CR invariant surfaces and hyperbolic equations

Jih-Hsin Cheng
Institute of Mathematics, Academia Sinica, Taipei

Sep 10-12, 2025

Workshop on GVP in Sub-Riemannian Geometry, Granada, Spain

CR structures on a contact 3-manifold

Let $M = \partial D \subset C^2$ where D is a strongly pseudoconvex domain.

- Contact bundle:

$$\xi := TM \cap J_{C^2} TM$$

- *CR* structure on M :

$$J : \xi \to \xi$$

defined by $J = J_{c^2}|_{\xi}$ (satisfying $J^2 = -Id$)

- On an abstract contact 3-manifold (M, ξ), a *CR* structure is an endomorphism $J : \xi \to \xi$ such that

$$J^2 = -I.$$

CR invariant surface area element

A surface $\Sigma \subset M$, a CR manifold of 3D.

- A CR invariant surface (area) element (or 2-form) is a 2-form dA_Σ defined on Σ such that

$$dA_{\Sigma_1} = \varphi^*(dA_{\Sigma_2})$$

for any CR diffeomorphism φ of M :

$$\begin{aligned} \varphi \quad &: \quad M \to M, \\ \varphi(\Sigma_1) \quad &= \quad \Sigma_2. \end{aligned}$$

- In early 90's (Math. Zeit., 1995), I discovered two such CR invariant surface elements dA_1 and dA_2 (via Cartan-Chern's method of admissible frames).

Basic quantities in pseudohermitian geometry (1)

- Let $\xi := \ker\theta$, the contact plane distribution.
  - S_Σ (**singular set**) $:= \{p \in \Sigma \mid \xi = T\Sigma \text{ at } p\}$.
  - On nonsingular set $\Sigma \backslash S_\Sigma$ we have the 1-dimensional foliation $\xi \cap T\Sigma$.
  - Take $e_1 \in \xi \cap T\Sigma$, unit w.r.t. the Levi metric $\frac{1}{2}d\theta(\cdot, J\cdot)$. Let $e_2 := Je_1$, the horizontal normal.
- Let $\Sigma = \partial\Omega$. Define *p*-**area** $\theta \wedge e^1$ (*p*-**mean curvature** H, resp.) by varying volume (*p*-area, resp.) along e_2 :

$$\begin{aligned} \delta_{e_2} \int_\Omega \theta \wedge d\theta &= 2\int_\Sigma \theta \wedge e^1. \\ \delta_{e_2} \int_\Sigma \theta \wedge e^1 &= -\int_\Sigma H\theta \wedge e^1. \end{aligned}$$

-where e^1, e^2, θ is a coframe dual to e_1, e_2, T (Reeb vector field defined by $\theta(T) = 1$, $d\theta(T, \cdot) = 0$).

Basic quantities in pseudohermitian geometry (2)

A deviation function α on Σ is defined so that

$$T + \alpha e_2 \in T\Sigma.$$

- As a set, $\mathbf{H}_1 = R^3$. Let x, y, t denote its coordinates. Take the contact form
$$\Theta := dt + xdy - ydx.$$
- Let $\mathring{e}_1 := \partial_x + y\partial_t$, $\mathring{e}_2 := \partial_y - x\partial_t$ be standard left invariant vector fields. Then define $\hat{J} : \xi(:= \ker \Theta) \to \xi$ (called CR structure) by
$$\hat{J}\mathring{e}_1 = \mathring{e}_2, \hat{J}\mathring{e}_2 = -\mathring{e}_1.$$
- So $(\mathbf{H}_1, \hat{J}, \Theta)$ is a (flat) pseudohermitian 3-manifold with Tanaka-Webster curvature W=0, torsion A_{11}=0.

dA_1 and dA_2 in pseudohermitian quantities

(Cheng-Yang-Zhang, 2017) Yongbing Zhang— USTC, Hefei
For a nonsingular surface $\Sigma \subset M = \mathbf{H}_1(W = A_{11} = 0)$, dA_1 and dA_2 read

$$\begin{aligned} dA_{\mathbf{1}} &= |e_1(\alpha) + \frac{1}{2}\alpha^2 + \frac{1}{6}H^2 + \frac{1}{4}W - ImA_{11}|^{3/2}\theta \wedge e^1, \\ dA_2 &= \{(T + \alpha e_2)(\alpha) + [\frac{2}{3}e_1(\alpha) + \frac{1}{3}\alpha^2]H + \frac{2}{27}H^3\}\theta \wedge e^1. \end{aligned}$$

