Blow-ups of minimal surfaces in the Heisenberg group

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September 12, 2025

Motivation

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- De Giorgi's regularity theory of the perimeter minimizer in $\mathbb{R}^n$ does not need any initial regularity.
- Aiming to reproduce De Giorgi's theory within the context of the Heisenberg group $\mathbb{H}^n$.

The Heisenberg group $\mathbb{H}^n$

- The Heisenberg group $\mathbb{H}^n$ is a nilpotent lie group: $\mathbb{C}^n \times \mathbb{R}$ with coordinates (z, t) and group law

$$(z,t) * (z', t') = \left(z + z', t + t' - 2\,\mathrm{Im}\langle z, z' \rangle\right),$$

- Lie algebra spanned by left-invariant vector fields :

$$X_k = \frac{\partial}{\partial x_k} + 2y_k \frac{\partial}{\partial t}, \quad Y_k = \frac{\partial}{\partial y_k} - 2x_k \frac{\partial}{\partial t}, \quad T = \frac{\partial}{\partial t},$$

  where $z_k = x_k + iy_k$, $k = 1, \ldots, n$.

- Horizontal subbundle H of $T\mathbb{H}^n$:

$$H(p) := H_p = \mathrm{Span}\{X_1(p), \ldots, X_n(p), Y_1(p), \ldots, Y_n(p)\}.$$

- Dilation: $\delta_\lambda(z, t) = (\lambda z, \lambda^2 t)$
- Box Norm: $\|p\|_\infty = \max\left\{|z|, |t|^{1/2}\right\}$.
- Homogeneous distance: $d(p, q) = \left\|p^{-1} * q\right\|_\infty$.

Horizontal perimeter

Definition

For a measurable set $E \subset \mathbb{H}^n$ and an open set $\Omega \subset \mathbb{H}^n$, the *H-perimeter* of E in Ω is defined as:

$$Per_H(E;\Omega) = \sup\left\{ \int_E \mathrm{div}_H V dzdt : V \in C_0^1(\Omega;\mathbb{R}^{2n}), \|V\|_\infty \leq 1 \right\},$$

where $V = \sum_{j=1}^n V_j X_j + V_{n+j} Y_j$ and $\mathrm{div}_H V = \sum_{j=1}^n X_j V_j + Y_j V_{n+j}$.

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- E is an *H-Caccioppoli set* in Ω if for any open set $A \subset \Omega$, $Per_H(E;A)$ is finite.

Horizontal inner normal

Definition

We can extend $\mu_E(A) := Per_H(E; A)$ as a Radon measure μ_E on Ω. By the Riesz representation theorem, there exists a Borel function $\nu_E : \Omega \to \mathbb{R}^{2n}$ (*horizontal inner normal*) such that $|\nu_E| = 1 \mu_E$ a.e. and the following formula

$$\int_\Omega \langle V, \nu_E \rangle \, d\mu_E = - \int_\Omega \mathrm{div}_H \, V dz dt$$

holds for any $V \in C_c^1 \left(\Omega; \mathbb{R}^{2n}\right)$.

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- When ∂E is smooth, $\nu_E(p) = \frac{P(-\nu(p))}{|P(\nu(p))|}$ for $p \in \partial E$, where P is the orthogonal projection onto the horizontal subbundle H.

H-perimeter minimizing

Let Q_r denote the homogeneous cube centered at 0 with radius r:

$$Q_r = \{(z,t) \in \mathbb{H}^n : |x_i| < r, |y_i| < r, |t| < r^2, i = 1, \dots, n\}.$$

Definition

A H-Caccioppoli set E is *H-perimeter minimizing* in Q_r if:

$$P_H(E, Q_r) \leq P_H(F, Q_r),$$

for any set $F \subset \mathbb{H}^n$ such that the symmetric difference $E \triangle F$ is a compact subset of Q_r.

