Gradient estimates for Mean Curvature Flow: from Riemannian to sub-Riemannian

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Granada - 11 September, 2025

Mean Curvature Flow

Assume that

- (N^n, g) is a n-dimensional Riemannian manifold;
- M^{n-1} is a smooth, compact $(n-1)$-dimensional manifold without boundary.

A smooth family of immersions $F(\cdot, t) : M^{n-1} \times [0, T) \to N^n$ evolves by **mean curvature flow** if

$$\frac{\partial F}{\partial t} = -H\nu$$

where $-H\nu$ is the mean curvature vector.

Let $\Omega \subset N^n$ be a bounded, open set with smooth, **mean convex** boundary, meaning that $H > 0$.

Under the flow, the evolving hypersurface $\Sigma_t := F(M^{n-1}, t)$ *remains mean convex* and thus moves monotonically inward.

⚠ Singularities may form and the flow cannot be classically prolonged.

Arrival time function

The ***Arrival Time Function*** $u : \overline{\Omega} \to \mathbb{R}$ records the time when the front Σ_t reaches a point $x \in \Omega$:

$$\Sigma_t = \{x \in \Omega \mid u(x) = t\}.$$

We can formulate the problem weakly as

$$\begin{cases} -1 = \Delta u - \nabla\nabla u\left(\frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|}\right) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

This formulation provides a weak continuation of the flow past singularities.

⚠ Equation (1) reduces to $H = \frac{1}{|\nabla u|} > 0$ meaningful only in the mean convex case.

Theorem (Evans-Spruck) If $\partial\Omega \subset \mathbb{R}^n$ is strictly mean convex then there exists a unique Lipshitz viscosity solution to (1).

Gradient estimate

A key tool for existence is the gradient estimate

$$|\nabla u|^2(x) \leq \max_{\partial\Omega} |\nabla u|^2$$

which is **not sharp**.

We are interested in a (possibly) **Sharp gradient estimate.** Why?

- Understand to what extent the theory of mean convex MCF can work outside of $\mathbb{R}^n$.
- Possible geometric interpretations.

Theorem (G.B., M. Fogagnolo, V. Franceschi)

$\partial\Omega \subset \mathbb{R}^n$ strictly mean convex, u arrival time function for the mean curvature flow. Then for any $x \in \Omega$

$$\left[|\nabla u|^2 + \frac{2u}{n-1}\right](x) \leq \max_{\partial\Omega} |\nabla u|^2.$$

Extinction time estimate

On smooth points of the evolving $\Sigma_t = \{u = t\}$

$$\frac{1}{H^2_{\Sigma_t}} + \frac{2t}{n-1} \leq \max_{\partial\Omega} \frac{1}{H^2}$$

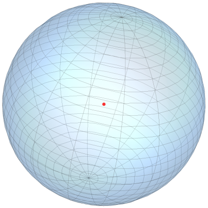
Corollary: Sharp extinction time estimate

The extinction time $T_{\max} = \max_{\overline{\Omega}} u$ satisfies

$$T_{\max} \leq \max_{\partial\Omega} \frac{n-1}{2H^2}.$$

Let S_R be the sphere of radius $R > 0$ in $\mathbb{R}^n$. Its mean curvature is $H = \frac{n-1}{R}$, and the $T_{\max}$ is

$$T_{\max} = \frac{R^2}{2(n-1)} = \max_{S_R} \frac{n-1}{2H^2}.$$



Gradient estimate for Ricci bounded from below

Let (N^n, g) be a Riemannian manifold with $\mathrm{Ric} > -k$. If u solves MCF in a smooth mean convex $\Omega \subset M$, then

$$e^{-2ku(x)}\Big(|\nabla u(x)|^2 - \tfrac{1}{k(n-1)}\Big) \leq \max_{\partial\Omega}\Big(|\nabla u|^2 - \tfrac{1}{k(n-1)}\Big) \quad \forall x \in \Omega.$$

Equality achieved on geodesic balls in the space form of constant curvature $\frac{k}{n-1}$.

Riemannian approximations. In each $\mathbb{H}_\varepsilon$ we have $\mathrm{Ric} > -k_\varepsilon$, hence the same inequality holds with k_ε. But as $\varepsilon \to 0$ we have $k_\varepsilon \to +\infty$, and

$$e^{-2k_\varepsilon u(x)}\Big(|\nabla u(x)|^2 - \tfrac{1}{k_\varepsilon(n-1)}\Big) \longrightarrow 0,$$

so the estimate *degenerates* and yields no useful uniform gradient bound.