–(W, A_{11} omitted in dA_2)

- Define two CR invariant energy functionals:

$$E_1(\Sigma) := \int_\Sigma dA_1, \ E_2(\Sigma) := \int_\Sigma dA_2.$$

-(CR version of the Willmore energy $\int_\Sigma H^2 dA$)

The transformation laws

The transformation laws of e_1, e_2, T, and α, H under the conformal change of contact form: $\tilde{\theta} = \lambda^2\theta$, $\lambda > 0$, read

- $\tilde{e}_1 = \lambda^{-1}e_1$,
- $\tilde{e}_2 = \lambda^{-1}e_2$,
- $\tilde{T} = \lambda^{-2}T + \lambda^{-3}(e_2(\lambda)e_1 - e_1(\lambda)e_2)$,
- $\tilde{\alpha} = \lambda^{-1}\alpha + \lambda^{-2}e_1(\lambda)$,
- $\tilde{H} = \lambda^{-1}H - 3\lambda^{-2}e_2(\lambda)$.
- $\tilde{W} = \lambda^{-2}(W - 4\lambda^{-1}\Delta_b\lambda)$
- $\tilde{A}_{11} = \lambda^{-2}(A_{11} + 2i\lambda^{-1}\lambda_{11} - 6i\lambda^{-2}\lambda_1\lambda_1)$
  $-dA_1$ and dA_2 are invariant under the conformal change of contact form, so are $E_1(\Sigma)$ and $E_2(\Sigma)$.

Minimizers for $E_\mathbf{1}$

- Type I/II shifted Heisenberg spheres: $(r^2 \pm \frac{\sqrt{3}}{2}\rho^2)^2 + 4t^2 = \rho^4$, $\rho > 0$ satisfy

$$H_{cr} = e_1(\alpha) + \frac{1}{2}\alpha^2 + \frac{1}{6}H^2 = 0, \tag{1}$$

- So they are minimizers for $E_\mathbf{1}(\Sigma) := \int_\Sigma dA_\mathbf{1}$.
- **Conjecture 1**: Type I and type II shifted Heisenberg spheres are the only closed minimizers for $E_\mathbf{1}$ (with zero energy) in $\mathbf{H}_1$.
  -Note that usual Heisenberg spheres do not satisfy equation (1).

Euler-Lagrange equation for E_1

The Euler-Lagrange equation for E_1 reads

$$0 = (e_1 + 3\alpha)(|H_{cr}|^{1/2}\mathfrak{f}) + sign(H_{cr})|H_{cr}|^{1/2}\mathfrak{g}$$

- where $H_{cr} := ... = e_1(\alpha) + \frac{1}{2}\alpha^2 + \frac{1}{6}H^2 \neq 0$ (by assumption),
- $|H_{cr}|\ \mathfrak{f} := ... = e_1(H)(e_1(\alpha) + \frac{1}{2}\alpha^2 + \frac{1}{3}H^2) + H(T + \alpha e_2)(H)$ $+\frac{3}{2}(T + \alpha e_2)(e_1(\alpha) + \frac{1}{2}\alpha^2) - \alpha H\{\frac{7}{2}e_1(\alpha) + \frac{5}{2}\alpha^2 + \frac{2}{3}H^2\}$
- $\mathfrak{g} := ... = 9(T + \alpha e_2)(\alpha) + 6H(e_1(\alpha) + \frac{1}{2}\alpha^2) + \frac{2}{3}H^3.$

dA_1 in S^3 and Clifford torus

- For $\Sigma \subset S^3$, the above CR invariant surface area form dA_1 in S^3 reads

$$dA_{\mathbf{S}^3} := |e_1(\alpha) + \frac{1}{2}\alpha^2 + \frac{1}{4}W + \frac{1}{6}H^2|^{3/2}\theta \wedge e^1$$

-where W is the (constant) Webster curvature of S^3.

