WORKSHOP ON GEOMETRIC VARIATIONAL PROBLEMS IN SUB-RIEMANNIAN GEOMETRY

Granada 10-12/09/2025

THE BEHAVIOUR OF AN ALMOST AREA-MINIMIZING SURFACE NEAR VERTICES

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Some terminology

- $\Omega \subset \mathbb{R}^n$ Open set
- $E \subset \Omega$ Measurable set
- We define the RELATIVE PERIMETER of E in Ω by

$$P(E;\Omega) := \sup\left\{ \int_E \operatorname{div}\varphi \;:\; \varphi \in C^1_c(\Omega;\mathbb{R}^n),\ |\varphi| \le 1 \right\}$$

ALMOST MINIMIZERS OF $P(*\,;\Omega)$

We say that E is an ALMOST MINIMIZER of $P(*\,;\Omega)$ if $\exists R>0$ s.t., if $x\in\bar{\Omega}$, $r\in(0,R)$ and $F\subset\Omega$ satisfies

$$E\Delta F\subset\subset B_r(x),$$

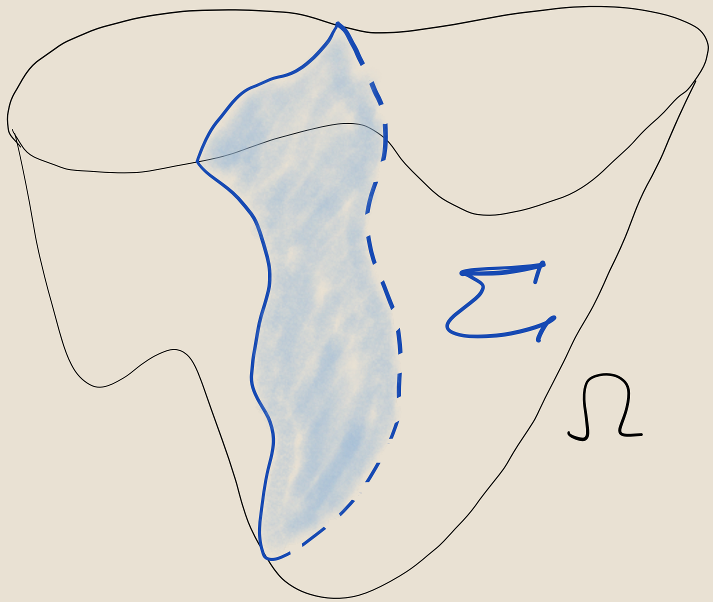
then one has

$$P(E;B_r(x)\cap\Omega)\leq P(F;B_r(x)\cap\Omega)+|E\Delta F|^{\frac{n-1}{n}}\psi(r)$$