Strongly H-periemeter minimizing

Let $\overline{Q}_r^{Y_1,+}$ the closure of the cube Q_r relative to the direction Y_1 as

$$\overline{Q}_r^{Y_1,+} = \{(z,t) \in \mathbb{H}^n : -r < y_1 \leq r,\ |x_1| < r,\ |t| < r^2, \\ \text{and } |x_i|, |y_i| < r \text{ for } i = 2, \dots, n\}.$$

Definition (Monti)

A H-Caccioppoli set E is *strongly H-perimeter minimizing* in some neighborhood U of 0, if for any $Q_r \subset U$,

$$P_H(E, Q_r) \leq P_H(F, Q_r), \tag{1}$$

for any set $F \subset \mathbb{H}^n$ such that $(E \triangle F) \cap \bar{Q}_r$ is a compact subset of $\overline{Q}_r^{Y_1,+}$.

Reduced boundary:

Definition

The *reduced boundary* $\partial_H^* E$ is the set of points p where,

$$\lim_{r \to 0} \frac{1}{\mu_E\left(B_r(p)\right)} \int_{B_r} \nu_E \, d\mu_E = \nu_E(p),$$

and $|\nu_E(p)| = 1$.

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Key facts.

- If ∂E is C^1, then $\partial_H^* E = \partial E$ and ν_E is the usual horizontal inner normal.
- $q \in \partial_{\mathbb{H}}^* E$ if and only if $\tau_p\left(q\right) \in \partial_{\mathbb{H}}^* \left(\tau_p E\right)$

Excess

Definition

Take any $p \in \mathbb{H}^n$, $r > 0$, and $v \in S^{2n}$. The *$v-$directional horizontal excess* of E in $B_r(p)$ is

$$\mathrm{Exc}(E, B_r(p), v) = \frac{1}{r^{2n+1}} \int_{B_r(p)} |\nu_E(p) - v| d\mu_E.$$

The *horizontal Excess* of E in $B_r(p)$ is

$$\mathrm{Exc}(E, B_r(p)) = \min_{v \in S^{2n}} \mathrm{Exc}(E, B_r(p), v).$$

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$$\mathrm{Exc}(E, B_r(p)) = \min_{v \in S^{2n}} \mathrm{Exc}(E, B_r(p), v).$$

- If $0 \in \partial^* E$, then we can find a sequence $r_h \to 0$ such that $\mathrm{Exc}(E, B_{r_h}, \nu_E(0)) \to 0$. The rescaled set $E_h = \delta_{1/r_h}(E)$ converges to some limit set E_0 in L^1 with $\mathrm{Exc}(E_0, B_1, \nu_E(0)) = 0$.

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- For $n \geq 2$: E_0 is a half-space;
  For $n = 1$: E_0 might be a non-planar Y-ruled graph. (t=2xy)

De Giorgi's method

- Lipschitz Approximation (Scheon and Simon): E is a perimeter minimizing set in $B_1(x)$, it can be approximate by some Lipschitz function f where the area of the difference is bounded by some constant times the excess.
- Approximating sequence: Pick a decreasing sequence $r_h \to 0$, a point x in the reduced boundary $\partial^* E$.
  Let f_{r_h} be the approximate function of $E_{r_h} = \frac{E-x}{r_h}$. When r_h is small, the excess of E_{r_h} in B_1 is small. Then f_{r_h} almost is flat and almost a minimizer of the Dirichlet functional

$$D(f) = \frac{1}{2} \int_D |\nabla f_r(x)|^2 dx$$

De Giorgi's method

- Harmonic approximation: $f_{r_h} \cdot C(Exc(E_{r_h}, B_1))$ weakly converges to some harmonic function f in $W^{1,2}$.

