT_0) = \varphi(FT) \ \text{ for all } \ T \in [U : Q],$$

equivalently

$$FT_0FTF = FTF \ \text{ and } \ FTFT_0F = FTF \ \text{ for all } \ T \in [U : Q]. \qquad (1)$$

From the first equality it follows that $(FT_0F - F)[U : Q]F = 0$, hence $FT_0F - F = 0$, and so $FT_0FT = FT$ for all $T \in [U : Q]$. Thus, $E_0 = FT_0$ is a left unit for the algebra $F[U:Q]$, and in particular E_0 is an idempotent. Note that the second equality in (1) can be rewritten as follows

$$FTE_0F = FTF \quad \text{for all} \quad T \in [U:Q]. \tag{2}$$

Given $T \in [U:Q]$ and $v \in V$, choose $G \in M(Q)$ such that $v = FG(q)$, and note that by using (2) we have

$$\varphi(FTE_0)(v) = FTE_0(v) = FTE_0FG(q) = FTFG(q) = FT(v) = \varphi(FT)(v).$$

Therefore, $\varphi(FTE_0) = \varphi(FT)$ for all $T \in [U:Q]$, and so $\varphi(F[U:Q]E_0) = \varphi(F[U:Q])$. Moreover, if $T \in [U:Q]$ satisfies $\varphi(FTE_0) = 0$, then $FTE_0F = 0$, and using (2) we see that $FTF = 0$, hence $FTFT_0 = 0$, and so $FTE_0 = 0$. Thus $\varphi_{|F[U:Q]E_0}$ is a one-to-one mapping. Taking into account the above assertions, we conclude that

$$F[U:Q]E_0 \cong \varphi(F[U:Q]E_0) = \varphi(F[U:Q]) \cong M_n(\Delta).$$

Thus, $F[U:Q]E_0$ has a minimal idempotent, say E, such that

$$E(F[U:Q]E_0)E \cong \Delta E \cong \Delta.$$

Since E_0 is the unit of $F[U:Q]E_0$, we see that $EE_0 = E_0E = E$, and so

$$EM(Q)E = EE_0M(Q)E_0E \subseteq EF[U:Q]E_0E \subseteq EM(Q)E.$$

Therefore,

$$EM(Q)E = E(F[U:Q]E_0)E \cong \Delta.$$

As a result, E is a minimal idempotent of $M(Q)$, and so $M(Q)$ has nonzero socle. □

2 Introducing and characterizing GMPI-algebras

Let A be an m.p. algebra with extended centroid C and central closure Q. Consider $\mathbf{X} = \{X_1, X_2, ...\}$ and form the coproduct $M(Q)_C\langle \mathbf{X} \rangle$ of the C-algebra $M(Q)$ and the free algebra $C\langle \mathbf{X} \rangle$. Let us denote by $M(Q)_C\langle \mathbf{X} \rangle'$ the left ideal of $M(Q)_C\langle \mathbf{X} \rangle$ generated by $\mathbf{X}$, that is, the set of all GPI (with coefficients in the multiplication algebra) that may be written as a linear combination of monomials ending in a variable. Consider the vector space over C given by $M(Q)_C\langle \mathbf{X}\rangle' \otimes_C Q$. Each element of this vector space will be called a GMPI (generalized multiplicative polynomial identity). If $s : M(Q)_C\langle \mathbf{X}\rangle \to M(Q)$ is a substitution homomorphism and if we denote by s' the restriction of s to $M(Q)_C\langle \mathbf{X}\rangle'$, then we can consider the canonical linear mapping $s' \otimes Id_Q : M(Q)_C\langle \mathbf{X}\rangle' \otimes_C Q \to M(Q) \otimes_C Q$. Since the evaluation mapping $\varepsilon : M(Q) \times Q \to Q$ given by $\varepsilon(F, q) = F(q)$ is bilinear, it determines, in a canonical way, a linear mapping $\widehat{\varepsilon} : M(Q) \otimes_C Q \to Q$. The linear mapping $\widehat{s} = \widehat{\varepsilon} \circ (s' \otimes Id_Q)$ will be called a substitution mapping. Since for $\Phi = \sum_{i=1}^n \phi_i \otimes q_i \in M(Q)_C\langle \mathbf{X}\rangle' \otimes_C Q$ we see that $\widehat{s}(\Phi) = \sum_{i=1}^n s(\phi_i)(q_i)$, it seems convenient to write the GMPI Φ as follows $\Phi = \sum_{i=1}^n \phi_i(q_i)$, even though this is a clear abuse of notation. Once one adopts this terminology, one can write the action of a substitution mapping in the following suggestive manner

