Journal of Algebra 349 (2012) 234–267

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

ε-Complemented algebras ☆

J.C. Cabello, M. Cabrera *, E. Nieto

Departamento de Análisis Matemático, Facultad de Ciencias, Universidad de Granada, 18071 Granada, Spain

ARTICLE INFO

Article history:
Received 22 June 2011
Available online 10 August 2011
Communicated by Efim Zelmanov

MSC:
primary 17A60
secondary 06A15

Keywords:
Semiprime algebras
Closure operations
Multiplication algebra
Complemented algebras

ABSTRACT

For a nonassociative algebra A, by considering A as a left module over its multiplication algebra $M(A)$, a closure operation (termed the ε-closure) appears on the lattice $\mathcal{I}_A$ of all ideals of A. For an ideal U of A, the ε-closure of U is the largest ideal of A which satisfies the same "multiplicative identities" as U. An algebra A is said to be ε-complemented if for every ε-closed ideal U of A there exists an ε-closed ideal V of A such that $A = U \oplus V$. What is termed the ε'-closure appears in a dual fashion in $\mathcal{I}_{M(A)}$ and the ε'-complementarity can be considered in $M(A)$. This paper provides different characterizations of both complementarities. Moreover, we determine the relation between these concepts, the classical complementarity, and the complementarity for the π-closure. We also develop a structure theory for ε-complemented algebras.

Introduction

A nonassociative algebra A is said to be *complemented* if, for each ideal U of A, there exists an ideal V of A such that $A = U \oplus V$. As is made clear below, if $M(A)$ denotes the multiplication algebra of A, then

$$M(A) \text{ complemented} \quad \Rightarrow \quad A \text{ complemented} \quad \Rightarrow \quad M(A) \text{ semiprime}.$$

Algebras with a semiprime multiplication algebra were first studied by Jacobson [11], who provided a complete description of $M(A)$ whenever A is a finite dimensional algebra with $M(A)$ semiprime.

☆ This research was partially supported by the Spanish MEC Project MTM2009-12067, and the Junta de Andalucía Grant FQM290.

* Corresponding author.
E-mail addresses: jcabello@ugr.es (J.C. Cabello), cabrera@ugr.es (M. Cabrera), enieto@ugr.es (E. Nieto).

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2011.07.009

This description shows that A and $M(A)$ are complemented. Therefore, for finite dimensional algebras, the three concepts related above coincide. Jacobson's pioneering paper was continued by Albert [1], who introduced his radical (see also [12, pp. 1090–1091]). Jacobson's results were also extended to algebras over a ring by Finston in [10]. Without any restriction on the dimension, a systematic study of algebras with zero annihilator and with a semiprime multiplication algebra was initiated in [2]. These algebras are precisely the multiplicatively semiprime algebras. Recall that an algebra A is said to be *multiplicatively semiprime* (in short, *m.s.p.*) whenever both A and $M(A)$ are semiprime algebras. Some results in [2] have recently been extended, avoiding the zero annihilator condition, in [3] and [5]. The cornerstone for the development of the theory of algebras with semiprime multiplication algebra is the ε-closure.

Let A be a nonassociative algebra and consider the complete lattice $\mathcal{I}_A$ of all ideals of A. The classical closure operation on $\mathcal{I}_A$ is the π*-closure* given by

$$\overline{U} = \mathrm{Ann}\big(\mathrm{Ann}(U)\big),$$

where, for each ideal U of A, $\mathrm{Ann}(U)$ denotes the *annihilator* of U in A, that is, the largest ideal V of A satisfying the conditions $UV = VU = 0$. By considering A as a left module over $M(A)$ for the evaluation action, the ε-closure in $\mathcal{I}_A$ and the ε'-closure in $\mathcal{I}_{M(A)}$ are obtained. The ε*-closure* $\widehat{U}$ of an ideal U of A is the largest ideal V of A satisfying $F(V) = 0$ for all $F \in M(A)$ such that $F(U) = 0$. The ε'*-closure* $\mathcal{P}^\vee$ of an ideal $\mathcal{P}$ of $M(A)$ is the largest ideal $\mathcal{Q}$ of $M(A)$ satisfying $\mathcal{Q}(a) = 0$ for all $a \in A$ such that $\mathcal{P}(a) = 0$. The ε-closure is stronger than the π-closure, and in the case in which $M(A)$ is semiprime, the same can be said for the ε'-closure.

Given an algebra A and a closure operation $\sim$ on $\mathcal{I}_A$, A is said to be $\sim$*-complemented* (resp. $\sim$*-quasicomplemented*) when, for each $\sim$-closed ideal U of A, there exists a $\sim$-closed ideal V of A such that $A = U \oplus V$ (resp. $A = (U \oplus V)^\sim$). The aim of this paper is to study the class of all ε-(quasi)complemented algebras. In its development, we determine the relationships between classical complementarity, (quasi)complementarity with respect to the ε and π closures in A, and (quasi)complementarity with respect to the ε' and π closures in $M(A)$. These relationships are summarized in the following diagram:

$$\begin{array}{ccc}
M(A) \text{ complemented} & & \\
\Downarrow & & \\
M(A)\ \varepsilon'\text{-complemented} & \Leftrightarrow & M(A)\ \pi\text{-complemented and } \varepsilon' = \pi \\
\Downarrow & & \\
A\ \varepsilon - \text{complemented} & \Leftarrow & A \text{ complemented and } \varepsilon' = \pi \\
\Downarrow & & \\
A\ \varepsilon\text{-quasicomplemented} & & \\
\Updownarrow & & \\
M(A)\ \varepsilon'\text{-quasicomplemented} & \Leftrightarrow & M(A) \text{ semiprime and } \varepsilon' = \pi.
\end{array}$$

We also show that ε-quasicomplemented algebras are those of the form $B_0 \oplus B$, where B_0 is a null algebra and B is an m.s.p. algebra, with B $\sharp$-unital whenever $B_0 \neq 0$. Moreover, the fact that an algebra $A \cong B_0 \oplus B$ (with B_0 and B as above) belongs to a subclass appearing in the above diagram can be characterized in terms of an additional property for B as follows

$$\begin{array}{ccc}
A\ \varepsilon\text{-complemented} & \Leftrightarrow & B\ \pi\text{-complemented} \\
A \text{ complemented with } \varepsilon' = \pi & \Leftrightarrow & B \text{ complemented}
\end{array}$$

$$M(A)\ \varepsilon'\text{-complemented} \quad \Leftrightarrow \quad M(B)\ \pi\text{-complemented}$$

$$M(A)\ \text{complemented} \quad \Leftrightarrow \quad B \cong \bigoplus_{i=1}^{n} B_i \text{ with } M(B_i) \text{ simple}.$$

These results allow us to take advantage of the structure theories for m.s.p. algebras [2] and for π-complemented algebras [4] in order to develop such a theory for the algebras under consideration in this paper.

Finally, we prove that, for finite dimensional algebras, the different types of complementarity agree, and that every finite dimensional ideal of an m.s.p. algebra is a complemented ideal. From this we deduce a nonassociative version of a result due to Lee and Wong [13, Theorem 1.7].

The first section is introductory and is intended to establish the notation used and to examine the relevant material on the π and ε closures and on m.s.p. algebras, which is needed later. The second section is devoted to establishing our main result for ε-quasicomplemented algebras, and to providing a structure theory for such algebras. In the third section, we address ε-complemented algebras, by refining the results obtained in the previous section. In the fourth section, we are concerned with algebras whose multiplication algebra is ε'-complemented, and prove that these are precisely the algebras A for which the evaluation action induces a lattice isomorphism from the lattice of all central idempotents in $M(A)$ onto the lattice of all ε-closed ideals of A. In the final section we restrict our attention to complemented algebras and to algebras whose multiplication algebra is complemented. The paper concludes by revisiting the finite dimensional case.

1. Preliminaries

This section contains the definitions, results and notation which are needed later. Our intention is to make the subsequent text as self-contained as possible. In this paper, we will deal with algebras over a fixed field $\mathbb{K}$ which are not necessarily associative.

1.1. Some outstanding closure operations in lattices of ideals

In this subsection, we discuss some specific closure operations on the lattice of all ideals of an algebra, as well as those on the lattice of all ideals of its multiplication algebra. Let us begin by reviewing the concept of closure operation on a complete lattice.

1.1. *Closure operations.* Let us recall that a map $x \mapsto \widetilde{x}$ of a complete lattice L into itself is called a *closure operation* if it satisfies:

(i) $x_1 \leqslant x_2 \Rightarrow \widetilde{x_1} \leqslant \widetilde{x_2}$, for all $x_1, x_2 \in L$,
(ii) $x \leqslant \widetilde{x}$, for every $x \in L$,
(iii) $\widetilde{x} = \widetilde{\widetilde{x}}$, for every $x \in L$,
(iv) $(\bigwedge \widetilde{x_i})^{\sim} = \bigwedge \widetilde{x_i}$, for every subset $\{x_i\}$ of L.

Note that (i), (ii), and (iii) imply (iv). An element $x \in L$ is called $\sim$*-closed* whenever $\widetilde{x} = x$. The set $L^{\sim}$ of all $\sim$-closed elements of L is a complete lattice for the meet and join operations given by

$$\bigsqcap x_i = \bigwedge x_i \quad \text{and} \quad \bigsqcup x_i = \Big(\bigvee x_i\Big)^{\sim}.$$

Moreover, if 0_L and 1_L respectively denote the smallest and the largest elements in the lattice L, then $\widetilde{0_L}$ and 1_L respectively are the smallest and the largest elements in the lattice $L^{\sim}$. An element $x \in L$ is called $\sim$*-dense* whenever $\widetilde{x} = 1_L$.

Galois connexions determine closure operations.