where $\psi(r)\geq 0$ and

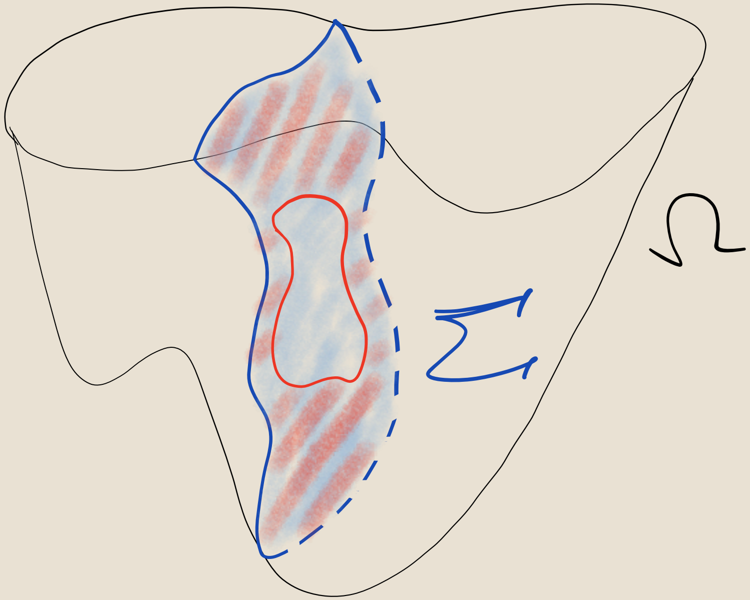
$$\psi(r)\to 0 \text{ as } r\to 0^+.$$

Almost minimizers of $P(*\,;\Omega)$

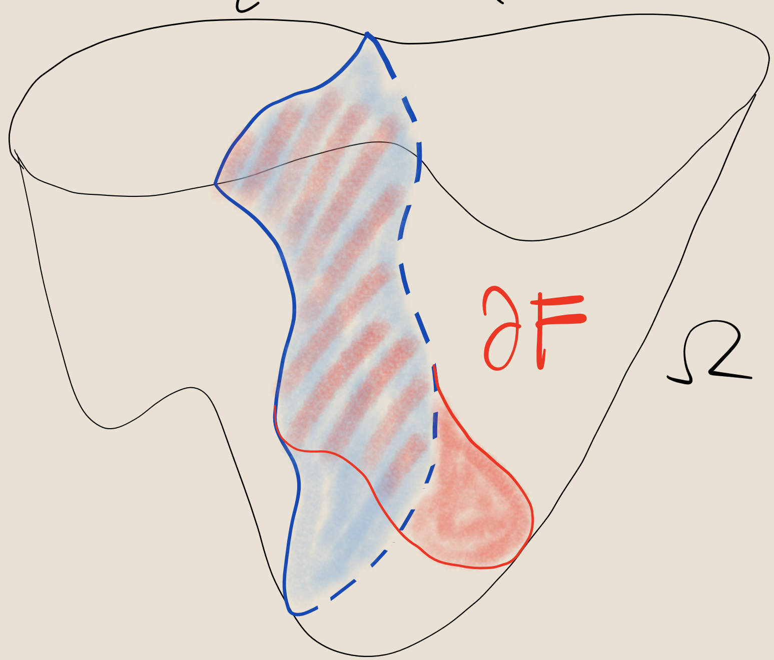


$\Sigma = \partial E$

$B_r(x) \subset\subset \Omega$

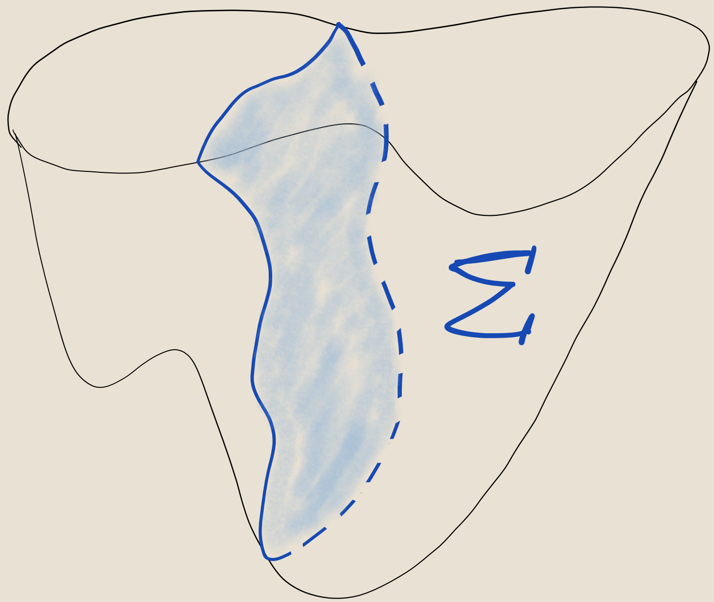


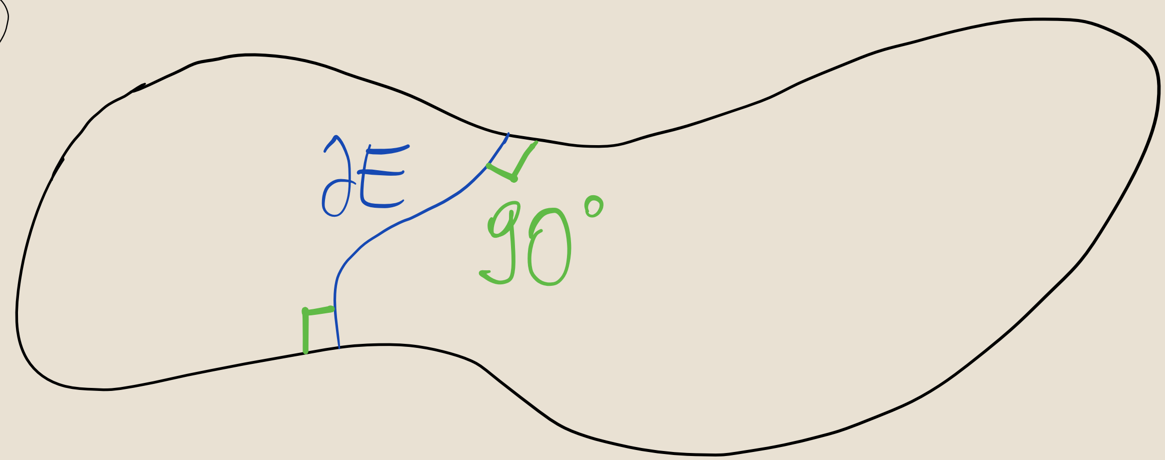
$B_r(x) \not\subset \Omega$



ALMOST MINIMIZERS OF $P(* \,;\, \Omega)$

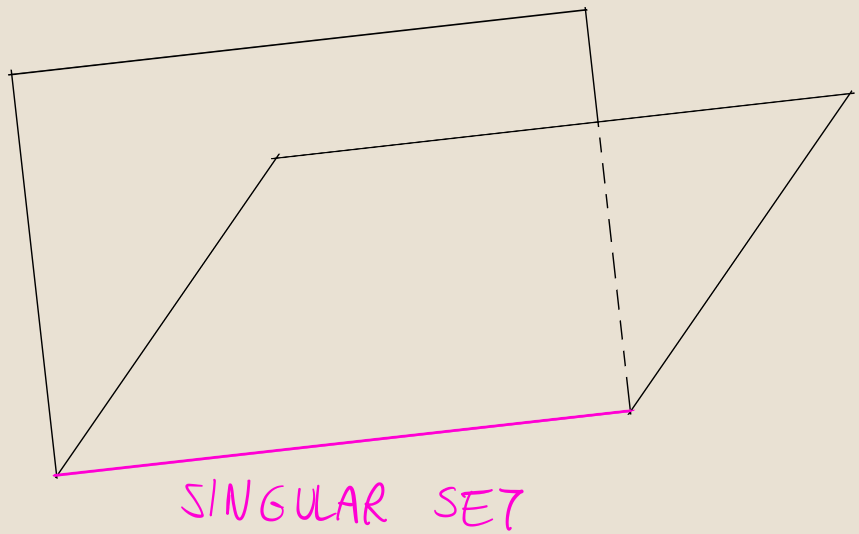
- If Ω is smooth $\Longrightarrow$ **Young's Law**



