The Bernstein problem for area-minimizing intrinsic graphs in the Sub-Riemannian Heisenberg group

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Workshop on
Geometrical Variational Problems in Sub-Riemannian Geometry
September 10-12, 2025, Granada

The Heisenberg group $\mathbb{H}^1$

$$\mathbb{H}^1 = (\mathbb{R}^3, \cdot, \{\delta_\lambda\}, H\mathbb{H}^1, d_{\mathbb{H}})$$

1. Group law.(not commutative) If $p = (x, y, z)$, $p' = (x', y', z') \in \mathbb{R}^3$,

$$p \cdot p' := \left(x + x', y + y', z + z' + \frac{1}{2}(xy' - yx') \right).$$

$$0 = (0, 0, 0) \equiv \text{ unit element,} \quad p^{-1} = -p$$

2. Dilations. $\delta_\lambda : \mathbb{R}^3 \to \mathbb{R}^3$ $(\lambda > 0)$ such that

$$\delta_\lambda(x, y, z) = (\lambda x, \lambda y, \lambda^2 z),$$

The Heisenberg group $\mathbb{H}^1$

3. Horizontal subbundle . The Lie algebra of left-invariant vector fields in $\mathbb{H}^1$, $\mathfrak{h}_1$, is generated by

$$X = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z}, \qquad Y = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z}, \qquad T = \frac{\partial}{\partial z}\,.$$

The only non-trivial commutator relation is

$$[X, Y] = T\,.$$

The Heisenberg group $\mathbb{H}^1$

The horizontal subbundle

$$H\mathbb{H}^1 \subset T\mathbb{H}^1 = T\mathbb{R}^3$$

where, given $p \in \mathbb{R}^3$,

$$H_p \equiv H_p\mathbb{H}^1 := \text{span}\{X(p), Y(p)\} \subset T_p\mathbb{H}^1 .$$

$T_p(\mathbb{H}^1)$ is equipped with a scalar product $< \cdot, \cdot >_p$. w.r. t. $X(p)$, $Y(p)$, T are orthonormal and let

$$|v|_p := \sqrt{< v, v >_p} \quad \text{if } v \in T_p(\mathbb{H}^1) .$$

The Heisenberg group $\mathbb{H}^1$

4. Metric in $\mathbb{H}^1$.

$$\|p\|_{\mathbb{H}} := \max\left\{\sqrt{|x|^2+|y|^2}, \sqrt{|z|}\right\} \qquad d_{\mathbb{H}}(p,q) := \|q^{-1}\cdot p\|_{\mathbb{H}}$$

Intrinsic regular surfaces

Definition

We say that $S \subset \mathbb{H}^1$ is an *intrinsic C^1-regular surface* if $\forall p \in S$ $\exists\ U$ neighborhood of p and a function $f \in C^1_{\mathbb{H}}(U)$ such that

$$S \cap U = \{p \in U : f(p) = 0\}; \qquad (i)$$

$$\nabla_{\mathbb{H}} f(p) := Xf(p)X(p) + Yf(p)Y(p) \qquad (ii)$$
$$\equiv (Xf(p), Yf(p)) \neq 0 \quad \forall p \in U\,.$$

where $f \in C^1_{\mathbb{H}}(U)$ in the sense of Folland and Stein, i.e. $f \in C^0(U)$ and $\exists\ Xf,\ Yf : U \to \mathbb{R}$ continuous.

Intrinsic regular surfaces

Remark: If S is a (Euclidean) C^1-regular surface in $\mathbb{H}^1 \equiv \mathbb{R}^3$, then $S \setminus \mathcal{C}(S)$ is intrinsic C^1- regular, denoting with $\mathcal{C}(S)$ the set of *characteristic points of S*. Viceversa there are intrinsic C^1-regular surfaces not C^1-regular.

The main reason for the introduction of intrinsic C^1-regular surfaces is that they can be used to give a useful notion of intrinsic rectifiability in $\mathbb{H}^1$, in contrast to (Euclidean) rectifiability, which does not work for this purpose (Ambrosio-Kirchheim, 2000).