1.2. *Galois connexions*. Recall that a *Galois connexion* between two complete lattices L and M is a pair of maps $x \mapsto x^*$ from L to M and $y \mapsto y^\diamond$ from M to L satisfying:

(i) $x_1 \leqslant x_2 \Rightarrow x_2^* \leqslant x_1^*$ and $y_1 \leqslant y_2 \Rightarrow y_2^\diamond \leqslant y_1^\diamond$, for all $x_i \in L$, $y_i \in M$ $(1 \leqslant i \leqslant 2)$.
(ii) $x \leqslant x^{*\diamond}$ and $y \leqslant y^{\diamond *}$, for all $x \in L$, $y \in M$.
(iii) $x^* = x^{*\diamond *}$ and $y^\diamond = y^{\diamond * \diamond}$, for all $x \in L$, $y \in M$.
(iv) $(\bigvee x_i)^* = \bigwedge x_i^*$ and $(\bigvee y_i)^\diamond = \bigwedge y_i^\diamond$, for all subsets $\{x_i\} \subseteq L$, $\{y_i\} \subseteq M$.
(v) $(0_L)^* = 1_M$ and $(0_M)^\diamond = 1_L$.

Note that (i) and (ii) imply (iii), (iv), and (v). If $L \underset{\diamond}{\overset{\star}{\rightleftarrows}} M$ is a Galois connexion, then the maps $\epsilon : L \to L$ and $\epsilon' : M \to M$ given respectively by

$$\epsilon(x) = x^{*\diamond} \quad \text{and} \quad \epsilon'(y) = y^{\diamond *}$$

are closure operations in L and M respectively, the map $x \mapsto x^*$ is an order-reversing bijection from L^ϵ onto $M^{\epsilon'}$, and its inverse is the map $y \mapsto y^\diamond$.

For an algebra A, the classical closure operation on $\mathcal{I}_A$ is the π-closure.

1.3. *π-Closure*. For S_1, S_2 subspaces of an algebra A, we denote by $S_1 S_2$ the subspace of A generated by all the products xy, for $x \in S_1$ and $y \in S_2$. For the sake of brevity, we write S^2 instead of SS. As usual, for each ideal U of A, the largest ideal V of A satisfying the conditions $UV = VU = 0$ is called the *annihilator* of U in A and is denoted by $\mathrm{Ann}_A(U)$. $\mathrm{Ann}(U)$ is usually written for $\mathrm{Ann}_A(U)$. The pairing

$$\mathcal{I}_A \underset{\mathrm{Ann}(.)}{\overset{\mathrm{Ann}(.)}{\rightleftarrows}} \mathcal{I}_A$$

is a Galois connexion (see [2, Proposition 1.3]). The *π-closure* is the closure associated with this Galois connexion, that is, the π-closure $\overline{U}$ of an ideal U of A is defined by

$$\overline{U} = \mathrm{Ann}\big(\mathrm{Ann}(U)\big).$$

Note that property (iii) in Definition 1.2 gives

$$\overline{\mathrm{Ann}(U)} = \mathrm{Ann}(\overline{U}) = \mathrm{Ann}(U), \quad \text{for every } U \in \mathcal{I}_A.$$

1.4. *The multiplication algebra*. Let A be an algebra and let $L(A)$ denote the algebra of all linear operators from A into A. For $a \in A$, L_a and R_a mean the operators of left and right, respectively, multiplication by a on A. $M(A)$ denotes the *multiplication algebra* of A, namely the subalgebra of $L(A)$ generated by the identity operator Id_A and the set $\{L_a, R_a\colon a \in A\}$. It is clear that A is a left $M(A)$-module for the evaluation action. The multiplication operators have the *extension property* [8, Proposition 3.1]: If B is a subalgebra of A, then for each $F \in M(B)$ there exists $T \in M(A)$ such that $T(x) = F(x)$ for all $x \in B$.

The *multiplication ideal* of A, denoted by $M^\sharp(A)$, is defined as the subalgebra of $L(A)$ generated by the set $\{L_a, R_a\colon a \in A\}$. It is clear that $M^\sharp(A)$ is an ideal of $M(A)$ and $M(A) = \mathbb{K} Id_A + M^\sharp(A)$. The algebra A is said to be a *$\sharp$-unital* algebra if $Id_A \in M^\sharp(A)$, that is whenever $M^\sharp(A) = M(A)$.

The following closures were introduced in [2].

1.5. *ε-Closure and ε'-closure.* Let $\mathcal{S}_A$ denote the complete lattice of all subspaces of an algebra A. For each $S \in \mathcal{S}_A$, we define

$$S^{\mathrm{ann}} = \{F \in M(A):\ F(x) = 0 \text{ for each } x \in S\}.$$

Analogously, for each $\mathcal{N} \in \mathcal{S}_{M(A)}$, we set

$$\mathcal{N}_{\mathrm{ann}} = \{a \in A:\ F(a) = 0 \text{ for each } F \in \mathcal{N}\}.$$

The *ε-closure* $S^\wedge$ of S is defined by

$$S^\wedge = (S^{\mathrm{ann}})_{\mathrm{ann}}.$$

Analogously, the *ε'-closure* $\mathcal{N}^\vee$ of $\mathcal{N}$ is defined by

$$\mathcal{N}^\vee = (\mathcal{N}_{\mathrm{ann}})^{\mathrm{ann}}.$$

The pair of maps $S \mapsto S^{\mathrm{ann}}$ and $\mathcal{N} \mapsto \mathcal{N}_{\mathrm{ann}}$ is a Galois connexion between $\mathcal{S}_A$ and $\mathcal{S}_{M(A)}$, as well as between $\mathcal{I}_A$ and $\mathcal{I}_{M(A)}$ (see [2, Proposition 1.7]). It is clear that the ε and ε' closures are the ones associated with this Galois connexion. The following properties of the ε-closure are relevant:

(1) *Continuity property* [2, Proposition 1.8]: If $F \in M(A)$, and if S is a subspace of A, then $F(S^\wedge) \subseteq F(S)^\wedge$. In consequence,

$$S_1^\wedge S_2^\wedge \subseteq (S_1 S_2)^\wedge, \quad \text{for all } S_1, S_2 \in \mathcal{S}_A.$$

(2) [5, Proposition 5.2(1)]: If U is an ε-closed ideal of an algebra A and $q : A \to A/U$ denotes the quotient map, then $q(V)$ is an ε-dense ideal of A/U whenever V is an ε-dense ideal of A.

Recall that an algebra A is said to be *semiprime* if 0 is the unique ideal U of A with $U^2 = 0$.

The relationships between the π-closure and the above were obtained in [2, Proposition 1.11]. In this paper, frequent use is made of them, often without explicit mention.

1.6. *Relationships between closures.* Let A be an algebra. Then:

(1) For any ideal U of A, we have:
 (i) $U^\wedge \subseteq \overline{U}$;
 (ii) $\mathrm{Ann}(U)^\wedge = \mathrm{Ann}(U^\wedge) = \mathrm{Ann}(U)$;
 (iii) $\overline{U^\wedge} = (\overline{U})^\wedge = \overline{U}$.
(2) If additionally $M(A)$ is semiprime, then for any ideal $\mathcal{P}$ of $M(A)$, we have:
 (i) $\mathcal{P}^\vee \subseteq \overline{\mathcal{P}}$;
 (ii) $\mathrm{Ann}(\mathcal{P})^\vee = \mathrm{Ann}(\mathcal{P}^\vee) = \mathrm{Ann}(\mathcal{P})$;
 (iii) $\overline{\mathcal{P}^\vee} = (\overline{\mathcal{P}})^\vee = \overline{\mathcal{P}}$.

Since we are only discussing the ε and π closures for an algebra A, in the case in which $\varepsilon = \pi$, it is possible to speak of closed ideals without risk of confusion, and following the notation of [2] we write $\mathcal{L}_A$ to denote the lattice of all closed ideals of A. Similarly, in the case in which A is an algebra such that $\varepsilon' = \pi$ in $M(A)$, we simply speak of closed ideals in $M(A)$, denoting the lattice of such ideals by $\mathcal{M}_{M(A)}$.

Relationships between closures become narrower in the following class of algebras.

1.7. *Multiplicatively semiprime algebras.* An algebra A is said to be *multiplicatively semiprime* or *m.s.p.* whenever both A and $M(A)$ are semiprime algebras. By [2, Theorem 2.6], for every algebra A the following assertions are equivalent:

(i) A is m.s.p.;
(ii) $\operatorname{Ann}(A) = 0$ and $M(A)$ is semiprime;
(iii) A is semiprime and $\varepsilon = \pi$;
(iv) $A = (U \oplus \operatorname{Ann}(U))^\wedge$, for every ideal U of A.

Semiprime associative algebras are examples of m.s.p. algebras [7, Section 4].

Given an algebra A, associated with each ideal U of A there is an ideal in $M(A)$ defined by

$$[U : A] := \{F \in M(A)\colon\ F(A) \subseteq U\}.$$

Recall that an algebra A is said to be *prime* if, for ideals U and V of A, the condition $UV = 0$ implies either $U = 0$ or $V = 0$.

Some particularly important properties of m.s.p. algebras are the following:

1.8. *Properties of m.s.p. algebras.* If A is an m.s.p. algebra, then:

(1) $\varepsilon' = \pi$ [2, Theorem 2.4];
(2) $\widehat{A^2} = A$ [2, Corollary 2.9];
(3) The map $U \mapsto [U : A]$ is a lattice isomorphism from $\mathcal{L}_A$ onto $\mathcal{M}_{M(A)}$ [2, Corollaries 2.8 and 2.5];
(4) If U is a closed ideal of A, then U and A/U are m.s.p. algebras. Moreover $\mathcal{L}_U = \{V \in \mathcal{L}_A\colon\ V \subseteq U\}$ [2, Theorem 2.11];
(5) If U is a proper closed ideal of A, then U is a maximal closed ideal of A if, and only if, A/U is a prime algebra [2, Corollary 2.12(2)].

Semiprimeness of $M(A)$ can be recognized through the ideals in the algebra A. The following statement summarizes several results (see, [5, Proposition 4.4] and [3, Propositions 3.4(2), 3.5 and Corollary 3.6(1)]).

1.9. *Algebras whose multiplication algebra is semiprime.* For every algebra A, the following assertions are equivalent:

(i) $M^\sharp(A)$ is semiprime;
(ii) $M(A)$ is semiprime;
(iii) $A = (U + [U : A]_{\mathrm{ann}})^\wedge$, for every ideal U of A.

