Forecasting with exponential smoothing methods and bootstrap

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***Abstract*— The Boot.EXPOS procedure is an algorithm that combines the use of exponential smoothing methods with the bootstrap methodology for obtaining forecasts. In previous works the authors have studied and analyzed the interaction between these two methodologies. The initial sketch of the procedure was developed, modified and evaluated until its final form designated as Boot.EXPOS.**

I. INTRODUCTION

A time series is a sequence of observations indexed by time, usually ordered in equally spaced intervals and correlated. In our days it is well known the importance of time series studies. These studies provide indicators about a country economy, the unemployment rate, the export and import product rates, etc. The most interesting and ambitious task in time series analysis is to forecast future values. Models are commonly fitted in order to predict future values of a time series.

Exponential smoothing methods (EXPOS[1]) are the most widely used forecasting methods. Exponential smoothing refers to a set of methods that can be used to model and to obtain forecasts. This is a versatile approach that continually updates a forecast emphasizing the most recent experience, that is, recent observations are given more weight than the older observations.

The bootstrap procedure [1] is a very popular methodology for independent data because of its simplicity and nice properties. It is a computer-intensive method that presents solutions in situations where the traditional methods fail or are very difficult to apply. Efron's bootstrap (IID bootstrap) has revealed inefficient in the context of dependent data, such as in the case of time series, where the dependence structure arrangement has to be kept during the resampling scheme. However, if the time series process is driven from iid innovations another way of resampling can be used; then the IID bootstrap can easily be extended to the dependent case. The autoregressive AR(p) is a commonly example of such a process. Because of the iid nature of the AR residuals, the IID bootstrap can easily be extended to the dependent case. This procedure is easy to apply, and leads to good theoretical behavior for estimates when the model is correct.

In previous works [2], [3], [4], [5], [6] the authors have studied and analyzed the relationship between EXPOS methods and bootstrap methodology. A procedure denominated Boot.EXPOS was constructed to forecast time series.

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[1]Designation in [11].

II. EXPONENTIAL SMOOTHING METHODS

Exponential smoothing refers to a set of forecasting methods, several of which are commonly used. The EXPOS is a procedure that continually updates a forecast emphasizing the most recent experience, that is, recent observations are given more weight than the older observations, see [10], [11], [12]. Many researchers have investigated and developed the EXPOS models in a total of 15 methods, table I [12], [13].

TABLE I

THE EXPONENTIAL SMOOTHING METHODS.

	Seasonal Component		
Trend Component	N (None)	A (Additive)	M (Multiplicative)
N (None)	N,N	N,A	N,M
A (Additive)	A,N	A,A	A,M
Ad (Additive damped)	Ad,N	Ad,A	Ad,M
M (Multiplicative)	M,N	M,A	M,M
Md (Multiplicative damped)	Md,N	Md,A	Md,M

The exponential smoothing methods have the corresponding statistical models. Hyndman *et al.* [13] presents a state space formulation for all models in the classification of table I. The state space model usually consists of two sets of equations: the observation equation (1) and the state equation (2),

$$y_t = \mathbf{w}'\mathbf{x}_{t-1} + \varepsilon_t, \quad (1)$$

$$\mathbf{x}_t = F\mathbf{x}_{t-1} + \mathbf{g}\varepsilon_t, \quad (2)$$

with $t = 1, 2, \cdots$, where y_t is the observation in time t, $\mathbf{x}_t$ is a "state vector" containing unobserved components (level, trend and seasonality), $\{\varepsilon_t\}$ is a white noise series and F, $\mathbf{g}$ and $\mathbf{w}$ are coefficients. The first equation (1) relates the observable time series value y_t to a random $k-$vector $\mathbf{x}_{t-1}$ of unobservable components from the previous period. $\mathbf{w}$ is a fixed $k-$vector. F is a fixed $k \times k$ matrix and $\mathbf{g}$ is a $k-$vector of smoothing parameters. For more details see [13]. The estimates of the exponential smoothing parameters are obtained by minimizing the mean squared error (MSE) of the one-step-ahead forecasts errors over the fitted period. The model selection is made using the Akaike's criterion (AIC). This model selection criterion is preferable when compared to other criteria because of the parsimonious model penalty, see [14] for more details.