De Giorgi's method

- Harmonic approximation: $f_{r_h} \cdot C(Exc(E_{r_h}, B_1))$ weakly converges to some harmonic function f in $W^{1,2}$.
- Decay Estimate: There exists a $\varepsilon_0 > 0$ such that for $x \in \partial E$,

$$\text{Exc}\,(E, B_r(x)) < \varepsilon_0 \Longrightarrow \text{Exc}\,(E, B_{kr}(x)) \leq \frac{1}{2}\,\text{Exc}\,(E, B_r(x))\,,$$

  for some $k \in (0,1)$ sufficiently small.
- Iterating the decay estimate, we get ∂E is locally a boundary of a $C^{1,\gamma}$ function g for some $\gamma \in (0,1)$.
- Schauder's estimate: By Schauder-type estimate, the reduced boundary $\partial^* E$ is $C^{1,\alpha}$ for any $\alpha \in (0, \frac{1}{2})$

Intrinsic graphs

Definition

Let $W = \{(z,t) \in \mathbb{H}^n, x_1 = 0\} = \mathbb{R}^{2n}$ be the vertical hyperplane, for $\varphi : W \to \mathbb{R}$, we define the *intrinsic graph* of φ as

$$\mathrm{gr}(\varphi) = \{w * \varphi(w) : w \in W\}$$

and the *intrinsic epigraph* of φ as

$$E_\varphi = \{(w * s) : w \in W, s > \varphi(w)\}$$

where $w * s := (z + se_1, t + 2y_1 s)$, $e_1 = (1, 0, \dots, 0) \in \mathbb{R}^{2n}$.

Intrinsic Lipschitz graph

Definition

An intrinsic graph $\mathrm{gr}(\varphi)$ is *L-intrinsic Lipschitz* with $L \in [0, \infty)$ if for all $p, q \in \mathrm{gr}(\varphi)$,

$$|x_1(p) - x_1(q)| \leq L d(p, q),$$

where $x_1(p)$ is the x_1-coordinate function.

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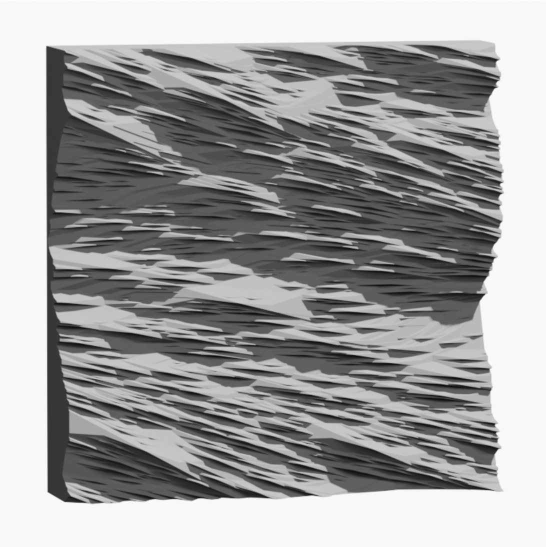
We define the intrinsic gradient of φ by

$$\nabla^{\varphi}\varphi = \left(X_2\varphi, \ldots, X_n\varphi, \frac{\partial\varphi}{\partial y_1} - 4\varphi\frac{\partial\varphi}{\partial t}, Y_2\varphi, \ldots, Y_n\varphi\right)$$

Theorem (Bigolin-Caravenna-Serra Cassano)

$gr(\varphi)$ is an intrinsic Lipschitz graph if and only if $\nabla_{\varphi}\varphi \in L^{\infty}_{\mathrm{loc}}\left(D; \mathbb{R}^{2n-1}\right)$ exists in the sense of distribution.