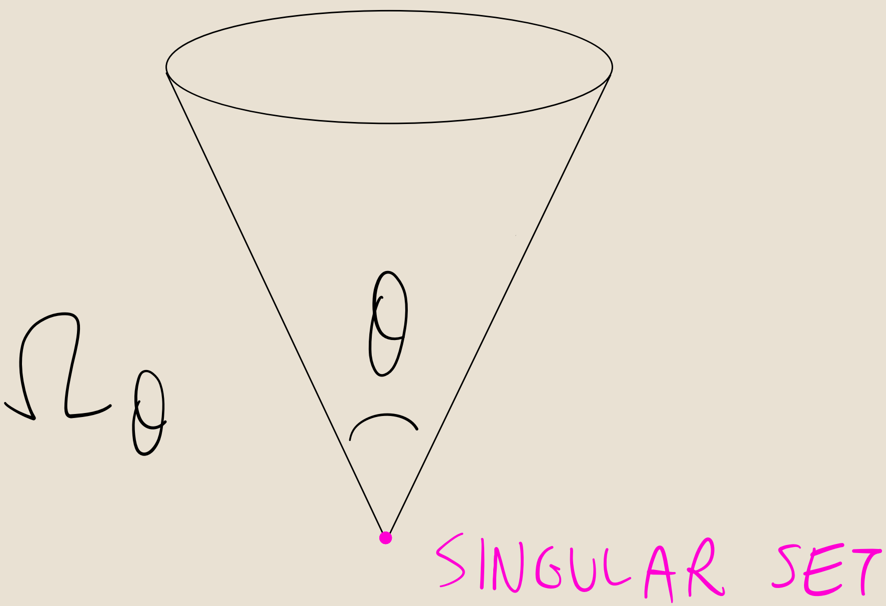


- And if Ω is non smooth ?

Two prototypical examples



WEDGE CONE

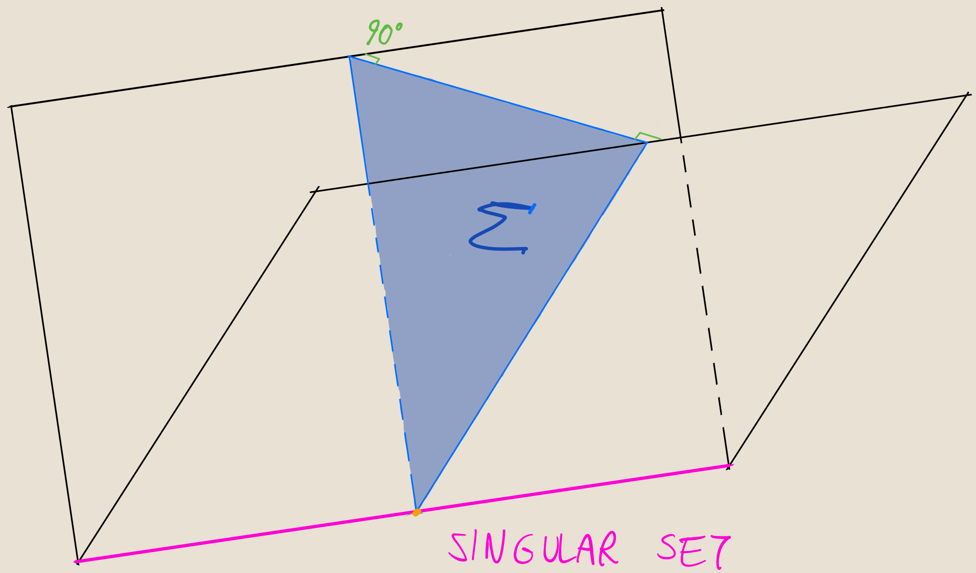


CIRCULAR CONE

$0 < \theta < \pi$

Two PROTOTYPICAL EXAMPLES

Can a free-boundary area-minimizing surface Σ intersect the singular set ?



yes

Hildebrandt, S.; Sauvigny, F., *Minimal surfaces in a wedge I.* Calc. Var. PDE, 1997.

Hildebrandt, S.; Sauvigny, F., *Minimal surfaces in a wedge II.* Archiv der Mathematik, 1997.

Hildebrandt, S.; Sauvigny, F., *Minimal surfaces in a wedge III.* J. Reine Angew. Math., 1999.