$$\widehat{s}(\sum_{i=1}^n \phi_i(q_i)) = \sum_{i=1}^n s(\phi_i)(q_i).$$

As usual, if Φ involves the variables $X_1, ..., X_n$, then we will write

$$\Phi = \Phi(X_1, ..., X_n).$$

Given $T_1, ..., T_n \in M(Q)$, $\Phi(T_1, ..., T_n)$ will denote the element $\widehat{s}(\Phi) \in Q$ that appears by considering any substitution homomorphism s such that $X_1 \mapsto T_1, ..., X_n \mapsto T_n$. We will say that a nonzero ideal $\mathcal{P}$ of $M(A)$ satisfies Φ whenever $\Phi(T_1, ..., T_n) = 0$ for all $T_1, ..., T_n \in \mathcal{P}$.

Let $\mathcal{B}$ be a C-basis of $M(Q)$. By [2, Lemma 1.4.5], a C-basis of $M(Q)_C\langle \mathbf{X}\rangle$ is constituted by all the monomials of the form

$$F_0 X_{i_1} F_1 X_{i_2} .... F_{n-1} X_{i_n} F_n$$

for all $n \in \mathbb{N} \cup \{0\}$, $F_0, F_1, ..., F_n \in \mathcal{B}$ and $X_{i_1}, X_{i_2}, ..., X_{i_n} \in \mathbf{X}$. As a consequence, a C-basis of $M(Q)_C\langle \mathbf{X}\rangle'$ is constituted by all the monomials of the form

$$F_0 X_{i_1} F_1 X_{i_2} .... F_{n-1} X_{i_n}$$

for all $n \in \mathbb{N}$, $F_0, F_1, ..., F_{n-1} \in \mathcal{B}$ and $X_{i_1}, X_{i_2}, ..., X_{i_n} \in \mathbf{X}$. Once a C-basis $\mathcal{A}$ of Q is fixed, according to the well-known theorem of description of a basis in a tensor product, we see that a C-basis of $M(Q)_C\langle \mathbf{X}\rangle' \otimes_C Q$ is constituted by all the monomials of the form

$$F_0 X_{i_1} F_1 X_{i_2} .... F_{n-1} X_{i_n}(a)$$

for all $n \in \mathbb{N}$, $F_0, F_1, ..., F_{n-1} \in \mathcal{B}$, $X_{i_1}, X_{i_2}, ..., X_{i_n} \in \mathbf{X}$, and $a \in \mathcal{A}$. The concepts of degree and height can be defined as in [2, pp. 214-215]. The same can be said of the operations of type A and B, and as in [2, Remark 6.1.5] we have:

If $\Phi \neq 0$ is a GMPI of degree n on a nonzero ideal $\mathcal{P}$ of $M(A)$, then there exists a multilinear GMPI Ψ of degree $\leq n$ on $\mathcal{P}$.

Another approach to the GMPI's can be made as follows: Since the algebra $M(Q)_C\langle\mathbf{X}\rangle$ is a C-algebra extension of $M(Q)$, we see that $M(Q)_C\langle\mathbf{X}\rangle$ is naturally a right $M(Q)$-module. Moreover, it is clear that Q is a left $M(Q)$-module for the action determined by the evaluation mapping. Then we can consider the tensor product $M(Q)_C\langle\mathbf{X}\rangle \otimes_{M(Q)} Q$. It seems clear that this space is isomorphic to $M(Q)_C\langle\mathbf{X}\rangle' \otimes_C Q$.

The following result is analogous to the GPI-theorem in [2, Theorem 6.1.6]. The proof will be similar to the one given there, which attributes it to Chuang [10].

Theorem 2.1. (GMPI-theorem). *Let A be an m.p. algebra with extended centroid C and central closure Q. Then there is a nonzero GMPI Φ on $\mathcal{P}$ for some nonzero ideal $\mathcal{P}$ of $M(A)$ if and only if $M(A)$ has a nonzero idempotent E such that $EM(A)$ is a minimal right ideal of $M(A)$ (hence $M(A)$ is a primitive algebra with nonzero socle) and $EM(A)E$ is a finite dimensional division algebra over C.*

Proof. ($\Leftarrow$). Since $dim_C(EM(A)E) = n < \infty$, it is well-known that $EM(A)E$ satisfies the standard polynomial in $n+1$ variables. If we choose $a \in A$ such that $E(a) \neq 0$, then it is clear that

$$\Phi(X_1, ..., X_{n+1}) = St_{n+1}(EX_1, ..., EX_{n+1})(E(a))$$

is a nonzero GMPI on A.