In this case, $\operatorname{Ann}(U^{\mathrm{ann}}) = U^{\mathrm{ann}}(A)^{\mathrm{ann}} = [\widehat{U} : A]$, for every ideal U of A.

1.2. Complemented algebras

This subsection begins by reviewing some well-known concepts and describing complemented algebras.

1.10. *Complemented algebras.* Let A be an algebra. An ideal U of A is said to be *complemented* in A if U is a direct summand of A, that is, if there exists an ideal V of A such that $A = U \oplus V$. The algebra A is said to be *complemented* if every ideal of A is complemented in A. Examples of complemented algebras are null algebras and decomposable algebras. Recall that a *null algebra* is an algebra with zero product; a *decomposable algebra* is an algebra that is isomorphic to a direct sum of simple algebras;

and a *simple algebra* is a non-null algebra lacking nonzero proper ideals. By regarding any algebra as a left module over its multiplication algebra, the standard characterization of completely reducible modules can be rewritten in this case as follows: for a non-null algebra A the following assertions are equivalent:

(i) A is complemented,
(ii) A is isomorphic to an algebra of the form $B_0 \oplus B$, where B_0 is a null algebra and B is a decomposable algebra.

We are interested in algebras A which are complemented with respect to a closure operation on $\mathcal{I}_A$.

1.11. *Complemented algebras with respect to a closure operation.* Let A be an algebra and let $\sim$ be a closure operation on $\mathcal{I}_A$. A $\sim$-closed ideal U of A is said to be *$\sim$-complemented* (resp. *$\sim$-quasicomplemented*) in A if there exists a $\sim$-closed ideal V of A such that

$$A = U \oplus V \quad \big(\text{resp. } A = (U \oplus V)^{\sim}\big).$$

In such a case, V is called a *$\sim$-complement* (resp. *$\sim$-quasicomplement*) of U. We say that A is a *$\sim$-complemented* (resp. *$\sim$-quasicomplemented*) algebra when every $\sim$-closed ideal of A is $\sim$-complemented (resp. $\sim$-quasicomplemented) in A. Clearly every $\sim$-complemented algebra is $\sim$-quasicomplemented.

Recall that a lattice L is said to be *complemented* if it has a smallest element 0 and a largest element 1, and each of its elements has a *complement*; i.e., for each $x \in L$, there exists $x' \in L$ such that $x \vee x' = 1$ and $x \wedge x' = 0$.

Note that, if $\sim$ is a closure operation on $\mathcal{I}_A$ for an algebra A and 0 is a $\sim$-closed ideal of A, then the fact that A is $\sim$-quasicomplemented is nothing but the lattice $(\mathcal{I}_A^{\sim}, \sqcap, \sqcup)$ is complemented.

1.12. *π-Quasicomplemented algebras.* By [4, Corollary 1.4], for every algebra A the following assertions are equivalent:

(i) A is π-quasicomplemented;
(ii) $A = \overline{U \oplus \operatorname{Ann}(U)}$, for every ideal U of A;
(iii) A is semiprime.

In this case, for each π-closed ideal U of A, $\operatorname{Ann}(U)$ is the unique π-quasicomplement of U.

Given an algebra A and a closure operation $\sim$ on $\mathcal{I}_A$, $\sim$ is said to be *additive* if $(U+V)^{\sim} = \widetilde{U} + \widetilde{V}$ for all $U, V \in \mathcal{I}_A$; equivalently, if $U + V \in \mathcal{I}_A^{\sim}$ for all $U, V \in \mathcal{I}_A^{\sim}$.

1.13. *π-Complemented algebras.* By [4, Propositions 3.3 and 4.3], for every algebra A the following assertions are equivalent:

(i) A is π-complemented;
(ii) $A = \overline{U} \oplus \operatorname{Ann}(U)$, for every ideal U of A;
(iii) A is semiprime and the π-closure is additive.

In this case, every π-closed ideal U of A is a π-complemented algebra, and

$$\mathcal{I}_A^{\pi} = \big\{V \oplus W\colon\ V \in \mathcal{I}_U^{\pi},\ W \in \mathcal{I}_{\operatorname{Ann}(U)}^{\pi}\big\}.$$

We will make frequent use of the following result [3, Theorem 2.8 and Proposition 2.6].

1.14. *Algebras with ε-quasicomplemented annihilator.* If A is an algebra with $\mathrm{Ann}(A) \neq 0$, then the following assertions are equivalent:

(i) $\mathrm{Ann}(A)$ is ε-quasicomplemented in A;
(ii) $M^\sharp(A)$ has a unit element;
(iii) $\mathrm{Ann}(A)$ is ε-complemented in A and A^2 is an ε-complement of $\mathrm{Ann}(A)$;
(iv) $A = \mathrm{Ann}(A) \oplus A^2$ and A^2 is a $\sharp$-unital algebra;
(v) A is isomorphic to an algebra of the form $B_0 \oplus B$, where B_0 is a nonzero null algebra and B is a $\sharp$-unital algebra.

In this case,

$$\mathcal{I}_A^\varepsilon = \{I, \mathrm{Ann}(A) \oplus I\colon\ I \in \mathcal{I}_{A^2}^\varepsilon\}. \tag{1}$$

As a consequence, A^2 is the unique ε-quasicomplement of $\mathrm{Ann}(A)$.

1.3. Structural results

This subsection sets out a brief exposition of the structure theories for π-(quasi)complemented algebras [4] and for m.s.p. algebras [2].

The study of the π-closure in a direct sum was carried out in [4, Propositions 2.2 and 4.4].

1.15. *π-Closure in a direct sum.* Let $\{A_i\}_{i\in I}$ be a nonempty family of nonzero algebras, and set $A = \bigoplus_{i\in I} A_i$. Then:

(1) If $\{U_i\}_{i\in I}$ is a family, where each U_i is an ideal of A_i, then

$$\mathrm{Ann}\Big(\bigoplus_{i\in I} U_i\Big) = \bigoplus_{i\in I} \mathrm{Ann}_{A_i}(U_i) \quad \text{and} \quad \overline{\bigoplus_{i\in I} U_i} = \bigoplus_{i\in I} \overline{U_i}.$$

(2) A is semiprime if, and only if, A_i is semiprime for all $i \in I$. In this case,

$$\mathcal{I}_A^\pi = \Big\{\bigoplus_{i\in I} U_i\colon\ U_i \in \mathcal{I}_{A_i}^\pi \text{ for every } i \in I\Big\}.$$

(3) A is π-complemented if, and only if, A_i is π-complemented for all $i \in I$.

1.16. *Radical and socle with respect to a closure operation.* Given an algebra A and a closure operation $\sim$ on $\mathcal{I}_A$, the set of all maximal $\sim$-closed ideals of A is denoted by $\mathbf{M}_A^\sim$, and the set

$$\sim\text{-}\mathrm{Rad}(A) := \bigcap_{M\in\mathbf{M}_A^\sim} M$$

is called the *$\sim$-radical* of A. A is said to be a *$\sim$-radical algebra* whenever $\mathbf{M}_A^\sim = \emptyset$. The set of all minimal $\sim$-closed ideals of A is denoted by $\mathbf{m}_A^\sim$, and the set

$$\sim\text{-}\mathrm{Soc}(A) := \sum_{B\in\mathbf{m}_A^\sim} B$$

is called the $\sim$-*socle* of A. A is said to be a $\sim$-*decomposable algebra* whenever

$$A = \left(\sim\text{-}\mathrm{Soc}(A)\right)^{\sim}.$$

When it is necessary to refer to one of the concepts introduced above for an algebra A in which $\varepsilon = \pi$, we adopt the convention of always using the letter π instead of ε, and follow the same criterion in $M(A)$ when $\varepsilon' = \pi$. Note that not writing any closure would lead to confusion.

Let A be an algebra. For a given nonempty subset $\mathcal{C}$ of $\mathcal{I}_A$, $h^{\mathcal{C}}(U)$ denotes the *hull* of an ideal U of A relative to $\mathcal{C}$, that is

$$h^{\mathcal{C}}(U) = \{V \in \mathcal{C}\colon\ U \subseteq V\}.$$

Recall that an ideal U of A is said to be *essential* if for every nonzero ideal V of A we have $U \cap V \neq 0$. Moreover, the algebra A is a *subdirect product* of a family of algebras $\{A_i\}_{i\in I}$ if there exists a monomorphism f from A into the full direct product $\prod_{i\in I} A_i$ such that, for every $i \in I$, $f_i = p_i \circ f$ maps onto A_i, where p_i is the canonical projection from $\prod_{i\in I} A_i$ onto A_i. When A contains an ideal U such that $f(U)$ is an essential ideal of $\prod_{i\in I} A_i$, A is said to be an *essential subdirect product*.

1.17. *Representation of semiprime algebras* [4, Proposition 2.6]. For every algebra A the following assertions are equivalent:

(i) A is semiprime;
(ii) A is an essential subdirect product of two algebras A_0 and A_1, where A_0 is a π-radical semiprime algebra and A_1 is a π-decomposable semiprime algebra.

In this case, $A_0 \cong A/\overline{\pi\text{-}\mathrm{Soc}(A)}$, $A_1 \cong A/\pi\text{-}\mathrm{Rad}(A)$, and

$$\mathcal{I}_A^{\pi} = \left\{U \cap V\colon\ U \in h^{\mathcal{I}_A^{\pi}}\left(\pi\text{-}\mathrm{Soc}(A)\right),\ V \in h^{\mathcal{I}_A^{\pi}}\left(\pi\text{-}\mathrm{Rad}(A)\right)\right\}.$$

1.18. *Decomposition of π-complemented algebras* [4, Proposition 4.5]. For every algebra A the following assertions are equivalent:

(i) A is π-complemented;
(ii) A is isomorphic to $A_0 \oplus A_1$, where A_0 is a π-radical π-complemented algebra and A_1 is a π-decomposable π-complemented algebra.

In this case, $A_0 \cong \pi\text{-}\mathrm{Rad}(A)$ and $A_1 \cong \overline{\pi\text{-}\mathrm{Soc}(A)}$.

Given a family of algebras $\{A_i\}_{i\in I}$, for each $J \subseteq I$, the *block-projection* $p_J : \prod_{i\in I} A_i \to \prod_{i\in I} A_i$ is the projection given by $p_J(a_i) = (b_i)$, where $b_i = a_i$ if $i \in J$ and $b_i = 0$ otherwise.