Intrinsic graphs (IG) in $\mathbb{H}^1$

Let

$$\mathbb{W} := \{(0, y, z) : \ y, z \in \mathbb{R}\} \ ,$$

$$\mathbb{V} := \{(x, 0, 0) : x \in \mathbb{R}\}$$

they are homogeneous subgroups of $\mathbb{H}^1$, i.e. they are subgroups and $\delta_\lambda(\mathbb{W}) \subseteq \mathbb{W}$ and $\delta_\lambda(\mathbb{V}) \subseteq \mathbb{V}$ for each $\lambda > 0$. Moreover, they are complementary, i.e.

$$\mathbb{R}^3 = \mathbb{W} \cdot \mathbb{V} \quad , \mathbb{W} \cap \mathbb{V} = \{0\}$$

IG in $\mathbb{H}^1$

More precisely, each $p = (x, y, z) \in \mathbb{H}^1$ can be written, in unique way, as

$$p = p_{\mathbb{W}} \cdot p_{\mathbb{V}}$$

with

$$p_{\mathbb{W}} = \left(0, y, z - \frac{x\,y}{2}\right) \in \mathbb{W},$$

$$p_{\mathbb{V}} = (x, 0, 0) \in \mathbb{V}\,.$$

IG in $\mathbb{H}^1$

A set $S \subset \mathbb{R}^3$ is a X- graph, or simply an intrinsic graph (IG), if there is $f: \omega \subset \mathbb{W} \equiv \mathbb{R}^2 \to \mathbb{V} \equiv \mathbb{R}$ such that, if $e_1 = (1,0,0)$,

$$S = G_f^X := \left\{ \Phi(A) := A \cdot f(A)\, e_1 \;:\; A \in \omega \right\}$$

where $\Phi : \omega \subset \mathbb{W} \to \mathbb{H}^1$,

$$\begin{aligned} \Phi(A) &:= A \cdot f(A)\, e_1 = \exp\left(f(A)X\right)(A) \\ &= \left(f(y,z), y, z - \frac{y}{2} f(y,z) \right) \text{ if } A = (0,y,z) \equiv (y,z) \in \omega\,. \end{aligned}$$

The X-subgraph of f is the set

$$E_f^X := \left\{ (0,y,z) \cdot s\, e_1 \in \omega \cdot \mathbb{R} e_1 : (0,y,z) \in \omega,\ s < f(y,z) \right\}.$$

Some features of IG

(i) $(\omega \subset)\mathbb{W}$ is a subgroup of $\mathbb{H}^1$.

(ii) Intrinsic graphs are invariant under both left-translations and dilations of the group.

(iii) An intrinsic C^1-regular surface can (locally) be represented as an intrinsic graph [Franchi-Serapioni-S.C., 1999].

(iv) There is an intrinsic C^1-regular X-graph S_0 such that

$$\mathrm{Hausdim}(S_0, d_H) = 3 \text{ and } \mathrm{Hausdim}(S_0, |\cdot|_{\mathbb{R}^3}) = 2.5\,,$$

[Kirchheim, S.C., 2004].

Functional spaces associated to X-graphs

The space of C^1-regular X-graphs

Let $\omega \subset \mathbb{W} \equiv \mathbb{R}^2$ be an open set and let

$$C^1_{\mathbb{W}}(\omega) := \left\{ f : \omega \to \mathbb{R} \,:\, G^X_f \text{ is intrinsic } C^1\text{-regular} \right\} .$$

Then the following are equivalent ([Ambrosio-S.C.-Vittone, 2006], [Citti-Manfredini, 2006], [Bigolin-S.C.,2010]):

(i) $f \in C^1_{\mathbb{W}}(\omega)$;

(ii) $f \in C^0(\omega)$ and $\exists \nabla^f f \in C^0(\omega)$ in distributional sense, where $\nabla^f f$ denotes the intrinsic gradient of f defined as

$$\nabla^f f := \partial_y f + \frac{1}{2}\, \partial_z(f^2) \quad \text{(Burgers' operator)} .$$

Functional spaces associated to X-graphs

Rmk.

(i) $C^1(\omega) \subsetneq C^1_{\mathbb{W}}(\omega)$;

(ii) $C^1_{\mathbb{W}}(\omega)$ is not a vector space.