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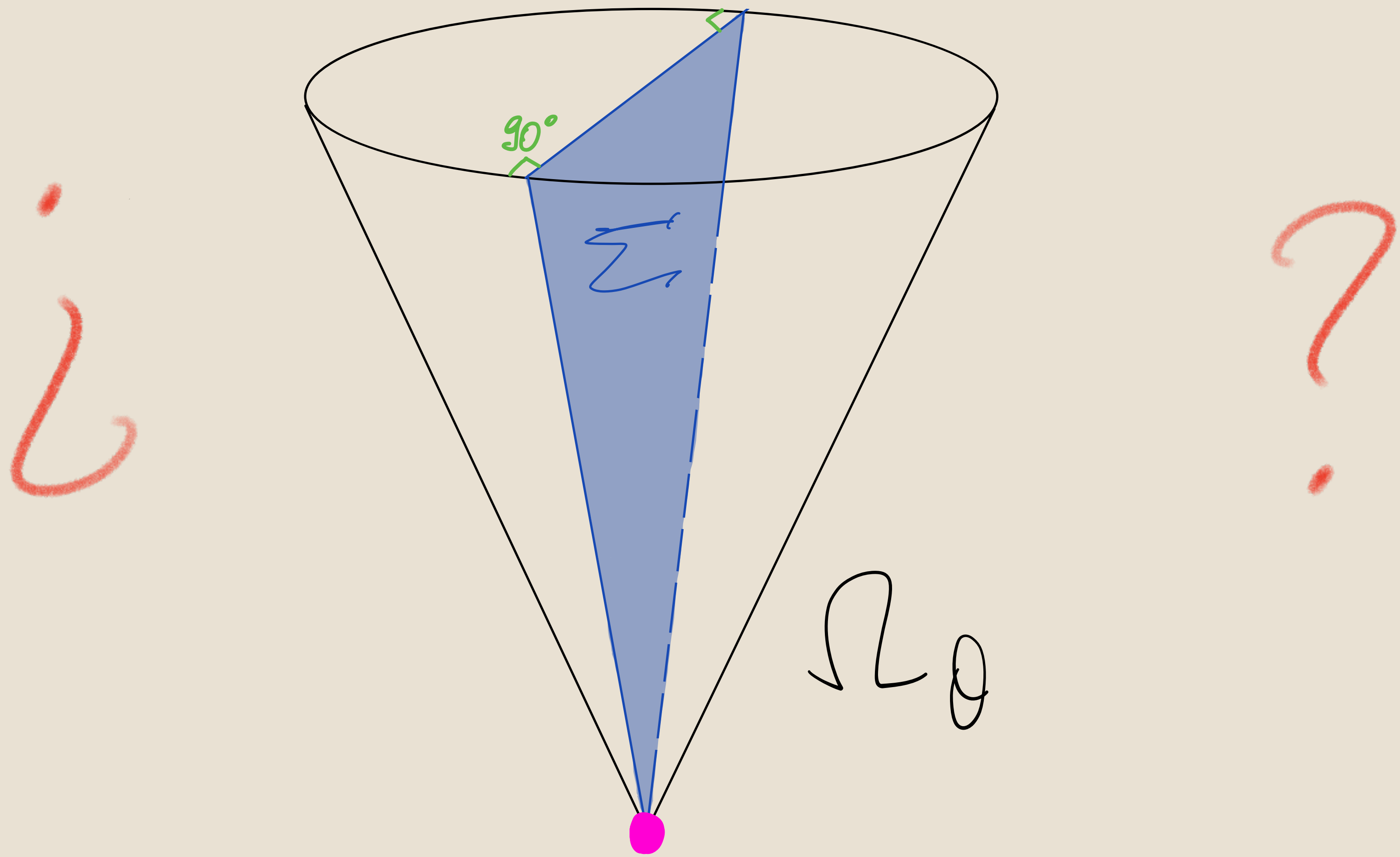
Two prototypical examples

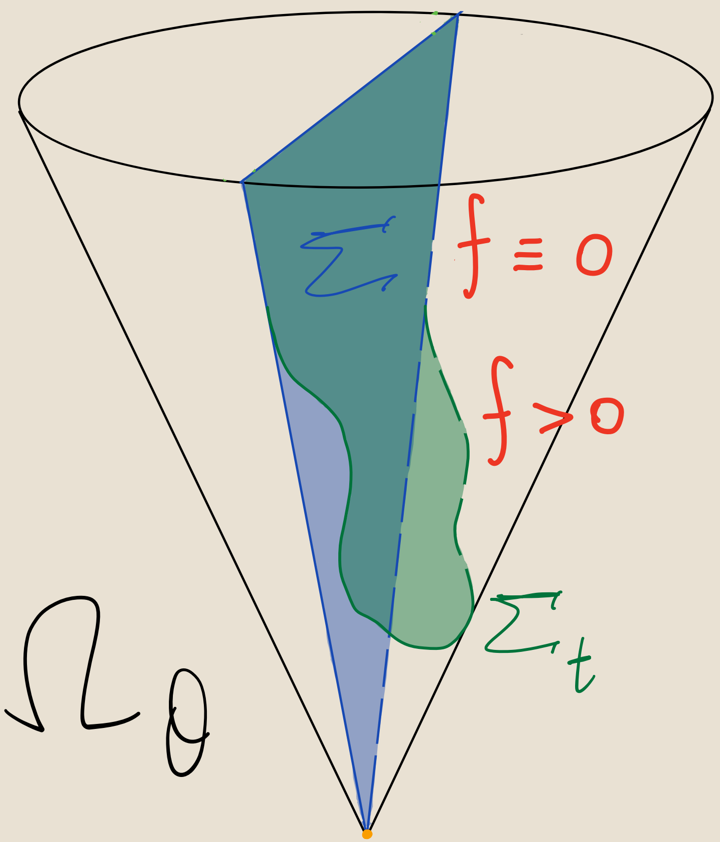
Can a free-boundary area-minimizing surface Σ intersect the singular set ?