($\Rightarrow$). Suppose that Φ is a nonzero GMPI on a nonzero ideal $\mathcal{P}$ of $M(A)$. We can assume that $\Phi = \Phi(X_1, ..., X_n)$ is multilinear of degree n. We fix a C-basis $\mathcal{A}$ of Q and a C-basis $\mathcal{B}$ of $M(Q)$, and we write Φ as a linear combination of basic monomials. By suitable reordering of the variables we may write

$$\Phi = F_0X_1F_1X_2....F_{n-2}X_{n-1}\Psi(X_n) + \sum \alpha_{m'}m' + \sum \beta_{m''}m'' + \sum \gamma_{m'''}m'''$$

for suitable $F_0, F_1, ..., F_{n-2} \in M(Q)$, $\alpha_{m'}, \beta_{m''}, \gamma_{m'''} \in C$, and where

(i) $\Psi(X_n)$ is a nonzero linear GMPI, that is

$$\Psi(X_n) = (\sum W_{G_i,a_i})(X_n)$$

for suitable $a_i \in \mathcal{A}$ and $G_i \in \mathcal{B}$.

(ii) $m' = F'_0 X_1 F'_1 X_2 ... F'_{n-2} X_{n-1} \Psi'(X_n)$ with $F'_0, F'_1, ..., F'_{n-2} \in \mathcal{B}$ and $(F'_0, F'_1, ..., F'_{n-2}) \neq (F_0, F_1, ..., F_{n-2})$, and

$$\Psi'(X_n) = (\sum W_{H_i,b_i})(X_n)$$

for suitable $b_i \in \mathcal{A}$ and $H_i \in \mathcal{B}$.

(iii) $m'' = F''_0 X_1 F''_1 X_2 ... F''_i X_n F''_{i+1} X_{i+1} ... F''_{n-1} X_{n-1}(c)$, with

$$F''_0, F''_1, ..., F''_{n-1} \in \mathcal{B} \text{ and } c \in \mathcal{A}.$$

(iv) m''' are monomials in which $X_1, X_2, ..., X_n$ appear in a different order.

Let $\mathcal{S}$ be the finite subset of $\mathcal{A}$ of all the elements that appear in the writing of Φ, and take V the C-linear envelope of $\mathcal{S}$. If $C\Psi(\mathcal{P}) \subseteq V$, then, by Theorem 1.5, we conclude the proof. Assume that $C\Psi(\mathcal{P}) \nsubseteq V$. Take $T \in \mathcal{P}$ such that $\Psi(T) \notin V$, and consider the GMPI

$$\Upsilon(X_1, ..., X_{n-1}) = \Phi(X_1, ..., X_{n-1}, T).$$

Now, fix a C-basis $\mathcal{A}'$ of Q containing $\mathcal{S} \cup \{\Psi(T)\}$ and write Υ as a linear combination of basic monomials for $\mathcal{A}'$ and $\mathcal{B}$. It is clear that one of these monomials is of the form

$$H = F_0 X_1 F_1 X_2 .... F_{n-2} X_{n-1}(\Psi(T)).$$

Now we will see that H cannot be canceled by any monomials. Indeed, the monomials m' are of the form $F'_0 X_1 F'_1 X_2 ... F'_{n-2} X_{n-1}(a')$ with $a' \in \mathcal{A}'$, and, since $(F'_0, F'_1, ..., F'_{n-2}) \neq (F_0, F_1, ..., F_{n-2})$, the GPI-part of these monomials is C-independent with the GPI-part of H, therefore they are C-independent with H. The monomials m'' determine monomials of the form $G'_0 X_1 G'_1 X_2 ... G'_{n-1} X_{n-1}(c)$ with $c \in \mathcal{S}$, and $G'_0, G'_1, ..., G'_{n-1} \in \mathcal{B}$, and, since $\Psi(T) \notin \mathcal{B}$, the Q-part of these monomials is C-independent with the Q-part of H, therefore they are C-independent with H. The GPI-part of monomials of type m''' is C-independent with the GPI-part of H, therefore they are C-independent with H. Thus Υ is a nonzero GMPI of degree $n-1$ satisfies by $\mathcal{P}$. Now, by induction, the proof is complete. □

The following result is analogous to [2, Corollary 6.1.7]. The proof will be similar to the one given there.