Functional spaces associated to X-graphs

Definition (Intrinsic Lipschitz X-graphs [Franchi,Serapioni.S.C., 2006])

We say that G_f^X is an intrinsic Lipschitz X-graph if $\exists\, L > 0$ s.t.

$$\|(\bar{p}^{-1} \cdot p)_{\mathbb{V}}\|_{\mathbb{H}} \leq L\,\|(\bar{p}^{-1} \cdot p)_{\mathbb{W}}\|_{\mathbb{H}} \quad \forall\, p, \bar{p} \in G_f^X$$

Let $\omega \subset \mathbb{W} \equiv \mathbb{R}^2$ be an open set and let

$$Lip_{\mathbb{W}}(\omega) := \left\{ f : \omega \to \mathbb{R} \,:\, G_f^X \text{ is intrinsic Lipschitz} \right\}.$$

Functional spaces associated to X-graphs

Then the following are equivalent [Bigolin,Caravenna, S.C, 2015]:

(i) $f \in Lip_{\mathbb{W},loc}(\omega)$;

(ii) $f \in C^0(\omega)$ and $\exists\, \nabla^f f \in L^\infty_{\rm loc}(\omega)$ in distributional sense.

Rmk.

(i) $Lip_{\rm loc}(\omega) \subsetneq Lip_{\mathbb{W},loc}(\omega)$;

(ii) $C^1_{\mathbb{W}}(\omega) \subsetneq Lip_{\mathbb{W},loc}(\omega) \subsetneq C^{0,1/2}_{loc}(\omega)$.

Functional spaces associated to X-graphs

Definition (Sobolev class of X-graphs [Monti,S.C.,Vittone, 2008])

Let $\omega \subset \mathbb{W} \equiv \mathbb{R}^2$ an open set. We say that a function $f \in L^1(\omega) \cap L^2(\omega)$ belongs to the class $W^{1,1}_{\mathbb{W}}(\omega)$ if $\exists\, (f_j)_j \subset C^1(\omega)$ and $w \in L^1(\omega)$ such that, as $j \to \infty$,

$$f_j \to f,\ f_j^2 \to f^2 \text{ and } \nabla^{f_j} f_j \to w \text{ in } L^1(\omega)\,.$$

Given $f \in W^{1,1}_{\mathbb{W},loc}(\omega)$, the distribution

$$\nabla^f f := \partial_y f + \frac{1}{2}\,\partial_z(f^2) = \ w \in L^1_{loc}(\omega)\,.$$

Functional spaces associated to X-graphs

Note that

(i) $Lip_{\mathbb{W},loc}(\omega) \subset W^{1,1}_{\mathbb{W},loc}(\omega)$ [Citti, Manfredini, Pinamonti,S.C.,2014];

(ii) $W^{1,1}_{loc}(\omega) \cap C^0(\omega) \subset W^{1,1}_{\mathbb{W},loc}(\omega)$ [Monti, S.C., Vittone, 2008].

Area$\equiv$ $\mathbb{H}$-perimeter measure

Given $E \subset \mathbb{R}^3$ measurable, $\Omega \subset \mathbb{R}^3$ open, we define the $\mathbb{H}$ or sub-Riemannian perimeter

$$|\partial E|_{\mathbb{H}}(\Omega) := \sup\left\{ \int_E \operatorname{div} V \, d\mathcal{L}^3 : V \in C_c^\infty(\Omega, H\mathbb{H}^1), |V(p)|_p \leq 1 \right\} .$$

Area functional for intrinsic graphs

Theorem (area of X-graphs, [Franchi-Serapioni-S.C., 2001], [Ambrosio-S.C.-Vittone, 2006],[Monti,S.C.,Vittone, 2008]

Let $\omega \subset \mathbb{W} \equiv \mathbb{R}^2$ be open,let $\Omega := \omega \cdot \mathbb{R}e_1$ and let $f \in W^{1,1}_{\mathbb{W}}(\omega)$. Then

$$|\partial E_f^X|_{\mathbb{H}}(\Omega) = \int_\omega \sqrt{1+|\nabla^f f|^2}\, d\mathcal{L}^2 \,. \qquad \text{(IAF)}$$

In particular, (IAF) holds for all $f \in W^{1,1}_{loc}(\omega) \cap C^0(\omega)$.