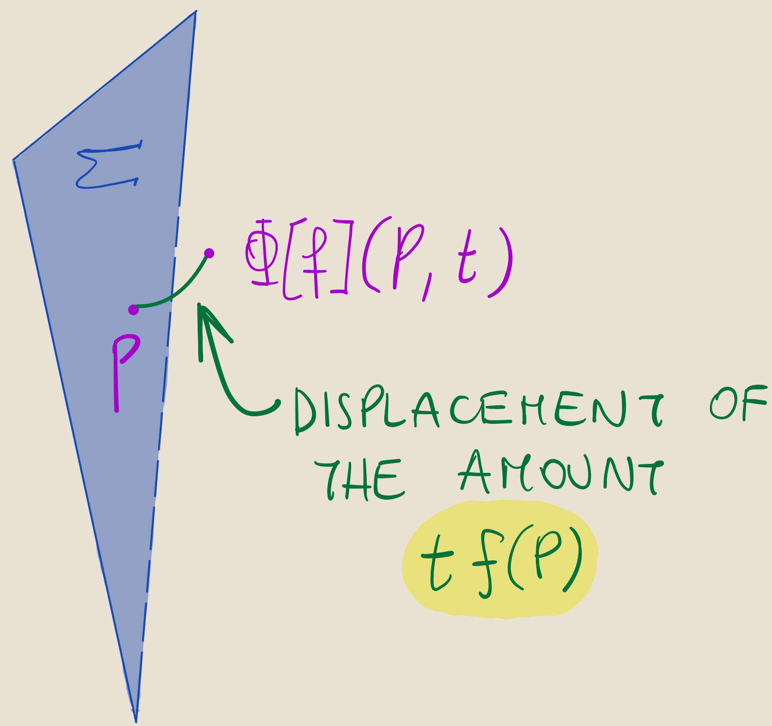


CONSTRUCTION OF A FIRST VARIATION FLOW

Fix $f \in \mathrm{Lip}(\Sigma;\mathbb{R})$, $\operatorname{spt} f \subset\subset \overline{\Sigma}$.

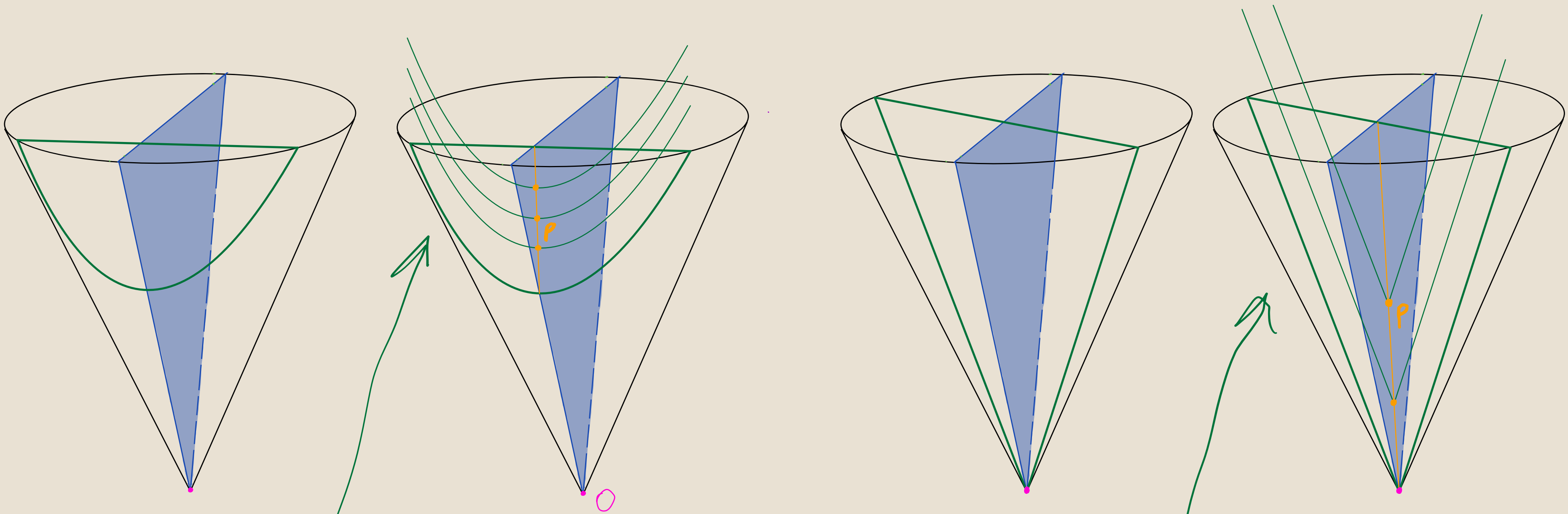
$$\Sigma \ni p \longmapsto \Phi[f](p,t) \in \Sigma_t$$





CONSTRUCTION OF A FIRST VARIATION FLOW

The curves:



VERTICAL LIFT OF THE SECTIONAL CURVES OF Ω_θ

STABILITY OF Σ

- The construction above can be extended to more general domains

Leonardi, G. P.; Vianello, G. *Stability of axial free-boundary hyperplanes in circular cones.* Preprint, 2025 (accepted on Calc. Var. PDE).