Corollary 2.2. *Let A be an m.p. algebra with extended centroid C and central closure Q. Then the following assertions are equivalent:*

(i) *There is a nonzero GMPI on $\mathcal{P}$ for some nonzero ideal $\mathcal{P}$ of $M(A)$.*

(ii) *$M(Q)$ is a primitive algebra with nonzero socle and there is a nonzero idempotent $E \in M(Q)$ such that $EM(Q)E$ is a finite dimensional division C-algebra.*

(iii) *$M(A)$ is GPI.*

(iv) *A is GMPI.*

Proof. (i) $\Rightarrow$ (ii). By the GMPI-theorem.

(ii) $\Rightarrow$ (iii). By the GMPI-theorem.

(iii) $\Rightarrow$ (iv). Let ϕ be a nonzero GPI on $M(A)$. Fix a C-basis $\mathcal{B}$ of $M(A)$ and write $\phi = \sum \alpha_m m$ of a linear combination of basic monomials. If such a monomial is given by $m = F_0 X_{i_1} F_1 X_{i_2} .... F_{n-1} X_{i_n} F_n$, then choose $q \in Q$ such that $F_n(q) \neq 0$. It is clear that $\Phi = \sum \alpha_m m(q)$ is a nonzero GMPI on $M(A)$.

(iv) $\Rightarrow$ (i). This implication is obvious. $\square$

References

[1] W. E. Baxter and W. S. Martindale III, Central closure of semiprime nonassociative rings. *Comm. Algebra* **7** (1979), 1103-1132.

[2] K. I. Beidar, W. S. Martindale III, and A. V. Mikhalev, *Rings with generalized identities.* Marcel Dekker, New York, 1996.

[3] J. C. Cabello and M. Cabrera, Structure theory for multiplicatively semiprime algebras. Preprint. University of Granada, 2003.

[4] J. C. Cabello, M. Cabrera, G. López, and W. S. Martindale 3rd, Multiplicative semiprimeness of skew Lie algebras. *Comm. Algebra* (to appear).

[5] M. Cabrera and A. A. Mohammed, Extended centroid and central closure of the multiplication algebra. *Comm. Algebra* **27** (1999), 5723-5736.

[6] M. Cabrera and A. A. Mohammed, Extended centroid and central closure of multiplicatively semiprime algebras. *Comm. Algebra* **29** (2001), 1215-1233.

[7] M. Cabrera and A. A. Mohammed, Totally multiplicatively prime algebras. *Proc. Royal Soc. Edinburg* **132A** (2002), 1145-1162.

[8] M. Cabrera and A. Rodríguez, Extended centroid and central closure of semiprime normed algebras. A first approach. *Comm. Algebra* **18** (1990), 2293-2326.

[9] M. Cabrera and A. R. Villena, Multiplicative-semiprimeness of nondegenerate Jordan algebras. To appear.

[10] C.-L. Chuang, A short proof of Martindale's theorem on GPIs. *J. Algebra* **151** (1992), 156-159.

[11] T. S. Erickson, W. S. Martindale III, and J. M. Osborn, Prime nonassociative algebras. *Pacific J. Math.* **60** (1975), 49-63.

[12] A. Fernández and M. I. Tocón, The local algebras of an associative algebra and their applications. Preprint. University of Málaga.

[13] W. S. Martindale 3rd, Prime rings satisfying a generalized polynomial identity. *J. Algebra* **12** (1969), 576-584.

[14] F. Montaner, Local PI theory of Jordan systems. *J. Algebra* **216** (1999), 302-327.

[15] F. Montaner, Local PI theory of Jordan systems II. *J. Algebra* **241** (2001), 473-514.

[16] J. M. Osborn, Varieties of algebras. *Adv. Math.* **8** (1972), 163-369.

[17] A. Rodríguez, Continuity of densely valued homomorphisms into H*-algebras. *Quart. J. Math. Oxford* **46** (1995), 107-118.

[18] L. H. Rowen, Nonassociative rings satisfying normal polynomial identities. *J. Algebra* **49** (1977), 104-111.

[19] L. H. Rowen, Polynomial identities of nonassociative rings, Part I: The general structure theory of nonassociative rings, with emphasis on polynomial identities and central polynomials, Part II: Fine points of the structure theory. *Illinois J. Math.* **22** (1978), 341-378, 521-540.

[20] L. H. Rowen, *Polynomial identities in ring theory.* Academic Press, New York, 1980.