Figures of an intrinsic Lipschitz graph



Lipschitz approximation

Theorem (Monti)

For $L > 0$, $n \geq 2$, there exists some constant $k > 1$ such that for any H-perimeter minimizing set E in B_{kr}, with $0 \in \partial^* E$, there exists an $L-$intrinsic Lipschitz function $\varphi : W \to \mathbb{R}$ such that

$$\mathcal{S}^{2n+1}\left((\mathrm{gr}(\varphi) \bigtriangleup \partial E) \cap B_r\right) \leq c(L,n)(kr)^{2n+1}\,\mathrm{Exc}\left(E, B_{kr}, X_1\right),$$

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- For $n \geq 2$, small excess $\Rightarrow$ intrinsic Lipschitz graph with slope proportional to $\sqrt{\mathrm{Exc}}$.
  For $n = 1$, you only have a universal bound on the Lipschitz constant.
- This theorem also holds for $n = 1$, but the Lipschitz constant L has to be suitably large.

Approximating sequence

Let $E \subset \mathbb{H}^n$ be a perimeter minimizing set in Q_1 with $0 \in \partial^* E$, $\nu_E(0) = X_1$. One can find a sequence of real numbers $r_h \to 0^+$ such that $E_h = \delta_{1/r_h}(E)$ satisfy the following properties:

1. $0 \in \partial^* E_h$, and $\nu_{E_h}(0) = X_1$.
2. Each set E_h is H-perimeter minimizing.
3. $\text{Exc}(E_h, B_1, X_1) = (\eta_h)^2 :=< \frac{1}{h}$ since excess is dilation invariant.
4. Each set E_h can be approximate by some L-intrinsic function $\varphi_h : W \to \mathbb{R}$.

Existence of limit

The sobolev space $W_H^{1,2}(D)$ is the set of all $\varphi \in L^2(D)$, such that the distributional derivatives in $\nabla_H \varphi$ are square integrable, where

$$\nabla_H \varphi = \left(X_2\varphi, \ldots, X_n\varphi, \frac{\partial \varphi}{\partial y_1}, Y_2\varphi, \ldots, Y_n\varphi \right).$$

Theorem (Monti)

Following from the construction above, there exists an open neighborhood $D \subset W$ of 0, real constants $\overline{\varphi_h}$, a limit function $\varphi \in W_H^{1,2}(D)$ and a selection of indices $k \to h_k$ such that as $k \to \infty$,

$$\begin{aligned} \frac{\varphi_{h_k} - \bar{\varphi}_{h_k}}{\eta_{h_k}} &\rightharpoonup \varphi && \text{weakly in } L^2(D), \\ \frac{\nabla^{\varphi_{h_k}} \varphi_{h_k}}{\eta_{h_k}} &\rightharpoonup \nabla_H \varphi && \text{weakly in } L^2\left(D; \mathbb{R}^{2n-1}\right). \end{aligned} \tag{2}$$

Properties of limit function

Theorem (revision)

- *If E is H-perimeter minimizing in some neighborhood of $0 \in \mathbb{H}^n$, the limit function φ solves the following equation weakly in $D_{1/4}$:*

$$\frac{\partial}{\partial y_1} \Delta_0 \varphi = 0, \tag{3}$$

where $\Delta_0 = \frac{\partial^2}{\partial y_1^2} + \sum_{i=2}^n X_i^2 + Y_i^2$.

- *If E is strongly H-perimeter minimizing in some neighborhood of $0 \in \mathbb{H}^n$, then the limit function φ solves the following equation weakly in $D_{1/4}$:*

$$\Delta_0 \varphi = 0 \tag{4}$$

Contact flows

- A diffeomorphism $\Psi : \mathbb{H}^n \to \mathbb{H}^n$ is a *contact map* if $d\Psi(H) \subset H$. Contact diffeomorphisms are the only diffeomorphisms that preserve the H-perimeter.

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- Examples:
  - Left translations: $(z,t) \mapsto (z+z', t+z' - 2\,\mathrm{Im}\langle z, z'\rangle)$.
  - Rotations: $(z,t) \mapsto (R(z), t)$.
  - Dilations: $(z,t) \mapsto \big(\lambda z, \lambda^2 t\big)$.
- Vertical Dilation $(z,t) \mapsto (z, \lambda t)$ is not contact.