- The Clifford torus $\subset S^3 \subset C^2$ defined by $\rho_1 = \rho_2 = \frac{\sqrt{2}}{2}$, $z_1 = \rho_1 e^{i\varphi_1}$, $z_2 = \rho_2 e^{i\varphi_2}$ satisfies the Euler-Lagrange equation for $E_{\mathbf{S}^3}(\Sigma) := \int_\Sigma dA_{\mathbf{S}^3}$ $(\alpha = 0,\ H = 0,\ ..)$.
- **[CCYZ25]**: The Clifford torus is not a minimizer among all surfaces of torus type for $E_{\mathbf{S}^3}$; which torus is a candidate (global) minimizer??

Pansu spheres in $\mathbf{H}_1$

- Describe a constant p-mean curvature surface (called a **Pansu sphere**) $S_\lambda \subset \mathbf{H}_1$ with $H = 2\lambda$ by

$$t = \pm\frac{1}{2\lambda^2}\left(\lambda r\sqrt{1-\lambda^2 r^2} + \cos^{-1}(\lambda r)\right)$$

(e.g., Leonardi-Masnou, 2005), $r = \sqrt{x^2+y^2} \leq \frac{1}{\lambda}$. Compute

$$\alpha = \frac{\sqrt{1-\lambda^2 r^2}}{r}$$

- Direct computation shows

Pansu spheres are not critical points of $E_{\mathbf{1}}$.

Examples of critical points for E_1

- Vertical planes $\Sigma := \{ax + by = c\} \subset \mathbf{H}_1$:
  - $T = \frac{\partial}{\partial t} \in T\Sigma \Longrightarrow \alpha \equiv 0;\ H = 0$
  - So $e_1(\alpha) + \frac{1}{2}\alpha^2 + \frac{1}{6}H^2 = 0 \Longrightarrow \Sigma$ is a minimizer for E_1 with zero energy.
- $\Sigma_c := \{t = cr^2\} \subset \mathbf{H}_1 \Longrightarrow \alpha = \frac{1}{r\sqrt{4c^2+1}}, H = -\frac{2c}{r\sqrt{4c^2+1}} \Longrightarrow$

$$e_1(\alpha) + \frac{1}{2}\alpha^2 + \frac{1}{6}H^2 = \frac{4c^2 - 3}{6(4c^2+1)r^2}$$

  - $= 0$ for $c = \pm\frac{\sqrt{3}}{2} \Longrightarrow \Sigma_{\pm\frac{\sqrt{3}}{2}}$ are minimizers for E_1 with zero energy.
  - for $c = 0, \Sigma_0$ is a critical point of higher energy level for E_1.
- Heisenberg spheres:$= \{r^4 + 4t^2 = \rho^4\}$ are critical points of higher energy level for E_1 (CR equivalent to $\{t = 0\}$ in S^3 through Cayley transform).

Euler-Lagrange equation for E_2

- Euler-Lagrange equation for E_2 on a surface $\Sigma \subset \mathbf{H}_1$:

$$
\begin{aligned}
0 \;=\; & He_1e_1(H) + 3e_1V(H) + e_1(H)^2 + \frac{1}{3}H^4 \\
& +3e_1(\alpha)^2 + 12\alpha^2 e_1(\alpha) + 12\alpha^4 \\
& -\alpha He_1(H) + 2H^2 e_1(\alpha) + 5\alpha^2 H^2.
\end{aligned} \tag{2}
$$

- where $V = T + \alpha e_2 \in T\Sigma$.

- Fact: in case $H = 0$,

$$
e_1(\alpha) + 2\alpha^2 = 0 \Leftrightarrow (2) \text{ holds}
$$

Examples of critical points for E_2

- Vertical planes: $ax + by = c \Longrightarrow \alpha = H = 0 \Longrightarrow$critical points of E_2.
- Surfaces foliated by $X = a\mathring{e}_1 + b\mathring{e}_2$
  $(\mathring{e}_1 := \partial_x + y\partial_t, \mathring{e}_2 := \partial_y - x\partial_t), a^2 + b^2 = 1$
  - $\Longrightarrow e_1 = X \Longrightarrow \nabla e_1 = \nabla X = 0 \Longrightarrow H = 0, \omega(T + \alpha e_2) = 0$
  - $\Longrightarrow$(integrability condition) $e_1(\alpha) + 2\alpha^2 = \omega(T + \alpha e_2) = 0$
  - $\Longrightarrow$critical points of E_2.
  - e.g. $X = \mathring{e}_1$ for $\{t = xy + c\}$, $X = \mathring{e}_2$ for $\{t = -xy + c\}$.