Right gradient estimate u_ε^δ

Consider the solution ${v_\varepsilon}^\delta$ in Ω_ε to the regularized equation (elliptic)

$$-1 = \Delta_\varepsilon {v_\varepsilon}^\delta - \sum_{i,j=1}^{3} \frac{{X^\varepsilon}_i {v_\varepsilon}^\delta {X^\varepsilon}_j {v_\varepsilon}^\delta}{|\nabla_\varepsilon {v_\varepsilon}^\delta|^2 + \delta^2} {X^\varepsilon}_i {X^\varepsilon}_j {v_\varepsilon}^\delta.$$

we search for a gradient estimates in order to get $\|{\nabla^\delta}_\varepsilon v_\varepsilon\|_{L^\infty(\partial\Omega)} \leq \tilde{C}$.

💡 We need a Bochner formula providing an elliptic operator for $|\tilde{\nabla}_\varepsilon u_\varepsilon^\delta|^2$, in order to apply the maximum principle.
Using **right-invariant** derivatives $(\tilde{X}_1,\ \tilde{X}_2,\ \tilde{X}_3)$, rather than left-invariant ones:

$$\Delta_\varepsilon(|\tilde{\nabla}_\varepsilon f|^2) = 2\left[|\tilde{\nabla}_\varepsilon(X_1 f)|^2 + |\tilde{\nabla}_\varepsilon(X_2 f)|^2 + \varepsilon^2|\tilde{\nabla}_\varepsilon(X_3 f)|^2\right] + 2\langle \tilde{\nabla}_\varepsilon f, \tilde{\nabla}_\varepsilon \Delta_\varepsilon f\rangle.$$

We find an elliptic operator such that $L|\tilde{\nabla}_\varepsilon u_\varepsilon^\delta|^2 \geq 0$, then by **Maximum Principle**

$$\|\tilde{\nabla}_\varepsilon u_\varepsilon^\delta\|_{L^\infty(\Omega_\varepsilon)} \leq C(n)\|\tilde{\nabla}_\varepsilon u_\varepsilon^\delta\|_{L^\infty(\partial\Omega_\varepsilon)}.$$

Gradient estimate — non-characteristic

Barrier argument: constructing a subsolution of the form

$$w = \lambda g(d), \quad g(z) = \log \frac{s_0}{s_0 - z}, \quad d(p) = d_E(p, \partial\Omega_\varepsilon)$$

we find that the derivatives are bounded, which yields

$$\|\tilde{\nabla}_\varepsilon u_\varepsilon^\delta\|_{L^\infty(\partial\Omega_\varepsilon)} \leq \mathrm{Lip}(w), \quad \forall \delta < \delta_0, \ \varepsilon < \bar{\varepsilon}.$$

⚠ This strategy requires finding $\bar{\varepsilon} > 0$ and $s_0 > 0$ such that

$$H_\varepsilon(\{p \in \mathbb{R}^3 : d(p) = s\}) > 0, \quad \forall 0 < \varepsilon < \bar{\varepsilon}, \ s < s_0.$$

This holds *only in the non-characteristic case*: for the Pansu sphere $H_\varepsilon \to 0$ near characteristic points .

Gradient estimate - axisymmetric

Under axisymmetric assumption holds that

$$|\tilde{\nabla}_\varepsilon u_\varepsilon^\delta|^2 = |\nabla_\varepsilon u_\varepsilon^\delta|^2$$

Using a specific case of the more general ***Brakke's Local Regularity Theorem*** we can pass to the limit

$$|\nabla_\varepsilon u_\varepsilon^\delta| \to |\nabla_\varepsilon u_\varepsilon|, \quad \text{as } \delta \to 0 \text{ on } \partial\Omega_\varepsilon.$$

Hence we obtain the **gradient estimate**

$$\|\nabla_\varepsilon u_\varepsilon\|_{L^\infty(\Omega_\varepsilon)} \le C(n)\|\nabla_\varepsilon u_\varepsilon\|_{L^\infty(\partial\Omega_\varepsilon)} \le C(n) \sup_{\partial\Omega_\varepsilon} \frac{1}{H} \le C,$$

exploiting the **Mean convexity** assumption.

THANK YOU!