Generating function of contact flow

- Given a contact vector field V_ψ generated by $\psi \in C^\infty(\Omega)$

$$V_\psi = \sum_{j=1}^{n} (Y_j\psi)X_j - (X_j\psi)Y_j - 4\psi T.$$

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$$V_\psi = \sum_{j=1}^{n} (Y_j\psi)X_j - (X_j\psi)Y_j - 4\psi T.$$

- Then there is a corresponding contact flow $\Psi : [-\delta, \delta] \times \Omega \to \mathbb{H}^n$, such that for any $s \in [-\delta, \delta]$, and $p \in \Omega$, the following relation holds:

$$\Psi'(s,p) = V_\psi(\Psi(s,p)),$$
$$\Psi(0,p) = p.$$

We say that the flow is generated by ψ.

Step 1: set up

- We assume the generating function $\psi \in C^\infty(\mathbb{H}^n)$ for the contact flow $\Psi : [-\delta, \delta] \times D_1 \to \mathbb{H}^n$ is of the form

$$\psi = \alpha + x_1\beta + \frac{1}{2}x_1^2\gamma,$$

where α, β, γ are smooth functions in $\mathbb{H}^n$ such that

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$$X_1\alpha = X_1\beta = X_1\gamma = 0 \text{ in } Q_{1/2}$$

- If E is H-perimeter minimizing in Q_1, we assume:
$$\alpha \in C_c^\infty\left(Q_{1/2}\right), \tag{5}$$
then $P_H(E_h, Q_1) \leq P_H(\Psi_s(E_h), Q_1)$.
- If E is strongly H-perimeter minimizing in Q_1, we assume
$$\alpha\left(y_1, z_2, \ldots, z_n, t\right) = \int_0^{y_1} \vartheta_0\left(s, z_2, \ldots, z_n, t\right) ds, \quad y_1 \in \mathbb{R},\ z_i \in \mathbb{C}, \tag{6}$$
where $\vartheta_0 \in C_c^\infty\left(D_{1/2}\right)$ and $X_1\vartheta_0 = 0$.

Step 2: First variation under contact flow

Theorem (Monti)

Let Ψ_s be the contact flow generated by V_ψ. We have the following first variation formula for any $s \in [-\delta, \delta]$,

$$\left| \Delta_s Per_H(E,\Omega) + s \int_\Omega \{4(n+1)T\psi + L_\psi(\nu_E)\} \, d\mu_E \right| \leq CP_H(E,\Omega)s^2, \tag{7}$$

where $\Delta_s Per_H(E,\Omega) = Per_H(\Psi_s(E), \Psi_s(\Omega)) - Per_H(E,\Omega)$*, and*

$$L_\psi\left(\sum_{j=1}^n x_j X_j + y_j Y_j\right) = \sum_{i,j=1}^n x_i x_j X_j Y_i \psi + x_j y_i (Y_i Y_j \psi - X_j X_i \psi) - y_i y_j Y_j X_i \psi$$

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$$\left| \Delta_s Per_H(E,\Omega) + s \int_\Omega \{4(n+1)T\psi + L_\psi(\nu_E)\}\, d\mu_E \right| \leq C P_H(E,\Omega) s^2, \tag{7}$$

where $\Delta_s Per_H(E,\Omega) = Per_H(\Psi_s(E), \Psi_s(\Omega)) - Per_H(E,\Omega)$, and

$$L_\psi\left(\sum_{j=1}^n x_j X_j + y_j Y_j\right) = \sum_{i,j=1}^n x_i x_j X_j Y_i \psi + x_j y_i (Y_i Y_j \psi - X_j X_i \psi) - y_i y_j Y_j X_i \psi$$

,

Minimality condition implies

$$\lim_{h\to\infty} \frac{1}{\eta_h} \int_D \left\{4(n+1)T\psi + L_\psi(\nu_{E_{\varphi_h}})\right\} dw = 0, \tag{8}$$