Critical points of E_2 continued

- $\Sigma_c := \{t = cr^2\} \subset \mathbf{H}_1 \Longrightarrow \alpha = \frac{1}{r\sqrt{4c^2+1}}, H = \frac{2c}{r\sqrt{4c^2+1}}$ (w.r.t. the defining function $t - cr^2$)$\Longrightarrow$

$$(2) \text{ holds} \Leftrightarrow c = \pm\frac{\sqrt{3}}{2}$$

- Heisenberg and Pansu spheres are not critical points of E_2.
- Question:

Do there exist closed surfaces which are critical points of E_2?

- Note that E_2 is unbounded from below and above in general.

Singular CR Yamabe problem

Let $\Sigma = \partial\Omega$, $\Omega \subset M$ where (M, J, θ) is a pseudohermitian 3-manifold.

- The singular CR Yamabe problem: Find u with $\tilde{\theta} = u^{-2}\theta$ s.t.

$$\begin{aligned} \tilde{W} &= -4 \text{ in } \Omega, \\ u &= 0 \text{ on } \Sigma, \ \ u > 0 \text{ in } \Omega \end{aligned} \tag{3}$$

  \- where $\tilde{W}$ is the Tanaka-Webster curvature w.r.t. $(J, \tilde{\theta})$; (3) has been solved for nonsingular Σ by Wei-Ting Kao in his 2025 PhD thesis of Taiwan University.

- Consider a formal solution to (3) near the boundary Σ : ([CYZ, 2017])

$$\begin{aligned} u(x, \rho) &= \rho + v(x)\rho^2 + w(x)\rho^3 \\ &\quad + z(x)\rho^4 + l(x)\rho^5 \log \rho \\ &\quad + h(x)\rho^5 + O(\rho^6) \end{aligned} \tag{4}$$

  \- where $x \in \Sigma$ is nonsingular and ρ is a suitably chosen defining function for Σ, which is justified by Kao's solution.

Coefficients and volume renormalization

- (3)+(4)$\Longrightarrow v(x) = -\frac{1}{6}H(x)$,
  - $w(x) = -\frac{2}{3}[e_1(\alpha) + 2\alpha^2 + \frac{1}{6}H^2 + curv\ \&\ torsion]$, $z(x)$ determined
- Volume renormalization:

$$\begin{aligned} Vol(\{\rho > \varepsilon\}) &:= \int_{\{\rho>\varepsilon\}} \tilde{\theta} \wedge d\tilde{\theta} \\ &= \int_{\{\rho>\varepsilon\}} u^{-4}\theta \wedge d\theta \\ &= c_0\varepsilon^{-3} + c_1\varepsilon^{-2} + c_2\varepsilon^{-1} + L \log \frac{1}{\varepsilon} + V_0 + o(1) \end{aligned}$$

- $v(x), w(x), z(x) \rightsquigarrow c_0, c_1, c_2$ and L as well ([CYZ, 2017]).

Log term coefficients L and l(x)

- Σ : a closed surface with no singular points$\Longrightarrow$

$$L = L(\Sigma) = 2E_2(\Sigma) \tag{5}$$

- Compute

$$\begin{aligned} \frac{d}{dt}|_{t=0} L(\Sigma_t) &= 10 \int_\Sigma l(x) h(x) \theta \wedge e^1, \\ \frac{d}{dt}|_{t=0} E_2(\Sigma_t) &= \int_\Sigma \mathcal{E}_2(x) h(x) \theta \wedge e^1 \end{aligned} \tag{6}$$

- where $\mathcal{E}_2 = \frac{4}{9}$[right quantity in (2)+curv term].

- (5) + (6)$\Longrightarrow$

$$l(x) = \frac{1}{5}\mathcal{E}_2(x)$$

- So $\mathcal{E}_2(x) \neq 0$ is an obstruction to the smoothness of solutions up to the boundary.