We will denote

$$\mathcal{A}_{\mathbb{W}}(f) := \int_\omega \sqrt{1+|\nabla^f f|^2}\, d\mathcal{L}^2 \text{ if } f \in W^{1,1}_{\mathbb{W},loc}(\omega)\,.$$

Area functional for intrinsic graphs

Rmk. The area functional for intrinsic graphs

$$Lip(\omega) \ni f \mapsto \mathcal{A}_{\mathbb{W}}(f) := \int_\omega \sqrt{1 + |\nabla^f f|^2}\, d\mathcal{L}^2 \text{ is not convex},$$

([Danielli, Garofalo, Nhieu, 2008]).

Minimal boundaries in $\mathbb{H}^1$

Inspired by De Giorgi's theory, we say that a set $E \subset \mathbb{R}^3$ is $\mathbb{H}$-perimeter minimizing in Ω, if it has locally finite $\mathbb{H}$-perimeter in Ω and

$$|\partial E|_{\mathbb{H}}(\Omega') \leq |\partial F|_{\mathbb{H}}(\Omega')$$

for every $F \subset \mathbb{R}^3$ with $E \Delta F := (E \setminus F) \cup (F \setminus E) \subset\subset \Omega'$, for any open set $\Omega' \subset\subset \Omega$.

Given $f : \omega \subset \mathbb{W} \equiv \mathbb{R}^2 \to \mathbb{R}$, we say that the graph G_f^X is area-minimizing in $\Omega = \omega \cdot \mathbb{R}e_1$ if the intrinsic subgraph E_f^X is $\mathbb{H}$-perimeter-minimizing in Ω.

BP for IG in $\mathbb{H}^1$.

The Bernstein problem (BP) for intrinsic graphs: Characterization of $f : \omega = \mathbb{R}^2 \to \mathbb{R}$ for which G_f^X is area-minimizing in $\Omega = \mathbb{R}^3$.
In particular, under which assumptions on f must G_f^X be a vertical plane? That is,

$$f(y,z) = \; ay + b \quad \forall\, (y,z) \in \mathbb{R}^2 \equiv \mathbb{W} \tag{IA}$$

for costants a, $b \in \mathbb{R}$.

Some results on BP for IG

Assume G_f^X is area-minimizig in $\mathbb{R}^3$. Then
it is a vertical plane:

- if $f \in C^2(\mathbb{R}^2)$: [Barone Adesi, S.C., Vittone, 2007], [Danielli, Garofalo,Nhieu,Pauls, 2009];
- if $f \in C^1(\mathbb{R}^2)$: [Galli, Ritoré, 2015];
- if $f \in Lip_{\mathrm{loc}}(\mathbb{R}^2)$: [Nicolussi Golo, S.C., 2019];
- if G_f^X is ruled by horizontal lines: [R. Young, 2022]
- if $f \in Lip_{\mathrm{loc}}(\mathbb{R}^2)$ in the sub-Finsler $\mathbb{H}^1$: [Giovannardi, Ritoré, 2024]

Some results on BP for IG

it need not be a vertical plane:

- if $f \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^2) \cap Lip_{\mathbb{W},loc}(\mathbb{R}^2)$ for each $p \in [1,2)$: [Monti, S.C., Vittone, 2008],…, [Nicolussi Golo, Ritoré, 2021].

Focus on some results on BP for IG

Theorem [Nicolussi Golo, S.C., 2019]

Let $f \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^2) \equiv Lip_{\mathrm{loc}}(\mathbb{R}^2)$ and suppose that G_f^X is stable as intrinsic graph, that is, for all $\varphi \in C_c^\infty(\omega)$, the following two conditions are satisfied:

1. $I_f(\varphi) = \frac{\mathrm{d}}{d\epsilon}\, \mathcal{A}_{\mathbb{W}}(f + \epsilon\varphi)|_{\epsilon=0} = 0$ (1st VF),
2. $II_f(\varphi) = \frac{\mathrm{d}^2}{d\epsilon^2}\, \mathcal{A}_{\mathbb{W}}(f + \epsilon\varphi)|_{\epsilon=0} \geq 0$ (2nd VF).

Then, for suitable a, $b \in \mathbb{R}$,

$$f(y,z) = \ a\,y + b \quad \forall\,(y,z) \in \mathbb{R}^2\,. \tag{IA}$$

In particular if $f \in Lip_{\mathrm{loc}}(\mathbb{R}^2)$ and G_f^X is area-minimizing in $\mathbb{R}^3$, then f satisfies (IA)

Rmk. Every area-minimizing intrinsic graph is stable; however, the converse need not be true.