1.19. *Description theorem for π-decomposable π-complemented algebras* [4, Theorem 4.9]. Let A be a nonzero algebra. Then the following assertions are equivalent:

(i) A is π-decomposable π-complemented;
(ii) There exists a nonempty family of nonzero prime algebras $\{A_i\}_{i\in I}$ such that A can be regarded as a subalgebra of $\prod_{i\in I} A_i$ containing $\bigoplus_{i\in I} A_i$, and $p_J(A) \subseteq A$ for all $J \subseteq I$.

In this case, $\mathcal{I}_A^{\pi} = \{p_J(A)\colon\ J \subseteq I\}$.

1.20. *Subdirect products of m.s.p. algebras.* Given two algebras A and B, each algebra epimorphism $q : A \to B$ gives rise to a map $q' : M(A) \to M(B)$ uniquely determined by the condition $q'(F) \circ q = q \circ F$

for all $F \in M(A)$. This map is an algebra epimorphism with kernel $[\mathrm{Ker}(q) : A]$, and hence induces a canonical isomorphism from $M(A)/[\mathrm{Ker}(q) : A]$ onto $M(B)$ (see, e.g., [1, Lemma 1]). Moreover, $q'(M^\sharp(A)) = M^\sharp(B)$. By [9, Proposition 2.2], if an algebra A is a subdirect product of the family of algebras $\{A_i\}_{i\in I}$ via the monomorphism f, then $M(A)$ is canonically a subdirect product of the family $\{M(A_i)\}_{i\in I}$ via the monomorphism $\widetilde{f}$ defined by $\widetilde{f}(F) = (f_i'(F))$ for all $F \in M(A)$. Accordingly, the subdirect product of a family of m.s.p. algebras is an m.s.p. algebra.

1.21. *Multiplicatively prime algebras.* An algebra A is said to be *multiplicatively prime* or *m.p.* whenever both A and $M(A)$ are prime algebras. By [6, Proposition 1], for every nonzero algebra A the following assertions are equivalent:

(i) A is an m.p. algebra;
(ii) A is non-null and $\mathcal{I}_A^\varepsilon = \{0, A\}$;
(iii) $\mathrm{Ann}(A) = 0$ and $M(A)$ is prime;
(iv) A is prime and $M(A)$ is semiprime.

A central result in the structure theory for m.s.p. algebras is the description and characterization of ε-decomposable algebras with zero annihilator given in [2, Theorems 3.7 and 3.8].

1.22. *Yood's Theorem.* For an algebra A with zero annihilator, the following assertions are equivalent:

(i) A is ε-decomposable;
(ii) A is a π-decomposable m.s.p. algebra;
(iii) ε-$\mathrm{Rad}(A) = 0$;
(iv) A is an essential subdirect product of a family of m.p. algebras.

2. ε-Quasicomplemented algebras

The aim of this section is to provide different characterizations and to give a representation theorem for ε-quasicomplemented algebras.

We begin with the following elemental result.

Lemma 2.1. *Let A be an algebra and U be an ideal of A. Then*:

(1) $[U : A]_{\mathrm{ann}} \subseteq \mathrm{Ann}(U)$;
(2) $V \subseteq [U : A]_{\mathrm{ann}}$, *for every ideal V of A such that $U \cap V = 0$*;
(3) *If $U \cap \mathrm{Ann}(A) = 0$ and there exists an ideal V of A such that $A = (U \oplus V)^\wedge$, then*

$$A = \big(U \oplus [U : A]_{\mathrm{ann}}\big)^\wedge = \big(U \oplus \mathrm{Ann}(U)\big)^\wedge.$$

Proof. (1) It is clear that the sets

$$L_U = \{L_x \colon x \in U\} \quad \text{and} \quad R_U = \{R_x \colon x \in U\}$$

are contained in $[U : A]$. From these facts it follows that

$$L_U\big([U : A]_{\mathrm{ann}}\big) = R_U\big([U : A]_{\mathrm{ann}}\big) = 0,$$

therefore $U[U : A]_{\mathrm{ann}} = [U : A]_{\mathrm{ann}}U = 0$, and hence $[U : A]_{\mathrm{ann}} \subseteq \mathrm{Ann}(U)$.

(2) Let V be an ideal of A such that $U \cap V = 0$. Since $[U : A](V) \subseteq U \cap V$, it follows that $[U : A](V) = 0$, and hence $V \subseteq [U : A]_{\mathrm{ann}}$.

(3) Assume that V is an ideal of A such that $A=(U\oplus V)^{\wedge}$. From the above parts in the statement it follows that $V\subseteq[U:A]_{\mathrm{ann}}\subseteq\mathrm{Ann}(U)$. Thus, we have

$$A=\big(U+[U:A]_{\mathrm{ann}}\big)^{\wedge}\quad\text{and}\quad A=\big(U+\mathrm{Ann}(U)\big)^{\wedge}.$$

From this last equality we see that

$$\mathrm{Ann}(A)=\mathrm{Ann}\big(U+\mathrm{Ann}(U)\big)=\mathrm{Ann}(U)\cap\overline{U}.$$

Therefore $U\cap\mathrm{Ann}(U)\subseteq\mathrm{Ann}(U)\cap\overline{U}\subseteq\mathrm{Ann}(A)$. Since we also assume that $U\cap\mathrm{Ann}(A)=0$, it follows that $U\cap\mathrm{Ann}(U)=0$, and hence $U\cap[U:A]_{\mathrm{ann}}=0$. Thus, we have proved that

$$A=\big(U\oplus[U:A]_{\mathrm{ann}}\big)^{\wedge}\quad\text{and}\quad A=\big(U\oplus\mathrm{Ann}(U)\big)^{\wedge},$$

as required. □

Corollary 2.2. *If A is an ε-quasicomplemented algebra, then $M(A)$ is semiprime.*

Proof. Given an ideal U of A and fixing an ε-quasicomplement V of $\widehat{U}$, from the inclusion $\widehat{U}\subseteq(U\oplus V)^{\wedge}$, we deduce that $A=(U\oplus V)^{\wedge}$. Therefore, by Lemma 2.1(2), we have $A=(U+[U:A]_{\mathrm{ann}})^{\wedge}$. Now, $M(A)$ is concluded to be a semiprime algebra because of 1.9. □

In order to characterize the ε-quasicomplemented algebras let us first consider the zero annihilator case.

Proposition 2.3. *Let A be an algebra. Then the following assertions are equivalent*:

(i) *A is ε-quasicomplemented with zero annihilator*;
(ii) *$A=(U\oplus\mathrm{Ann}(U))^{\wedge}$, for every ideal U of A*;
(iii) *A is m.s.p.*

In this case, for each closed ideal U of A, $\mathrm{Ann}(U)$ is the unique ε-quasicomplement of U.

Proof. The equivalence between assertions (ii) and (iii) was noted in 1.7. Since the implication (ii) ⇒ (i) is clear, it is only necessary to show that (i) ⇒ (ii). Given an ideal U of A, since $\widehat{U}$ is ε-quasicomplemented in A, by Lemma 2.1(3), we see that $A=(\widehat{U}\oplus\mathrm{Ann}(U))^{\wedge}$. Finally, from the inclusion $\widehat{U}\subseteq(U\oplus\mathrm{Ann}(U))^{\wedge}$, it follows that $A=(U\oplus\mathrm{Ann}(U))^{\wedge}$.

Now, assume that A satisfies the equivalent conditions in the statement. Given a closed ideal U of A, from (ii) it is clear that $\mathrm{Ann}(U)$ is an ε-quasicomplement of U. On the other hand, if V is an ε-quasicomplement of U, then

$$\mathrm{Ann}(U)\cap\mathrm{Ann}(V)=\mathrm{Ann}\big((U\oplus V)^{\wedge}\big)=\mathrm{Ann}(A)=0,$$

and so $\mathrm{Ann}(U)\subseteq\overline{V}=V$. Finally, by Lemma 2.1(1)–(2), we can confirm that $V=\mathrm{Ann}(U)$. Thus $\mathrm{Ann}(U)$ is the unique ε-quasicomplement of U. □

For an ideal U of an algebra A, for simplicity of notation, we will write $[U:A]^{\sharp}$ instead of $[U:A]\cap M^{\sharp}(A)$. If U is a complemented ideal of A and $p_U:A\to U$ denotes the natural projection, then note that the epimorphism $p'_U:M(A)\to M(U)$ is given by $p'_U(F)(x)=F(x)$ for all $F\in M(A)$ and $x\in U$. Thus, p'_U is determined by the evaluation in the elements of U.

Proposition 2.4. *Let A be an algebra. If U is a complemented ideal of A, then the evaluation in the elements of U determines an algebra monomorphism from $[U:A]$ into $M(U)$, which takes $[U:A]^\sharp$ onto $M^\sharp(U)$.*

Proof. Suppose that U is an ideal of A such that $A = U \oplus V$ for an ideal V of A. Note that, for every $F \in [U:A]$, we have $F(V) \subseteq U \cap V$, and hence $F(V) = 0$. Therefore, for $F \in [U:A]$, the condition $p'_U(F) = 0$ yields that $F(A) = F(U \oplus V) = F(U) = p'_U(F)(U) = 0$, and hence $F = 0$. Thus the map p'_U induces an algebra monomorphism from $[U:A]$ into $M(U)$. Moreover, it is clear that $p'_U(L_x) = L_x^U$ and $p'_U(R_x) = R_x^U$ for all $x \in U$, where, to avoid any confusion, L_x^U and R_x^U denote the operators of left and right (respectively) multiplication by x on U. Therefore $p'_U([U:A]^\sharp)$ is a subalgebra of $M^\sharp(U)$ containing L_x^U and R_x^U for all $x \in U$, and hence $p'_U([U:A]^\sharp) = M^\sharp(U)$. $\square$

Let us now centre our attention on the nonzero annihilator case. Note that if A is an algebra with $\mathrm{Ann}(A) \neq 0$, then $M^\sharp(A)$ is a proper ideal of $M(A)$, and therefore $M(A) = \mathbb{K}\, Id_A \oplus M^\sharp(A)$.