Back to E_1

$E_1 = 0 \iff$

$$e_1(\alpha) + \frac{1}{2}\alpha^2 + \frac{1}{6}H^2 = 0. \tag{7}$$

- For a graph $(x, y, u(x, y))$ in $\mathbf{H}_1$, $e_1 = -\frac{u_y+x}{D}\partial_x + \frac{u_x-y}{D}\partial_y + (\cdot)\partial_t$, $\alpha = \frac{-1}{D}$, $D := \sqrt{(u_x - y)^2 + (u_y + x)^2}$ and

$$H = \frac{1}{D}(\Delta u - \frac{u_x - y}{D}D_x - \frac{u_y + x}{D}D_y).$$

  – Equation (7) reads as

$$\begin{aligned} 0 &= F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) \quad (8) \\ &\equiv \frac{u_y + x}{D}D_x - \frac{u_x - y}{D}D_y - \frac{1}{2} - \frac{1}{6}(\Delta u - \frac{u_x - y}{D}D_x - \frac{u_y + x}{D}D_y)^2. \end{aligned}$$

  – Compute

$$F_{u_{xx}}F_{u_{yy}} - \frac{1}{4}F_{u_{xy}}^2 = -\frac{1}{4} < 0 \implies \text{hyperbolic.}$$

Two characteristic directions

Write

$$\cos\theta \quad : \quad = \frac{u_x - y}{D}, \quad \sin\theta := \frac{u_y + x}{D}$$
$$N \quad : \quad = \cos\theta\partial_x + \sin\theta\partial_y, \quad N^\perp := \sin\theta\partial_x - \cos\theta\partial_y.$$

- Two characteristic directions are $N^\perp$ and $N - \frac{1}{3}DHN^\perp$:

$$(N - \frac{1}{3}DHN^\perp) \circ N^\perp$$
$$= \quad F_{u_{xx}}\partial_x^2 + F_{u_{xy}}\partial_x\partial_y + F_{u_{yy}}\partial_y^2 \text{ mod lower order terms.}$$

- Principle of bicharacteristic directions (??)– The value of u at p is uniquely determined by the value of u on the region surrounded by respective integral curves of two characteristic directions and the initial curve.

Uniqueness of E_1-minimizers for the initial-value problem

Recall

$$\cos\theta \ := \ \frac{u_x - y}{D}, \ \sin\theta := \frac{u_y + x}{D}$$

- **[CCYZ25]** Suppose that u, v are two smooth E_1-minimizers having the same initial values H, α and θ on a non-characteristic smooth curve w.r.t. u, lying on a "non-characteristic region".
- Then $u = v$ on a neighborhood of the initial curve.

Rotationally symmetric E_1-minimizers

Equation (7) is reduced to a non-linear ODE of second order

$$\frac{1}{3}r^4 u_{rr}^2 - \left(\frac{4}{3}ru_r^3 + 2r^3 u_r\right)u_{rr} + \frac{1}{3}\frac{u_r^6}{r^2} + u_r^4 - r^4 = 0.$$

- **[CCYZ25]** The complete, rotationally symmetric E_1-minimizers are divided into four classes up to CR transformations on the Heisenberg group $\mathbf{H}_1$:
  (1) $u(r) = \frac{\sqrt{3}}{2}r^2$,
  (2) $u(r) = -\frac{\sqrt{3}}{2}r^2$,
  (3) Type 1 shifted Heisenberg spheres: $(r^2 + \frac{\sqrt{3}}{2}\rho^2)^2 + 4t^2 = \rho^4$ for $\rho > 0$,
  (4) Type 2 shifted Heisenberg spheres: $(r^2 - \frac{\sqrt{3}}{2}\rho^2)^2 + 4t^2 = \rho^4$ for $\rho > 0$.

Closed solutions to $E_1 = 0$

- Conjecture (global): The type I and type II shifted Heisenberg spheres $(r^2 \pm \frac{\sqrt{3}}{2}\rho^2)^2 + 4t^2 = \rho^4$, $\rho > 0$ are the only closed solutions to (7), i.e.

$$e_1(\alpha) + \frac{1}{2}\alpha^2 + \frac{1}{6}H^2 = 0.$$

–with no assumption on rotational invariance.

Thanks for your attention