Focus on some results on BP for IG

A *ruled surface* in $\mathbb{H}^1$ is surface $S \subset \mathbb{H}^1$ that can be written as union of horizontal line segments with endpoints in ∂S.

Theorem [R. Young, 2022]

An entire area-minimizing ruled intrinsic graph in $\mathbb{H}^1$ is a vertical plane.

Example of area-minimizing IG not vertical plane

A counterexample [Monti, S.C., Vittone,2008]

Let

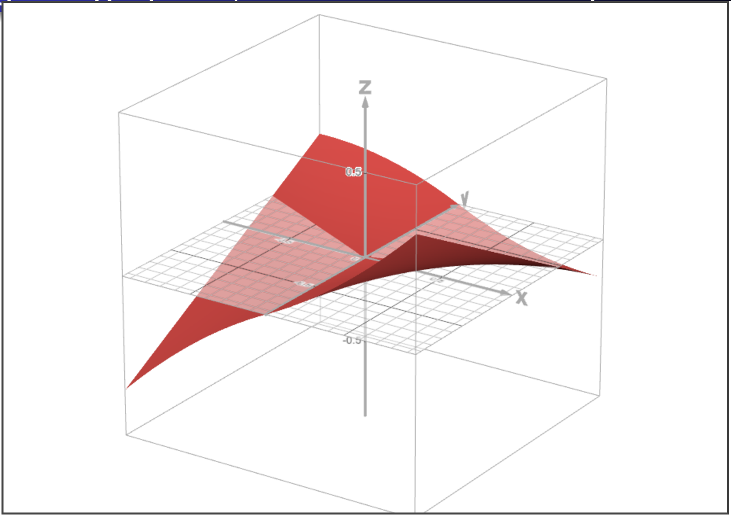
$$f(y,z) := 2\frac{z}{|z|}\sqrt{|z|} \quad \text{if } (y,z) \in \mathbb{R}^2 \equiv \mathbb{W}\,.$$

Then

- $f \in W^{1,p}_{\rm loc}(\mathbb{R}^2) \cap C^0(\mathbb{R}^2)$ for each $p \in [1,2)$;
- $f \in Lip_{\mathbb{W},\rm loc}(\mathbb{R}^2)$;
- G_f^X is a cone, area-minimizing in $\mathbb{R}^3$ and it is not a ruled surface.

https://www.desmos.com/3d/bmmo8t6lsy?lang=it

Figure: graph G_f^X in the counterexample



Examples of stable, not area-minimizing IG

Example 1

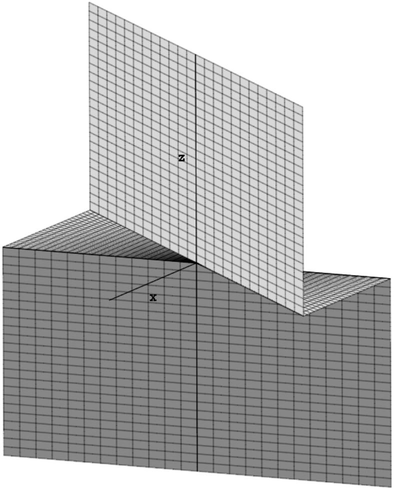
Let $f : \mathbb{R}^2 \to \mathbb{R}$ be

$$f(y,z) := \begin{cases} 0 & z \leq 0 \\ \frac{2z}{y} & 0 < z \leq \frac{y^2}{2} \\ y & z > \frac{y^2}{2}. \end{cases}$$

Then $f \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^2) \cap C^1_{\mathbb{W}}(\mathbb{R}^2 \setminus \{0\}) \cap Lip_{\mathrm{loc}}(\mathbb{R}^2 \setminus \{0\})$ with $1 \leq p < 3$ such that:

- G^X_f is a stable cone [Nicolussi Golo, S.C.,2019], but not area-minimizing in $\mathbb{R}^3$ [Young, 2022];
- $f \in Lip_{\mathbb{W},\mathrm{loc}}(\mathbb{R}^2)$.