Proposition 2.5. *If A is an algebra with $\mathrm{Ann}(A) \neq 0$ and $\mathrm{Ann}(A)$ is ε-quasicomplemented in A, then the evaluation in the elements of A^2 determines an algebra isomorphism from $M^\sharp(A)$ onto $M(A^2)$ that allows us to regard $M(A^2)$ as an ideal of $M(A)$ and to write*

$$\mathcal{I}^{\varepsilon'}_{M(A)} = \left\{\mathcal{P}, \mathbb{K}\,(Id_A - Id_{A^2}) \oplus \mathcal{P}\colon\ \mathcal{P} \in \mathcal{I}^{\varepsilon'}_{M(A^2)}\right\}.$$

Proof. By 1.14, $A = \mathrm{Ann}(A) \oplus A^2$, A^2 is a $\sharp$-unital algebra, and

$$\mathcal{I}^{\varepsilon}_A = \left\{I, \mathrm{Ann}(A) \oplus I\colon\ I \in \mathcal{I}^{\varepsilon}_{A^2}\right\}.$$

Since A^2 is $\sharp$-unital, we have $M(A^2) = M^\sharp(A^2)$. On the other hand, by Proposition 2.4, the evaluation in the elements of A^2 induces an algebra isomorphism from $[A^2:A]^\sharp$ onto $M^\sharp(A^2)$. Note also that $[A^2:A] = M^\sharp(A)$, and consequently $[A^2:A]^\sharp = M^\sharp(A)$. Therefore, $M^\sharp(A) \cong M(A^2)$.

Regarding $M(A^2)$ as a subalgebra of $M(A)$, we will now prove that, for each $I \in \mathcal{I}^{\varepsilon}_{A^2}$, we have

$$I^{\mathrm{ann}} = \mathbb{K}(Id_A - Id_{A^2}) \oplus I^{\mathrm{ann}^\sharp} \quad \text{and} \quad \big(\mathrm{Ann}(A) \oplus I\big)^{\mathrm{ann}} = I^{\mathrm{ann}^\sharp},$$

where I^{ann} denotes the annihilator in $M(A)$ of I, and $I^{\mathrm{ann}^\sharp}$ denotes the annihilator in $M(A^2)$ of I. Given $F \in I^{\mathrm{ann}}$, by writing

$$F = \lambda(Id_A - Id_{A^2}) + T$$

for $\lambda \in \mathbb{K}$ and $T \in M(A^2)$, we see that $0 = F(x) = \lambda(Id_A - Id_{A^2})(x) + T(x) = T(x)$ for every $x \in I$, and hence $T \in I^{\mathrm{ann}^\sharp}$. Thus, we obtain the inclusion $I^{\mathrm{ann}} \subseteq \mathbb{K}(Id_A - Id_{A^2}) \oplus I^{\mathrm{ann}^\sharp}$. Since the converse inclusion is obvious, the first equality is proved. Now, taking into account that $\mathrm{Ann}(A)^{\mathrm{ann}} = M^\sharp(A) = M(A^2)$, we see that

$$\big(\mathrm{Ann}(A) \oplus I\big)^{\mathrm{ann}} = \mathrm{Ann}(A)^{\mathrm{ann}} \cap I^{\mathrm{ann}} = M\big(A^2\big) \cap \big(\mathbb{K}(Id_A - Id_{A^2}) \oplus I^{\mathrm{ann}^\sharp}\big) = I^{\mathrm{ann}^\sharp},$$

and the second equality is also proved. Since $\mathcal{I}^{\varepsilon'}_{M(A)} = \{U^{\mathrm{ann}}\colon\ U \in \mathcal{I}^{\varepsilon}_A\}$, from the equalities above and the description of $\mathcal{I}^{\varepsilon}_A$, the statement follows. $\square$

Lemma 2.6. *If A is an algebra with $\mathrm{Ann}(A) \neq 0$ and $\mathrm{Ann}(A)$ is ε-quasicomplemented in A, then the following assertions are equivalent*:

(i) *A is an ε-quasicomplemented algebra*;
(ii) *A^2 is an ε-quasicomplemented algebra.*

In this case, the ε-quasicomplements in both algebras A and A^2 are unique. Moreover, for each ε-closed ideal U of A, if V denotes its ε-quasicomplement in A and W denotes the ε-quasicomplement of $U \cap A^2$ in A^2, then $V = \mathrm{Ann}(A) \oplus W$ if $U \subseteq A^2$ and $V = W$ otherwise.

Proof. We begin by noting that, by 1.14, we have $A = \mathrm{Ann}(A) \oplus A^2$ and

$$\mathcal{I}_A^\varepsilon = \{I, \mathrm{Ann}(A) \oplus I \colon I \in \mathcal{I}_{A^2}^\varepsilon\}.$$

As a consequence, for each ideal P of A^2, the ε-closure $P^\wedge$ of P in A agrees with the ε-closure of P in A^2.

(i) $\Rightarrow$ (ii). For a given ε-closed ideal I of A^2, since I is an ε-closed ideal of A, we can assert the existence of an ε-closed ideal V of A such that $A = (I \oplus V)^\wedge$. Moreover, since A^2 is ε-closed in A, it follows that $V \nsubseteq A^2$, and hence $V = \mathrm{Ann}(A) \oplus J$ for an ε-closed ideal J of A^2. Now, from the equality $A = (I \oplus \mathrm{Ann}(A) \oplus J)^\wedge$, we deduce that $A = (\mathrm{Ann}(A) \oplus (I \oplus J)^\wedge)^\wedge$, that is $(I \oplus J)^\wedge$ is an ε-quasicomplement of $\mathrm{Ann}(A)$. Therefore, again by 1.14, we conclude that $A^2 = (I \oplus J)^\wedge$, that is J is an ε-quasicomplement of I in A^2. Thus A^2 is ε-quasicomplemented.

(ii) $\Rightarrow$ (i). For each ε-closed ideal I of A^2, there exists an ε-closed ideal J of A^2 such that $A^2 = (I \oplus J)^\wedge$. Therefore

$$A = \mathrm{Ann}(A) \oplus A^2 = \mathrm{Ann}(A) \oplus (I \oplus J)^\wedge,$$

and hence $A = (\mathrm{Ann}(A) \oplus I \oplus J)^\wedge$. Thus I and $\mathrm{Ann}(A) \oplus I$ are ε-quasicomplemented in A. Now, keeping in mind the description of $\mathcal{I}_A^\varepsilon$, we confirm that A is ε-quasicomplemented.

Now, assume that A satisfies the equivalent conditions in the statement. By Proposition 2.3, the ε-quasicomplements in the algebra A^2 are unique. From this fact, and taking into account the above arguments, the uniqueness and the description of the ε-quasicomplements in the algebra A follow. □

Proposition 2.7. *Let A be an algebra with $\mathrm{Ann}(A) \neq 0$. Then the following assertions are equivalent*:

(i) *A is ε-quasicomplemented*;
(ii) *$M^\sharp(A)$ is a semiprime algebra with a unit element*;
(iii) *$A = \mathrm{Ann}(A) \oplus A^2$ and A^2 is a $\sharp$-unital m.s.p. algebra.*

In this case, every ε-closed ideal U of A has a unique ε-quasicomplement U' in A. Precisely, $U' = \mathrm{Ann}(U)$ when $U \subseteq A^2$ and $U' = \mathrm{Ann}(U) \cap A^2$ otherwise.

Proof. (i) $\Rightarrow$ (ii). By Corollary 2.2, $M(A)$ is semiprime. Therefore $M^\sharp(A)$ is semiprime because of 1.9. On the other hand, since $\mathrm{Ann}(A) = M^\sharp(A)_{\mathrm{ann}}$, we see that $\mathrm{Ann}(A)$ is ε-closed, and consequently ε-quasicomplemented in A. Now, by 1.14, we conclude that $M^\sharp(A)$ has a unit element.

(ii) $\Rightarrow$ (iii). By 1.14, $\mathrm{Ann}(A)$ is ε-quasicomplemented in A, A^2 is a $\sharp$-unital algebra, and $A = \mathrm{Ann}(A) \oplus A^2$. Moreover, by Proposition 2.5, $M^\sharp(A) \cong M(A^2)$, and consequently $M(A^2)$ is a semiprime algebra. Since clearly $\mathrm{Ann}_{A^2}(A^2) = 0$, by 1.7, we conclude that A^2 is m.s.p.

(iii) $\Rightarrow$ (i). Again by 1.14 $\mathrm{Ann}(A)$ is ε-quasicomplemented in A. Moreover, by Proposition 2.3, A^2 is an ε-quasicomplemented algebra. Therefore, by Lemma 2.6, A is ε-quasicomplemented.

Now, assume that A satisfies the equivalent conditions in the statement. By Lemma 2.6, every ε-closed ideal of A has a unique ε-quasicomplement in A. Precisely, for a given ε-closed ideal U of A, if V denotes its ε-quasicomplement in A and W denotes the ε-quasicomplement of $U \cap A^2$ in A^2, then $V = \mathrm{Ann}(A) \oplus W$ if $U \subseteq A^2$ and $V = W$ otherwise. Note that, by 1.14(1), either $U \subseteq A^2$ or

$U = \mathrm{Ann}(A) \oplus (U \cap A^2)$. In any case, by 1.15(1), we have $\mathrm{Ann}(U) = \mathrm{Ann}(A) \oplus \mathrm{Ann}_{A^2}(U \cap A^2)$. Since, by Proposition 2.3, $W = \mathrm{Ann}_{A^2}(U \cap A^2)$, from the above it follows that $V = \mathrm{Ann}(U)$ when $U \subseteq A^2$ and $V = \mathrm{Ann}(U) \cap A^2$ otherwise. $\square$

The relation between the quasicomplementarity for the ε and ε' closures is derived from the following general result.