III. ABOUT BOOTSTRAP METHODOLOGY

Among resampling techniques, bootstrap is perhaps the most popular one. It is a computational method for estimating the distribution of an estimator or test statistic by resampling from the data. Under conditions that hold in a wide variety of applications, the bootstrap provides approximations to distributions of statistics, coverage probabilities of confidence intervals and accurate rejection probabilities of tests. The procedure was devised for an i.i.d. situation and it usually fails for dependent observations.

In context of stationary time series two different bootstrap methods have been proposed. Perhaps the best-known for time-series data is the block bootstrap. However, if the time series process is driven from i.i.d. innovations another way of resampling can be considered. The classical bootstrap derived for i.i.d. samples can easily be extended to the dependent case.

Another procedure, the sieve bootstrap, was proposed by Bühlmann (1997) [15] for dependent observations and extended by Alonso *et al.* [16], [17] for constructing prediction intervals in stationary time series. In a few words, the sieve bootstrap considers first an autoregressive process that is fitted to a stationary time series. Considering a model-based approach, which resamples from approximately i.i.d. residuals, the classical bootstrap methodology was applied to the centered residuals. Following Bühlmann [15] and Lahiri [18], validity and accuracy of IID-innovation bootstrap is well studied.

In previous works, Cordeiro and Neves [2], [3], [4], [5], [6] studied and analyzed the possibility of joining EXPOS methods and the bootstrap methodology. From those studies the idea behind the sieve bootstrap, suggested the connection of those two procedures.

IV. Computational procedure for prediction

A first computational algorithm was constructed using four models for fitting to the time series: single exponential smoothing, Holts linear and Holt-Winters with additive and multiplicative seasonality. Nowadays it considers thirty exponential smoothing methods and it consists of an automatic procedure in R language. This procedure was named Boot.EXPOS. The idea is to select the most adequate EXPOS model by using the AIC criterion and obtain the residuals. The error component is isolated and investigated regarding its stationarity using the Augmented Dickey-Fuller test. If it is not compatible with this hypothesis, data transformation is required. If there is some stationarity evidence, the residual sequence is filtered by an autoregressive model, autoregressive coefficients are estimated and innovations are obtained. In the context of AR models the bootstrap can be conducted by resampling the centered residuals and then generating a data set, using the estimated coefficients and the resampled residuals. The EXPOS fitted values and the reconstructed series are used to obtain a sample path of the data. Forecasts are obtained using the initial EXPOS model. The bootstrap process is repeated B times and information is kept into a matrix. An "optimal" point forecast is obtained by taking the average of each column.

A. Algorithm sketck

For a given time series $\{y_1, \cdots, y_n\}$ select the "best" EXPOS model (Table I) using the AIC criterion.

Any good model should yield residuals that do not show a significant pattern. It is rare to discuss white noise in this context because there is frequently some pattern left in the residuals, see [11]. In order to model such left-over patterns and in case of stationarity, an autoregressive model is used to filter the EXPOS residuals series. Thus, in order to apply the residual-based bootstrap, a stationary series is required.

The algorithm that joins the EXPOS methods with the bootstrap approach is summarized as follows:

Step 0: Select an EXPOS model by AIC criterion, obtain the exponential smoothing constants $\theta_0 = (\alpha, \beta, \gamma, \phi)$, the $\hat{\boldsymbol{y}} = \{\hat{y}_1, \cdots, \hat{y}_n\}$ and the residual sequence $\{r_1, \cdots, r_n\}$;

Boot.EXPOS

Step 1: Fit an AR(p) to the residual sequence using the AIC criterion;
Step 2: Obtain the AR residuals;
For B replicates
Step 3: Resample the centered residuals;
Step 4: Obtain a new series by recursion using the resampld series and the autoregressive coefficients from **Step 1**;
Step 5: Join the fitted values $\hat{\boldsymbol{y}}$ (**Step 0**) to the previous series;
Step 6: Forecast the initial series using the selected model and θ_0 estimated in **Step 0**.

Statistical tests, transformations and differentiation are prepared for analysis of stationarity of the random part before the AR ajustment is done (**Step 1** of Boot.EXPOS). The computational work is performed using the R 2.14.1 [7] and packages forecast [8] and tseries [9]. A new function Boot.EXPOS is implemented.

V. Remarks

In this article the authors propose the use of the Boot.EXPOS procedure to forecast time series. Based on past and recent empirical results in forecasting using the Boot.EXPOS procedure, it has revealed a good option in obtaining forecasts. This suggests that the "optimal" combination of EXPOS methods and bootstrap can provide more accurate forecasts.

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