Figure: graph G_f^X in example 1



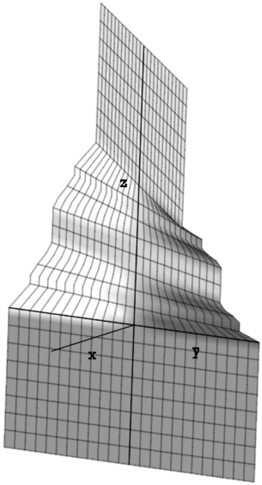
Examples of stable, not area-minimizing IG

Example 2

There exists $f \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2) \cap C^1_{\mathbb{W}}(\mathbb{R}^2) \cap Lip_{\mathrm{loc}}(\mathbb{R}^2 \setminus (\{0\} \times \mathbb{R}))$ such that G^X_f is :

- not a vertical plane;
- a ruled surface;
- stable [Nicolussi Golo, S.C.,2019], but not area-minimizing in $\mathbb{R}^3$ [Young, 2022].

Figure: graph G_f^X in example 2



A new result on BP for IG

Theorem [Nicolussi Golo, S.C., Vedovato, 2025]

Let $f \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^2)$ with $p > 4$ and suppose that:

- $\exists k > 0$ such that $\exp(k|\partial_z f|) \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$;
- G_f^X is stable.

Then G_f^X is a vertical plane.
In particular, if G_f^X is area minimizing in $\mathbb{R}^3$, then it is a vertical plane

Some open problems

Question. What about the Bernstein problem for intrinsic graphs when f only has standard Sobolev regularity or $C^1_{\mathbb{W}}$-regularity. In other words, is an entire are-minimizing G_f^X still a vertical plane if

- $f \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^2) \cap C^0(\mathbb{R}^2)$ for $p \in [2, \infty)$

  or

- $f \in C^1_{\mathbb{W}}(\mathbb{R}^2)$?

1st VF: minimal surface equation for i.g.

Assume $f \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^2) \cap C^0(\mathbb{R}^2)$ and $\varphi \in C^\infty_c(\mathbb{R}^2)$:

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}\bigg|_{\epsilon=0} \mathcal{A}_{\mathbb{W}}(f+\epsilon\varphi) = \int_{\mathbb{R}^2} \frac{\nabla^f f}{\sqrt{1+(\nabla^f f)^2}} (\partial_y \varphi + \partial_t(f\varphi))\, \mathrm{d}\mathcal{L}^2$$

$$= \int_{\mathbb{R}^2} \frac{\nabla^f f}{\sqrt{1+(\nabla^f f)^2}} (\nabla^f)^* \varphi \, \mathrm{d}\mathcal{L}^2 = 0 \qquad \text{(IMSE)}$$

where $(\nabla^f)^*\varphi := -(\nabla^f \varphi + \partial_t f \varphi)$ (adjoint of ∇^f).
If $f \in C^2(\mathbb{R}^2)$, then

$$0 = \int_{\mathbb{R}^2} \frac{\nabla^f f}{\sqrt{1+(\nabla^f f)^2}} (\nabla^f)^* \varphi \, \mathrm{d}\mathcal{L}^2$$

$$= \int_{\mathbb{R}^2} \nabla^f \left(\frac{\nabla^f f}{\sqrt{1+(\nabla^f f)^2}} \right) \varphi \, \mathrm{d}\mathcal{L}^2$$

1st VF: minimal surface equation for i.g.

Therefore, if $f \in C^2(\mathbb{R}^2)$,

$$\nabla^f \left(\frac{\nabla^f f}{\sqrt{1 + (\nabla^f f)^2}} \right) = \frac{\nabla^f \nabla^f f}{(1 + (\nabla^f f)^2)^{3/2}} = 0 \quad \text{in } \mathbb{R}^2. \qquad \text{(IMSE)}$$

Remark. If $f \in C^2(\mathbb{R}^2)$ satisfies (IMSE):

- need not be an intrinsic plane [Danielli,Garofalo,Nhieu, 2008];
- $\nabla^f f$ is constant along the integral curves of $\nabla^f = \partial_y + f\partial_t$: this implies that G_f^X is ruled by horizontal straight lines.

2nd VF for i.g.