Lemma 2.8. *Let $L \underset{\diamond}{\overset{\star}{\rightleftarrows}} M$ be a Galois connexion between complete lattices, and let ϵ and ϵ' denote the associated closure operations on L and M, respectively. If $x \in L^\epsilon$ is complemented with x' as a complement in L^ϵ, then x^* is complemented with x'^* as a complement in $M^{\epsilon'}$. Accordingly, if L^ϵ is a complemented lattice, then $M^{\epsilon'}$ is a complemented lattice.*

Proof. Suppose that $x, x' \in L^\epsilon$ satisfy $x \sqcup x' = 1_{L^\epsilon}$ and $x \sqcap x' = 0_{L^\epsilon}$. Since $0_{L^\epsilon} = x \sqcap x' = x \wedge x' = x^{*\diamond} \wedge x'^{*\diamond} = (x^* \vee x'^*)^\diamond$, it follows that $1_{M^{\epsilon'}} = (0_{L^\epsilon})^* = (x^* \vee x'^*)^{\diamond *} = \epsilon'(x^* \vee x'^*) = x^* \sqcup x'^*$. On the other hand, we have $(0_M)^\diamond = 1_L = 1_{L^\epsilon} = x \sqcup x' = \epsilon(x \vee x') = (x \vee x')^{*\diamond}$, and hence

$$0_{M^{\epsilon'}} = \epsilon'(0_M) = \bigl(x \vee x'\bigr)^{*\diamond *} = \bigl(x \vee x'\bigr)^* = x^* \wedge x'^* = x^* \sqcap x'^*.$$

Therefore x^* is complemented with x'^* as a complement in $M^{\epsilon'}$, and the proof is complete. $\square$

We are now in a position to prove the main result of this section, which relates the quasicomplementarity for the different closures without any distinction of cases.

Theorem 2.9. *Let A be an algebra. Then the following assertions are equivalent*:

(i) *A is ε-quasicomplemented*;
(ii) *$M(A)$ is ε'-quasicomplemented*;
(iii) *$M(A)$ is semiprime and $\varepsilon' = \pi$*;
(iv) *$A = (U \oplus [\widehat{U} : A]_{\mathrm{ann}})^\wedge$, for every ideal U of A*;
(v) *$A = \mathrm{Ann}(A) \oplus \widehat{A^2}$ and $\widehat{A^2}$ is an m.s.p. algebra*;
(vi) *A is isomorphic to an algebra of the form $B_0 \oplus B$, where B_0 is a null algebra and B is an m.s.p. algebra, with B $\sharp$-unital whenever $B_0 \neq 0$.*

In this case, each ε-closed ideal of A has a unique ε-quasicomplement in A, the map $U \mapsto [U : A]$ is a lattice isomorphism from $\mathcal{I}_A^\varepsilon$ onto $\mathcal{M}_{M(A)}$, and its inverse is the map $\mathcal{P} \mapsto \mathrm{Ann}(\mathcal{P})_{\mathrm{ann}}$.

Proof. We begin by noting that the equivalence (i) $\Leftrightarrow$ (ii) follows immediately from Lemma 2.8. Now, we will first prove the equivalence between assertions (i), (iii), and (iv), and secondly the equivalence between assertions (i), (v), and (vi).

(i) $\Rightarrow$ (iii). When $\mathrm{Ann}(A) = 0$, by Proposition 2.3, we see that A is m.s.p., and hence $M(A)$ is semiprime and $\varepsilon' = \pi$ because of 1.8(1). In the case $\mathrm{Ann}(A) \neq 0$, by Proposition 2.7, A^2 is a $\sharp$-unital m.s.p. algebra. Moreover, by Proposition 2.5, $M(A^2)$ can be regarded as a subalgebra of $M(A)$, and we have

$$\mathcal{I}_{M(A)}^{\varepsilon'} = \bigl\{\mathcal{P}, \mathbb{K}(Id_A - Id_{A^2}) \oplus \mathcal{P}\colon\ \mathcal{P} \in \mathcal{M}_{M(A^2)}\bigr\}.$$

By considering the decomposition $M(A) = \mathbb{K}(Id_A - Id_{A^2}) \oplus M(A^2)$, and taking into account 1.15(2), we deduce that $M(A)$ is semiprime and

$$\mathcal{I}^{\pi}_{M(A)} = \left\{\mathcal{P}, \mathbb{K}(Id_A - Id_{A^2}) \oplus \mathcal{P}\colon \mathcal{P} \in \mathcal{M}_{M(A^2)}\right\}.$$

Therefore, we have $\mathcal{I}^{\pi}_{M(A)} = \mathcal{I}^{\varepsilon'}_{M(A)}$, and consequently $\varepsilon' = \pi$.

(iii) ⇒ (iv). Given an ideal U of A, from 1.12 we deduce that

$$M(A) = \overline{U^{\mathrm{ann}} \oplus \mathrm{Ann}\big(U^{\mathrm{ann}}\big)}.$$

Since, by assumption, $\varepsilon' = \pi$, this equality can be rewritten as follows

$$M(A) = \big(U^{\mathrm{ann}} \oplus \mathrm{Ann}\big(U^{\mathrm{ann}}\big)\big)^{\vee}.$$

Now, by 1.9, we see that $M(A) = (U^{\mathrm{ann}} \oplus [\widehat{U} : A])^{\vee}$. Finally, by using Lemma 2.8, we obtain $A = (\widehat{U} \oplus [\widehat{U} : A]_{\mathrm{ann}})^{\wedge}$, and hence we also have

$$A = \big(U \oplus [\widehat{U} : A]_{\mathrm{ann}}\big)^{\wedge}.$$

(iv) ⇒ (i). This implication is obvious.

(i) ⇒ (v). When $\mathrm{Ann}(A) = 0$, by Proposition 2.3, we see that A is m.s.p., and hence $A = \widehat{A^2}$ because of 1.8(2). Thus $\widehat{A^2}$ is an m.s.p. algebra. When $\mathrm{Ann}(A) \neq 0$, by Proposition 2.7, we see that $A = \mathrm{Ann}(A) \oplus A^2$ and A^2 is a $\sharp$-unital m.s.p. algebra. Finally, according to 1.14, we have $\widehat{A^2} = A^2$.

(v) ⇒ (vi). When $\mathrm{Ann}(A) = 0$, this implication is clear. Assume that $\mathrm{Ann}(A) \neq 0$. Then, by 1.14, we have $\widehat{A^2} = A^2$, and A^2 is $\sharp$-unital.

(vi) ⇒ (i). When $B_0 = 0$, this implication follows from Proposition 2.3. When $B_0 \neq 0$, since B is $\sharp$-unital, we see that $B^2 = B$. Therefore, $A = \mathrm{Ann}(A) \oplus A^2$ and A^2 is a $\sharp$-unital m.s.p. algebra. Now, by applying Proposition 2.7, we conclude that A is ε-quasicomplemented.

Finally, suppose that A satisfies the equivalent conditions in the statement. The uniqueness of the ε-quasicomplements is assured, with the different cases distinguished by Propositions 2.3 and 2.7. Moreover, since the inverse maps $\mathcal{I}^{\varepsilon}_A \underset{\cdot\mathrm{ann}}{\overset{\cdot\mathrm{ann}}{\rightleftarrows}} \mathcal{M}_{M(A)}$ and the inverse maps $\mathcal{M}_{M(A)} \underset{\mathrm{Ann}(.)}{\overset{\mathrm{Ann}(.)}{\rightleftarrows}} \mathcal{M}_{M(A)}$ are order-reversing bijections, we see that the inverse maps $\mathcal{I}^{\varepsilon}_A \underset{\mathrm{Ann}(.)_{\mathrm{ann}}}{\overset{\mathrm{Ann}(.^{\mathrm{ann}})}{\rightleftarrows}} \mathcal{M}_{M(A)}$ are lattice isomorphisms. Now, by 1.9, $\mathrm{Ann}(U^{\mathrm{ann}}) = [U : A]$ for every $U \in \mathcal{I}^{\varepsilon}_A$, and the proof is complete. □

Remark 2.10. By comparing the description of the ε-quasicomplements in Propositions 2.3 and 2.7 with the one obtained in Theorem 2.9(iv) we have the following: if U is an ε-closed ideal of an ε-quasicomplemented algebra A, then $[U : A]_{\mathrm{ann}} = \mathrm{Ann}(U)$ when $U \subseteq \widehat{A^2}$ and $[U : A]_{\mathrm{ann}} = \mathrm{Ann}(U) \cap \widehat{A^2}$ otherwise. On the other hand, by using 1.15(1),

$$\mathrm{Ann}(U) = \mathrm{Ann}(A) \oplus \big(\mathrm{Ann}(U) \cap \widehat{A^2}\big).$$

As a consequence, we see that $\mathrm{Ann}(U) \subseteq \mathrm{Ann}(A) + [U : A]_{\mathrm{ann}}$. Finally, taking into account the inclusion $[U : A]_{\mathrm{ann}} \subseteq \mathrm{Ann}(U)$ given in Lemma 2.1(1), we deduce that $\mathrm{Ann}(U) = \mathrm{Ann}(A) + [U : A]_{\mathrm{ann}}$.

The two conditions in clause (iii) in Theorem 2.9 are independent of each other, even in an associative context.

Example 2.11. The $\varepsilon' = \pi$ condition, even in a finite dimensional context, does not imply the semiprimeness of the multiplication algebra. Let A be the two-dimensional algebra with generator $\{e_0, e_1\}$ given by the relations

$$e_0^2 = e_0 e_1 = e_1 e_0 = 0, \quad \text{and} \quad e_1^2 = e_0.$$

It is immediately verifiable that

$$\operatorname{Ann}(A) = \mathbb{K}e_0 = A^2, \qquad \mathcal{I}_A = \{0, \operatorname{Ann}(A), A\}, \qquad M^\sharp(A) = \mathbb{K}L_{e_1},$$

and

$$\mathcal{I}_{M(A)} = \{0, M^\sharp(A), M(A)\}.$$

Note also that $\mathcal{I}_{M(A)} = \mathcal{I}^{\varepsilon'}_{M(A)} = \mathcal{I}^{\pi}_{M(A)}$, and hence $\varepsilon' = \pi$. However, $M^\sharp(A)^2 = 0$, and so $M(A)$ is not semiprime.