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- We define the AREA FUNCTION

$$A[f](t) := \mathcal{H}^{n-1}\big(\Phi[f](\operatorname{spt} f, t)\big)$$

STABILITY OF Σ

- RIGHT FIRST VARIATION of Σ:

$$\underline{\partial_t} A[f](0^+) := \liminf_{t \to 0^+} \frac{A[f](t) - A[f](0)}{t} .$$

STABILITY OF Σ

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$$\underline{\partial_t} A[f](0^+) := \liminf_{t \to 0^+} \frac{A[f](t) - A[f](0)}{t}.$$

- RIGHT SECOND VARIATION of Σ:

$$\underline{\partial_t^2} A[f](0^+) := 2 \liminf_{t \to 0^+} \frac{A[f](t) - A[f](0)}{t^2}.$$

when $\underline{\partial_t} A[f](0^+) = 0$ ...

Stability of Σ

It can be showed that, for every f,

$$\underline{\partial_t} A[f](0^+) = 0,$$

$$\underline{\partial_t^2} A[f](0^+) = \int_{\Sigma} |\nabla f|^2 \, d\mathcal{H}^{n-1} - \frac{1}{\tan(\theta/2)} \int_{\partial\Sigma} \frac{f^2(x)}{|x|} \, d\mathcal{H}^{n-2}(x).$$

STABILITY OF Σ

Hence, we have stability iff

$$\int_{\Sigma} |\nabla f|^2 \, d\mathcal{H}^{n-1} \geq \frac{1}{\tan(\theta/2)} \int_{\partial\Sigma} \frac{f(x)}{|x|} \, d\mathcal{H}^{n-2}(x) \quad (K)$$

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- $n = 3$: (K) fails, because $\int_{\partial\Sigma} \frac{f^2(x)}{|x|} \underset{f(0)\neq 0}{\sim} \int_{\mathbb{R}} 1/|x| = +\infty$
- $n \geq 4$: (K) holds with an optimal constant $c(n)$.

Dávila, J.; Dupaigne, L.; Montenegro, M. *The extremal solution of a boundary reaction problem.* Commun. Pure Appl. Anal.7(2008), no.4, 795–817.

STABILITY OF Σ

Hence the situation is the following:

- $n=3$, $\forall\, 0<\theta<\pi$: STABILITY ✗
- $n\geq 4$, $\dfrac{1}{\tan(\theta/2)} > c(n)$: STABILITY ✗
- $n\geq 4$, $\underbrace{\dfrac{1}{\tan(\theta/2)}}_{\theta \,\geq\, 2\arctan(1/c(n)) \,=:\, \theta^*} \leq c(n)$: STABILITY ✓

Vertex-skipping Theorem

- When $n=3$, something more general holds.

Vertex-skipping Theorem

- When $n=3$, something more general holds.
- We proved in particular the following

Theorem [Leonardi, G. P.; Vianello, G. *A Vertex-skipping property for almost-minimizers of the relative perimeter in convex sets.* Trans. Amer. Math. Soc., 2025.]:

- $\Omega \subset \mathbb{R}^3$ open, convex
- $E \subset \Omega$ almost-minimizer of $P(*;\Omega)$
- $p \in \partial\Omega$ VERTEX for $\partial\Omega$

Then $p \notin \Sigma_r := \overline{\partial E \cap \Omega}$.