Theorem [Monti, S.C., Vittone, 2008]

Let $f \in W^{1,1}_{\mathrm{loc}}(\omega) \cap C^0(\omega)$ and assume that G^X_f is area minimizing in $\Omega = \omega \cdot \mathbb{R}e_1$. Then, for each $\varphi \in C^\infty_c(\omega)$,

$$0 \le II_f(\varphi) = \frac{\mathrm{d}^2}{d\epsilon^2}\,\mathcal{A}_{\mathbb{W}}(f + \epsilon\varphi)\big|_{\epsilon=0} =$$

$$\int_\omega \frac{(1+(\nabla^f f)^2)\left(((\nabla^f)^*\varphi)^2 + 2\varphi\partial_t\varphi\nabla^f f\right) - (\nabla^f f(\nabla^f)^*\varphi)^2}{(1+(\nabla^f f)^2)^{3/2}}\,\mathrm{d}\mathcal{L}^2$$

From the Euclidean approach to the Lagrangian one

The strategy: 1st and 2nd VF meant in the Lagrangian point of view rather than in the Eulerian one.

Theorem 1[Nicolussi Golo,S.C.,Vedovato,2025]

Let $f \in W^{1,q}_{\text{loc}}(\mathbb{R}^2)$ with $q > 2$ and $\exists k > 0$ such that $\exp(k|\partial_z f|) \in L^1_{\text{loc}}(\mathbb{R}^2)$. Assume it satisfies (IMSE) . Then

(i) $f,\ \nabla^f f \in C^0(\mathbb{R}^2) \cap W^{1,p}_{\text{loc}}(\mathbb{R}^2)\ \forall\, p \geq 1$;

(ii) if $\chi(s,\tau) := \dfrac{\nabla^f f(0,\tau)}{2} s^2 + f(0,\tau)s + \tau$ and $\Psi(s,\tau) := (s, \chi(s,\tau))$, then

$$\Psi : \mathbb{R}^2 \to \mathbb{R}^2 \text{ is a local } p \text{ bi-Sobolev homeomorphism } \forall\, p \geq 1\,;$$

(iii) $\partial_s \chi(s,\tau) = f(s, \chi(s,\tau))\ \forall\,(s,\tau) \in \mathbb{R}^2$. In particular $\nabla^f f$ is constant along integral curves of ∇^f.

From the Euclidean approach to the Lagrangian one

The new tool for showing Thm. 1 is the following

Theorem [Ambrosio,Nicolussi Golo,S.C.,2023]

Let $I, J \subset \mathbb{R}$ be open intervals with I of length $\ell > 0$. Suppose that $f \in W^{1,1}_{loc}(I \times J) \cap C^0(\bar{I} \times \bar{J})$ and that

$$\int_{I \times J} \exp\left(\frac{\ell p^2}{p-1} |\partial_z f(y,z)|\right) dydz < \infty.$$

Then, $\forall\, (y_0, z_0) \in I \times J$, $\exists\, \Psi : \tilde{\omega} \subset I \times J \to \omega \subset I \times J$ p-bi-Sobolev homeomorphism with $\tilde{\omega} := (y_0 - \epsilon, y_0 + \epsilon) \times (z_0 - \epsilon, z_0 + \epsilon)$ and satisfying

1. $\Psi(s,\tau) = (s, \chi(s,\tau))$ for all $(s,\tau) \in \tilde{\omega}$,
2. $$\begin{cases} \partial_s \chi(s,\tau) & = f(s, \chi(s,\tau)) \quad \forall\, (s,\tau) \in \tilde{\omega} \\ \chi(y_0, \tau) & = \tau \quad \forall\, \tau \in (z_0 - \epsilon, z_0 + \epsilon) \end{cases}.$$

From the Euclidean approach to the Lagrangian one

Theorem 2 [Nicolussi Golo,S.C.,Vedovato,2025]

Let $f \in W^{1,q}_{\mathrm{loc}}(\mathbb{R}^2)$ with $q > 4$ and $\exists k > 0$ such that $\exp(k|\partial_z f|) \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$.Assume it satisfies (IMSE) and 2nd VF.

(i) For suitable constants $a, b \in \mathbb{R}$,

$$\nabla^f f(0,\tau) = a \quad f(0,\tau) = b \quad \forall \tau \in \mathbb{R}\,;$$

(iI) if $\chi(s,\tau) := \frac{a}{2}s^2 + bs + \tau$ and $\Psi := (s, \chi(s,\tau))$, then $\Psi : \mathbb{R}^2 \to \mathbb{R}^2$ is a homeomorphism and

$$f(s, \chi(s,\tau)) = a\,s + b \quad \forall (s,\tau) \in \mathbb{R}^2$$

In particular G^X_f is a vertical plane.