Recall that any algebra A without a unit element can be embedded in another algebra which does possess a unit element. The *unitization* of A over $\mathbb{K}$, denoted by A^1, is the algebra consisting of the vector space $\mathbb{K} \times A$ with the product defined by

$$(\lambda, x)(\mu, y) = (\lambda\mu, xy + \lambda y + \mu x).$$

It is a matter of routine to verify that $\mathbf{1} := (1, 0)$ is the unit element of A^1, and that the map $x \mapsto (0, x)$ allows us to regard A as a subalgebra of A^1 in such a way that $A^1 = \mathbb{K}\mathbf{1} \oplus A$.

For a given algebra A, A^u denotes the *unital hull* of A. Namely, A^u is the unitization of A if A lacks a unit element, and $A^u = A$ otherwise. As usual, if A is an associative algebra, then, for all $a, b \in A^u$, $M_{a,b}$ denotes the two-sided multiplication operator on A defined by $M_{a,b}(x) = axb$ for all $x \in A$. Note that

$$M(A) = \left\{ \sum_{i=1}^{n} M_{a_i,b_i} \colon n \in \mathbb{N},\ a_i, b_i \in A^u\ (1 \leqslant i \leqslant n) \right\}.$$

Example 2.12. There are algebras whose multiplication algebra is prime and $\varepsilon' \neq \pi$. Let $\mathbf{X}$ be a countably infinite set (of "formal variables"), let $M(\mathbf{X})'$ be the free monoid generated by $\mathbf{X}$, and let $B = \mathbb{K}\langle\mathbf{X}\rangle'$ be the nonunital free associative algebra over $\mathbb{K}$ generated by $\mathbf{X}$. Fix $\mathbf{x} \in \mathbf{X}$ and set $I = M^\sharp(B)(\mathbf{x}^2)$, that is I is the ideal of B consisting of the linear hull of the set N of all words of length $\geqslant 3$ containing $\mathbf{x}^2$ as a subword. Define $A = B/I$. For simplicity of notation, for each formal variable $\mathbf{y}$, we will continue writing $\mathbf{y}$ instead of $\mathbf{y} + I$. Therefore, the algebra A can be regarded as the vector space over $\mathbb{K}$ generated by all words p in $M(\mathbf{X})' \backslash N$ taking into account the usual product of words and the fact that pq must be zero when $p, q \in M(\mathbf{X})'$ satisfies $pq \in N$.

Let $F, G \in M(A) \backslash \{0\}$. Denote by $M(\mathbf{X})$ the free monoid with a unit element generated by $\mathbf{X}$, and write

$$F = \sum_{i=1}^{n} \lambda_i M_{p_i,q_i} \quad \text{and} \quad G = \sum_{j=1}^{m} \mu_j M_{r_j,s_j},$$

where $\lambda_i, \mu_j \in \mathbb{K} \backslash \{0\}$, and $p_i, q_i, r_j, s_j \in M(\mathbf{X}) \backslash N$ satisfying the conditions:

(1) If $i \neq i'$, then either $p_i \neq p_{i'}$ or $q_i \neq q_{i'}$.
(2) If $j \neq j'$, then either $r_j \neq r_{j'}$ or $s_j \neq s_{j'}$.

Let us fix $\mathbf{y} \in \mathbf{X} \backslash \{\mathbf{x}\}$ which is different from the variables involved in the words p_i, q_i, r_j, s_j. The above conditions (1)–(2) yield

$$p_i \mathbf{y} r_j \mathbf{y} s_j \mathbf{y} q_i \neq p_{i'} \mathbf{y} r_{j'} \mathbf{y} s_{j'} \mathbf{y} q_{i'} \quad \text{whenever } (i, j) \neq \left(i', j'\right),$$

and consequently

$$F M_{\mathbf{y},\mathbf{y}} G(\mathbf{y}) = \sum_{\substack{1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant m}} \lambda_i \mu_j p_i \mathbf{y} r_j \mathbf{y} s_j \mathbf{y} q_i \neq 0.$$

Therefore, $FM(A)G \neq 0$. Hence $M(A)$ is prime. However, it is clear that $\mathbf{x}^2 \in \operatorname{Ann}(A) \cap A^2$, hence $\operatorname{Ann}(A) \cap A^2 \neq 0$, and so, by Proposition 2.7, A is not ε-quasicomplemented. Therefore, by Theorem 2.9, $\varepsilon' \neq \pi$.

In view of assertion (vi) in Theorem 2.9, the structure theory for ε-quasicomplemented algebras is subject to that of ($\sharp$-unital) m.s.p. algebras. This section concludes by providing such a theory.

Let A be a subdirect product of a family of algebras $\{A_i\}_{i \in I}$ via the monomorphism f. The algebra A is said to be a *$\sharp$-unital subdirect product* if $\widetilde{f}(M^\sharp(A))$ is a unital subalgebra of $\prod_{i \in I} M(A_i)$, that is, if there exists $E \in M^\sharp(A)$ such that $f_i'(E) = Id_{A_i}$ for all $i \in I$.

Lemma 2.13. *Let $\{A_i\}_{i \in I}$ be a nonempty family of nonzero algebras. If A is a subdirect product of $\{A_i\}_{i \in I}$, then the following assertions are equivalent*:

(i) *A is a $\sharp$-unital algebra*;
(ii) *A is a $\sharp$-unital subdirect product.*

In this case, A_i is $\sharp$-unital, for all $i \in I$.

Proof. (i) $\Rightarrow$ (ii). It is clear that $f_i'(Id_A) = Id_{A_i}$ for all i. Thus, A is a $\sharp$-unital subdirect product.

(ii) $\Rightarrow$ (i). Let $E \in M^\sharp(A)$ such that $\widetilde{f}(E) = (Id_{A_i})$. Since the map $\widetilde{f} : M(A) \to \prod_{i \in I} M(A_i)$ is a monomorphism satisfying $\widetilde{f}(Id_A) = (Id_{A_i})$, it follows that $E = Id_A$, and so A is a $\sharp$-unital algebra.

Finally, assume that A satisfies the equivalent assertions in the statement. Given $i \in I$, since $f_i' : M(A) \to M(A_i)$ is an epimorphism satisfying $f_i'(M^\sharp(A)) = M^\sharp(A_i)$, it follows that A_i is $\sharp$-unital. $\square$

Corollary 2.14. *If A is a $\sharp$-unital algebra, then every complemented ideal of A is a $\sharp$-unital algebra.*

The m.s.p. version of 1.17 is as follows.

Theorem 2.15. *Let A be an algebra. Then the following assertions are equivalent*:

(i) *A is ($\sharp$-unital) m.s.p.*;
(ii) *A is an essential ($\sharp$-unital) subdirect product of two algebras A_0 and A_1, where A_0 is a π-radical m.s.p. algebra and A_1 is a π-decomposable m.s.p. algebra.*

In this case, $\mathcal{L}_A = \{U \cap V\colon\ U \in h^{\mathcal{L}_A}(\pi\text{-}\mathrm{Soc}(A)),\ V \in h^{\mathcal{L}_A}(\pi\text{-}\mathrm{Rad}(A))\}$.

Proof. We first prove the equivalence without considering the $\sharp$-unital condition in the two sentences. Implication (ii) $\Rightarrow$ (i) is a direct consequence of 1.20. Implication (i) $\Rightarrow$ (ii) and the description of $\mathcal{L}_A$ follow from 1.17 by considering the algebras $A_0 = A/\overline{\pi\text{-}\mathrm{Soc}(A)}$ and $A_1 = A/\pi\text{-}\mathrm{Rad}(A)$. Note that, by 1.8(4), both algebras are m.s.p. Finally, note that the equivalence adding $\sharp$-unital to both sentences follows from the previous lemma. $\square$

Now, on account of Lemma 2.13 we can complete the equivalence (ii) $\Leftrightarrow$ (iv) in 1.22 as follows:

Theorem 2.16. *For an algebra A, the following conditions are equivalent*:

(i) *A is ($\sharp$-unital) π-decomposable m.s.p.*;
(ii) *A is an essential ($\sharp$-unital) subdirect product of a family of m.p. algebras.*

3. ε-Complemented algebras

Our principal objective in the present section is to refine the results obtained in the previous one for ε-complemented algebras. As previously, we start by addressing the zero annihilator case.

Proposition 3.1. *For every algebra A the following assertions are equivalent*:

(i) *A is an ε-complemented algebra with zero annihilator*;
(ii) *$A = \widehat{U} \oplus \mathrm{Ann}(U)$, for every ideal U of A*;
(iii) *A is a π-complemented m.s.p. algebra.*

In this case, every closed ideal U of A is an ε-complemented algebra with zero annihilator, and $\mathcal{L}_A = \{V \oplus W\colon\ V \in \mathcal{L}_U,\ W \in \mathcal{L}_{\mathrm{Ann}(U)}\}$.

Proof. (i) $\Rightarrow$ (ii). For a given ideal U of A, $\mathrm{Ann}(U)$ is the unique ε-complement of $\widehat{U}$ because of Proposition 2.3. Thus $A = \widehat{U} \oplus \mathrm{Ann}(U)$.

(ii) $\Rightarrow$ (iii). By Proposition 2.3, A is m.s.p. Now, by 1.7, $\varepsilon = \pi$, and hence condition (ii) can be rewritten as follows: $A = \overline{U} \oplus \mathrm{Ann}(U)$, for every ideal U of A. Finally, by 1.13, we can assert that A is π-complemented.

(iii) $\Rightarrow$ (i). Since A is m.s.p., by 1.7, we have $\mathrm{Ann}(A) = 0$ and $\varepsilon = \pi$. Moreover, since A is π-complemented, the equality $\varepsilon = \pi$ yields that A is ε-complemented.

Now, assume that A satisfies the equivalent conditions in the statement and fix a closed ideal U of A. By 1.8(4), U is an m.s.p. algebra, and consequently the proof is concluded by invoking 1.13. $\square$

Example 3.2. Unital π-decomposable ε-quasicomplemented algebras may not be ε-complemented. Let c_{qc} be the algebra of all sequences on $\mathbb{K}$ which are quasi-constant endowed with the coordinatewise algebra operations. It is clear that c_{qc} is a unital subdirect product of a family of copies of $\mathbb{K}$, and hence is a unital π-decomposable ε-quasicomplemented algebra (see Theorem 2.16). However, c_{qc} is not π-complemented because

$$U = \big\{\{a_n\} \in c_{qc}\colon\ a_{2n} = 0 \text{ for all } n \in \mathbb{N}\big\}$$

is a π-closed ideal of c_{qc} with

$$\mathrm{Ann}(U) = \big\{\{a_n\} \in c_{qc}\colon\ a_{2n-1} = 0 \text{ for all } n \in \mathbb{N}\big\}$$

and $c_{qc} \neq U \oplus \mathrm{Ann}(U)$.