VERTEX-SKIPPING THEOREM

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- $\Omega \subset \mathbb{R}^3$ open, convex
- $E \subset \Omega$ almost-minimizer of $P(\ast\,;\Omega)$
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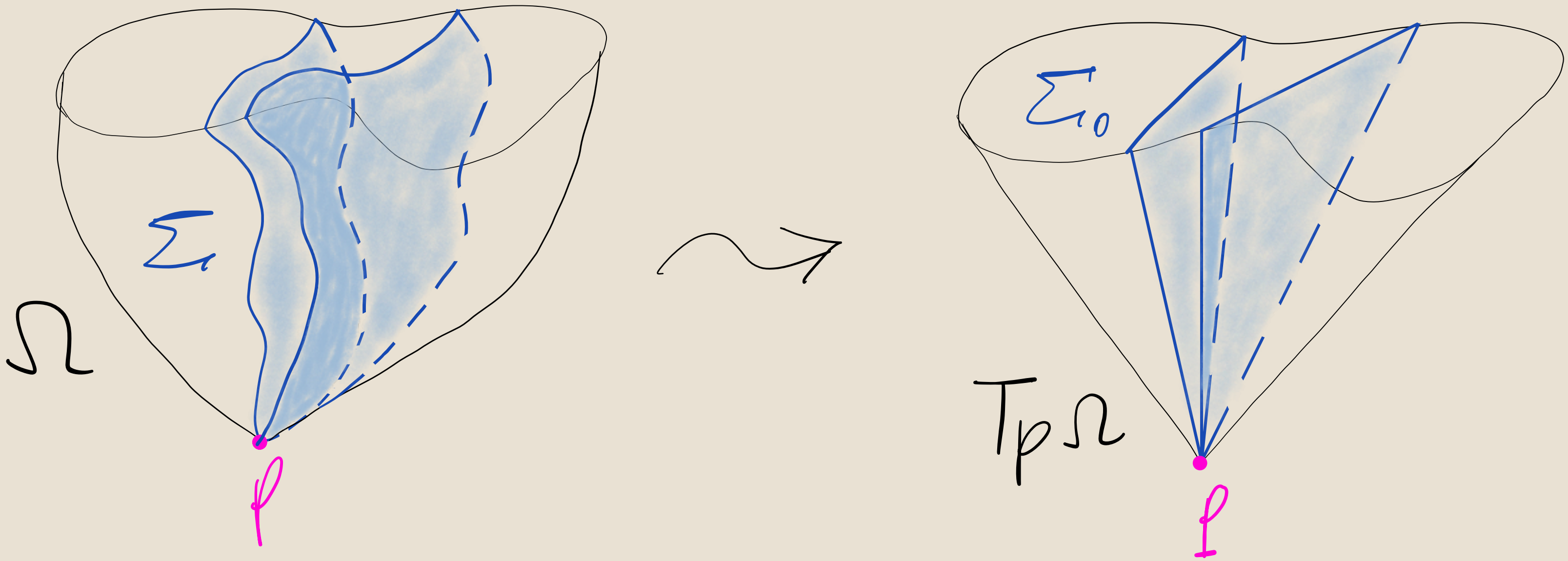
Then $$p \notin \Sigma := \overline{\partial E \cap \Omega}.$$

The tangent cone $T_p\Omega$ to Ω at p does not contain lines.

PROOF OF THE VS: MAIN STEPS

We argue by contradiction

- Blow-up :



KEY TOOL

Leonardi, G. P.; Vianello, G. *Free-boundary Monotonicity for Almost-Minimizers of the Relative perimeter.* Interfaces and Free Boundaries, 2025.

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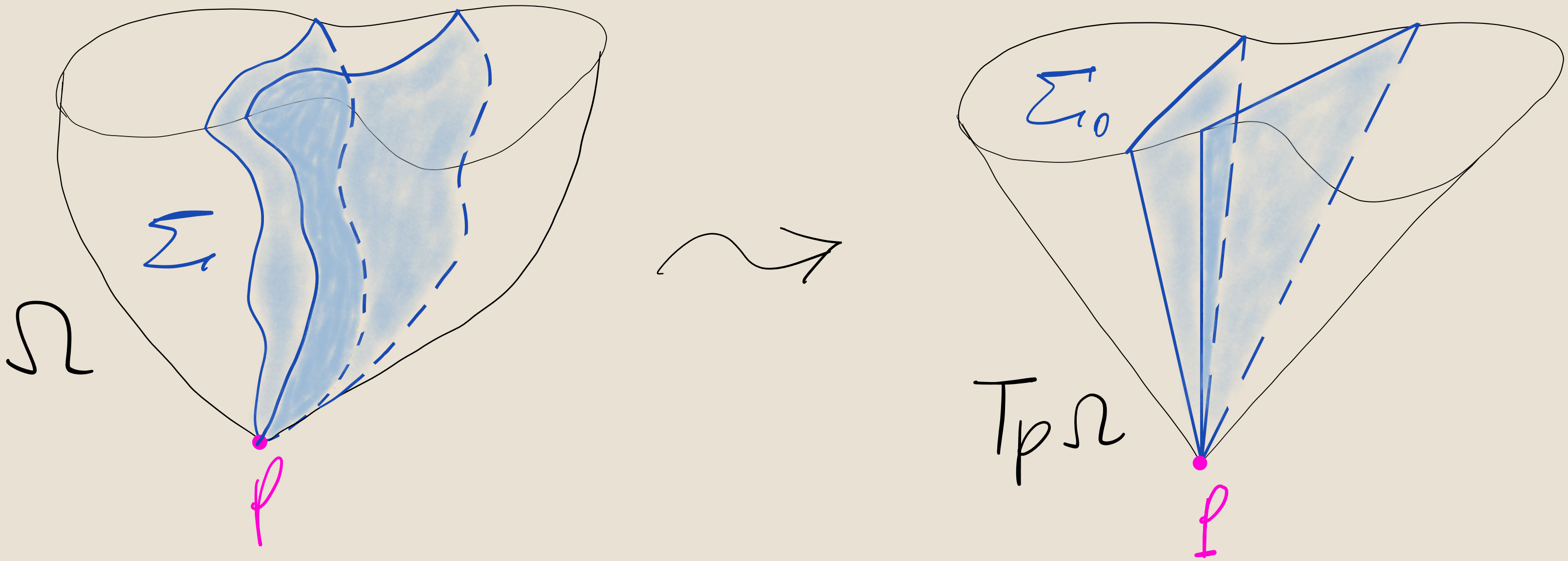