Step 3: Further Simplification

Theorem (Citti,Manfredini,Pinamonti,Serra Cassano)

The horizontal inner normal $\nu_{E_\varphi} = (\nu_{X_1}, \ldots, \nu_{X_n}, \nu_{Y_1}, \ldots \nu_{Y_n})$ *is of the form:*

$$\nu_{E_\varphi}(w \cdot \varphi(w)) = \left(\frac{1}{\sqrt{1 + |\nabla^\varphi \varphi(w)|^2}}, \frac{-\nabla^\varphi \varphi(w)}{\sqrt{1 + |\nabla^\varphi \varphi(w)|^2}} \right).$$

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- Separate $L_\psi(\nu_{E_{\varphi_h}}) = L_1 + L_2$, where L_1 is a quadratic form that contains ν_{X_1} and L_2 is the rest of L_ψ

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- Separate $L_\psi(\nu_{E_{\varphi_h}}) = L_1 + L_2$, where L_1 is a quadratic form that contains ν_{X_1} and L_2 is the rest of L_ψ
- Since φ_h is intrinsic $L-$Lipschitz, we can assume $\|L_2\| \leq A\|\nabla^{\varphi_h}\varphi_h\|$ for some constant A, then

$$\lim_{h\to\infty} \frac{1}{\eta_h} \int_D |L_2|\, dw \leq \lim_{h\to\infty} \frac{1}{\eta_h} \int_D A\, |\nabla^{\varphi_h}\varphi_h|^2\, dw \leq \lim_{h\to\infty} \frac{A}{\eta_h} c_0 \eta_h^2 = 0$$

Step 4: Expanding the quadratic form

- In L_1, when we set $\beta = \gamma = 0$, for quadratic term that is of the form $Y_i Y_1(\psi)\nu_{X_1}\nu_{Y_i} = Y_j Y_1(\alpha)\nu_{X_1}\nu_{Y_i}$ for $i = 1, \cdots n$. we have,

$$\lim_{h\to\infty} \frac{1}{\eta_h} \int_D \frac{(Y_i Y_1 \alpha)\nu_{X_1}\nu_{Y_i}}{\eta_h} = \int_D Y_i \alpha_{y_1} \cdot Y_i \varphi$$

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- The error occurred when Monti was expanding the term $X_j X_1 \psi$. It should be γ instead of $x_1 X_1 \gamma$ when $j = 1$. The later will vanish in the limit, which caused the final integral in his proof to miss a term $\gamma\varphi_{y_1}$.

Final results

- The final equation is of the form:

$$\int_D (4(n-1)\beta_t + \gamma_{y_1})\varphi - \varphi_{y_1} Y_1^2 \alpha + \gamma\varphi_{y_1} -$$

$$- \sum_{i=2}^{n} [(X_i\alpha_{y_1} + Y_i\beta)X_i\varphi + (Y_i\alpha_{y_1} - X_i\beta)Y_i\varphi] dw = 0.$$

Final results

- The final equation is of the form:

$$\int_D (4(n-1)\beta_t + \gamma_{y_1})\varphi - \varphi_{y_1} Y_1^2 \alpha + \gamma \varphi_{y_1} - \\ - \sum_{i=2}^{n} [(X_i \alpha_{y_1} + Y_i \beta) X_i \varphi + (Y_i \alpha_{y_1} - X_i \beta) Y_i \varphi] dw = 0.$$

- Letting $\beta = \gamma = 0$, by integration apart, we obtain

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- If E is strongly perimeter minimizing, $\alpha_{y_1} = \vartheta$ for any test function $\vartheta \in C_c^\infty(D_{1/2})$, then we have

$$0 = \int_D \vartheta \Delta_0 \varphi dw.$$

Open questions

- Quantitative rates for the convergence to the blow-up profile.
- Possible decay estimate from this PDE.
- Higher-step Carnot groups: does an analogue of $\partial_{y_1}\Delta_0\varphi = 0$ persist?