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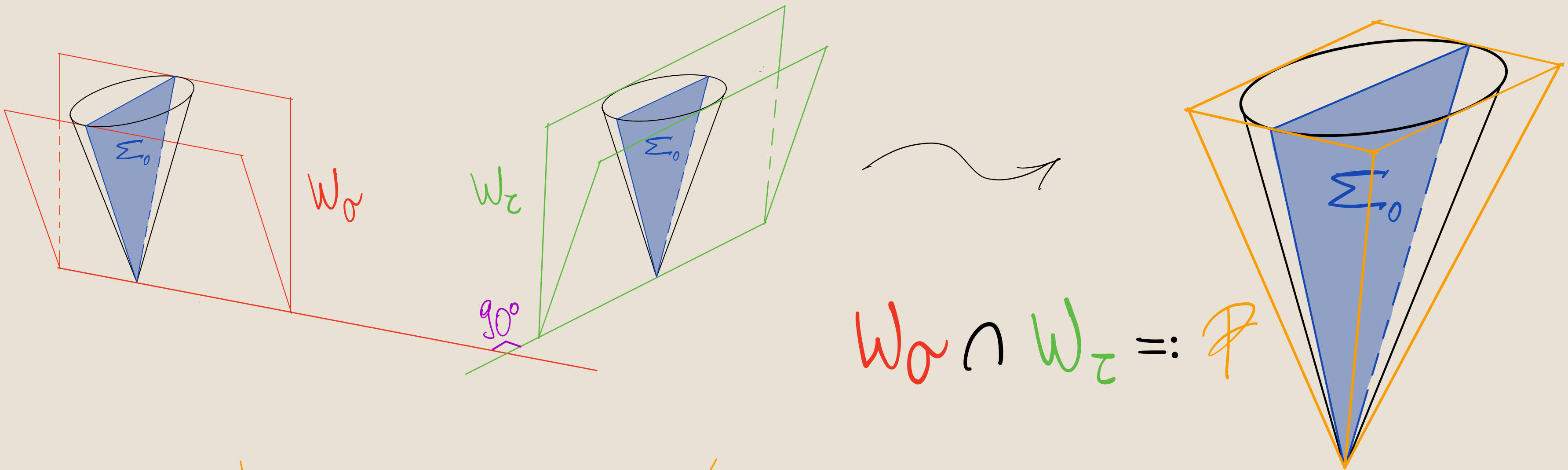
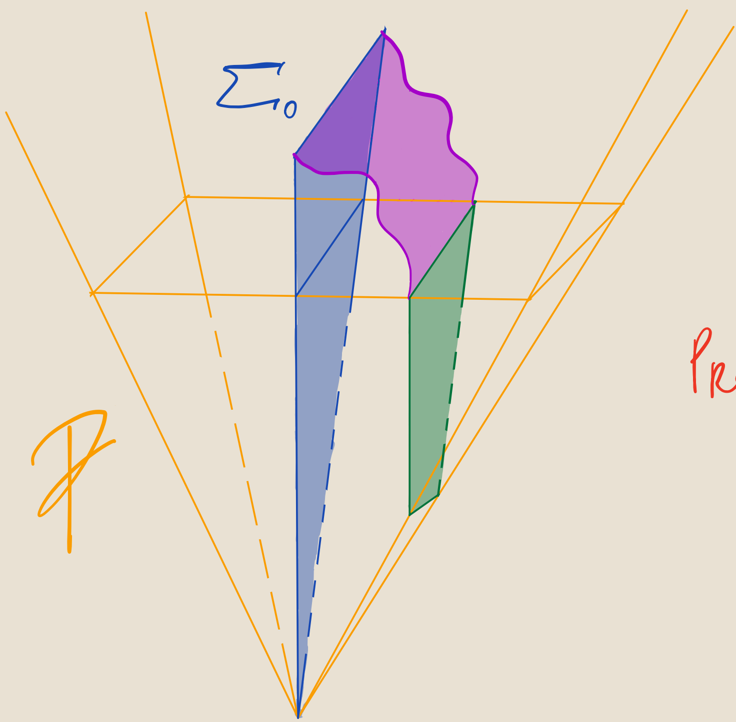
Leonardi, G. P.; Vianello, G. *Free-boundary Monotonicity for Almost-Minimizers of the Relative perimeter.* Interfaces and Free Boundaries, 2025.

- # Connected components of $\Sigma_0 = 1$. ⚠

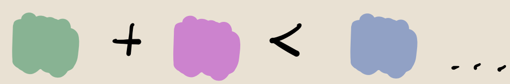
PROOF OF THE VS: MAIN STEPS

- Reduction to the case of a PYRAMID CONE $\mathcal{P}$.





PROBLEM: FIND A SUITABLE CONNECTION S.T.



Open questions

- Let $\theta \geq \theta^*$. Is Σ only stable or even area-minimizing inside Ω_θ ?

$n \geq 5$

Vianello, G. *Area-minimality of axial free-boundary hyperplanes in circular cones via calibration.* In preparation.

↑ Based on a calibration technique:

Lawlor, G. *A sufficient criterion for a cone to be area-minimizing*, Memoirs of the American Mathematical Society, 1991.

Morgan, F. *Area-minimizing surfaces in cones*, Communications in Analysis and Geometry, 2002.

$n = 4$?

- General capillary surfaces ?

Thank you for
the attention