Proposition 3.3. *Let $\{A_i\}$ be a nonempty family of nonzero algebras, and set $A = \bigoplus_{i\in I} A_i$. Then*

$$M^\sharp(A) = \bigoplus_{i\in I} [A_i : A]^\sharp \cong \bigoplus_{i\in I} M^\sharp(A_i).$$

Proof. For each $F \in [A_i : A] \cap (\sum_{j\neq i}[A_j : A])$, we have

$$F(A) \subseteq A_i \cap \left(\sum_{j\neq i} A_j\right) = 0,$$

and hence $F = 0$. Therefore $\sum_{i\in I}[A_i : A] = \bigoplus_{i\in I}[A_i : A]$, and in particular $\sum_{i\in I}[A_i : A]^\sharp = \bigoplus_{i\in I}[A_i : A]^\sharp$. Moreover, for a given $a \in A$, by writing $a = a_{i_1} + \cdots + a_{i_n}$ for suitable $a_{i_k} \in A_{i_k}$ and $i_k \in I$, we find that $L_a = \sum_{k=1}^n L_{a_{i_k}}$ and $R_a = \sum_{k=1}^n R_{a_{i_k}}$ belong to $\bigoplus_{i\in I}[A_i : A]^\sharp$. As a consequence we deduce that $M^\sharp(A) = \bigoplus_{i\in I}[A_i : A]^\sharp$. Finally, by Proposition 2.4, we can confirm that $M^\sharp(A) \cong \bigoplus_{i\in I} M^\sharp(A_i)$. $\square$

Proposition 3.4. *Let A be a semiprime algebra without a unit element and let A^1 denote the unitization of A over $\mathbb{K}$. Then*

$$\mathrm{Ann}_{A^1}(U) \cap A = \mathrm{Ann}_A(U \cap A),$$

for every ideal U of A^1. Accordingly, $\mathcal{I}_A^\pi = \{U \cap A\colon U \in \mathcal{I}_{A^1}^\pi\}$.

Proof. Assume that U is an ideal of A^1. The inclusion $\mathrm{Ann}_{A^1}(U) \cap A \subseteq \mathrm{Ann}_A(U \cap A)$ is clear. On the other hand, by semiprimeness, we have $\mathrm{Ann}_A(U \cap A) \cap U \cap A = 0$. Since

$$\mathrm{Ann}_A(U \cap A)U + U\,\mathrm{Ann}_A(U \cap A) \subseteq \mathrm{Ann}_A(U \cap A) \cap U \cap A,$$

it follows that $\mathrm{Ann}_A(U \cap A)U = U\,\mathrm{Ann}_A(U \cap A) = 0$, and hence

$$\mathrm{Ann}_A(U \cap A) \subseteq \mathrm{Ann}_{A^1}(U) \cap A,$$

as required. $\square$

Proposition 3.5. *Let A be an m.s.p. algebra. Consider the following conditions*:

(1) *$M(A)$ is π-complemented*;
(2) *A is π-complemented*;
(3) *$M^\sharp(A)$ is π-complemented.*

Then the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) *hold.*

Proof. Recall that, by 1.8(3), the map $U \mapsto [U : A]$ is a lattice isomorphism from $\mathcal{L}_A$ onto $\mathcal{M}_{M(A)}$.

(1) $\Rightarrow$ (2). For a given $U \in \mathcal{L}_A$, since $M(A)$ is assumed to be π-complemented, we have $M(A) = [U : A] \oplus [V : A]$ for a suitable $V \in \mathcal{L}_A$. By writing $Id_A = E + F$ for $E \in [U : A]$ and $F \in [V : A]$, we see that $a = Id_A(a) = E(a) + F(a) \in U + V$ for every $a \in A$, and hence $A = U + V$. Moreover, from $[U \cap V : A] \subseteq [U : A] \cap [V : A] = 0$, it follows that $U \cap V = 0$. Therefore $A = U \oplus V$. Thus A is π-complemented.

(2) $\Rightarrow$ (3). Note that $\mathcal{I}^\pi_{M^\sharp(A)} = \{\mathcal{P} \cap M^\sharp(A)\colon \mathcal{P} \in \mathcal{M}_{M(A)}\}$. (When A is not $\sharp$-unital, identify $M(A)$ with the unitization of $M^\sharp(A)$ and apply Proposition 3.4.) Therefore, given $\mathcal{P} \in \mathcal{I}^\pi_{M^\sharp(A)}$, there exists

$U \in \mathcal{L}_A$ such that $\mathcal{P} = [U : A]^\sharp$. Since A is assumed be π-complemented, we have $A = U \oplus V$ for a suitable $V \in \mathcal{L}_A$. Now, by Proposition 3.3, it follows that $M^\sharp(A) = [U : A]^\sharp \oplus [V : A]^\sharp$. Thus, $M^\sharp(A)$ is π-complemented. $\square$

Corollary 3.6. *Let A be an algebra with zero annihilator. Then*:

(1) *If $M(A)$ is π-complemented, then A is ε-complemented*;
(2) *If in addition A is $\sharp$-unital, then $M(A)$ is π-complemented if, and only if, A is ε-complemented.*

Proof. (1) If $M(A)$ is π-complemented, then, by 1.13, we can assert that $M(A)$ is semiprime, and hence A is m.s.p. because of 1.7. Now, by Proposition 3.5, A is π-complemented, and hence, by Proposition 3.1, A is ε-complemented.

(2) Assume that A is $\sharp$-unital. If A is ε-complemented, then, by Proposition 3.1, A is π-complemented m.s.p., and hence, by Proposition 3.5, $M(A)$ is π-complemented. The converse was proved in part (1) without the $\sharp$-unital condition. $\square$

Let us now examine the case of ε-complemented algebras with nonzero annihilator. Note that, arguing as in Lemma 2.6, the following result can be proved.

Lemma 3.7. *If A is an algebra with* $\mathrm{Ann}(A) \neq 0$ *and* $\mathrm{Ann}(A)$ *is ε-quasicomplemented in A, then the following assertions are equivalent*:

(i) *A is an ε-complemented algebra*;
(ii) *A^2 is an ε-complemented algebra.*

Proposition 3.8. *Let A be an algebra with* $\mathrm{Ann}(A) \neq 0$. *Then the following assertions are equivalent*:

(i) *A is ε-complemented*;
(ii) *$M^\sharp(A)$ is a π-complemented algebra with a unit element*;
(iii) $A = \mathrm{Ann}(A) \oplus A^2$ *and A^2 is a $\sharp$-unital π-complemented m.s.p. algebra.*

In this case, every ε-closed ideal U of A is an ε-complemented algebra, and $\mathcal{I}_A^\varepsilon = \{V \oplus W\colon\ V \in \mathcal{I}_U^\varepsilon,\ W \in \mathcal{I}_{U'}^\varepsilon\}$, where U' is the ε-complement of U in A.

Proof. (i) $\Rightarrow$ (ii). By 1.14 and Lemma 3.7, $A = \mathrm{Ann}(A) \oplus A^2$ and A^2 is a $\sharp$-unital ε-complemented algebra. Therefore, by Corollary 3.6(2), we can assert that $M(A^2)$ is a π-complemented algebra. Finally, since by Proposition 2.5 $M^\sharp(A) \cong M(A^2)$, we can conclude that $M^\sharp(A)$ is a π-complemented algebra with a unit element.

(ii) $\Rightarrow$ (iii). By Proposition 2.7, A is ε-quasicomplemented, $A = \mathrm{Ann}(A) \oplus A^2$, and A^2 is a $\sharp$-unital m.s.p. algebra. Since, by Proposition 2.5, $M^\sharp(A) \cong M(A^2)$, it follows that $M(A^2)$ is π-complemented. Finally, A^2 is π-complemented because of Proposition 3.5.

(iii) $\Rightarrow$ (i). By Proposition 2.7, A is ε-quasicomplemented. On the other hand, by Proposition 3.1, A^2 is ε-complemented. Finally, by Lemma 3.7, we conclude that A is ε-complemented.

Now, let us assume that A satisfies the equivalent conditions in the statement, take into account the relation between the ε-closed ideals of A and A^2 given in 1.14(1), and fix an ε-closed ideal U of A. When $U \subseteq A^2$, by Proposition 3.1 and Corollary 2.14, we can confirm that U is a $\sharp$-unital ε-complemented algebra and $\mathcal{L}_{A^2} = \{V \oplus W\colon\ V \in \mathcal{L}_U,\ W \in \mathcal{L}_{\mathrm{Ann}_{A^2}(U)}\}$. When $U \nsubseteq A^2$, we see that $U = \mathrm{Ann}(A) \oplus I$ for some $I \in \mathcal{L}_{A^2}$, and so the above argument allows us to assert that I is a $\sharp$-unital ε-complemented algebra and $\mathcal{L}_{A^2} = \{V \oplus W\colon\ V \in \mathcal{L}_I,\ W \in \mathcal{L}_{\mathrm{Ann}_{A^2}(I)}\}$. Keeping in mind Proposition 3.1, it is confirmed that I is a π-complemented m.s.p. algebra. Moreover, since I is $\sharp$-unital, it follows that

$$I = M(I)(I) = M^\sharp(I)(I) = I^2 = \big(\mathrm{Ann}(A) \oplus I\big)^2 = U^2.$$

Now, from implication (iii) ⇒ (i) in the statement, we see that U is an ε-complemented algebra. Moreover, we have $\mathcal{I}_U^\varepsilon = \{K, \operatorname{Ann}(A) \oplus K\colon\ K \in \mathcal{L}_I\}$ (by 1.14(1)). In summary, for any case, it can be asserted that U is an ε-complemented algebra and that a description of $\mathcal{I}_U^\varepsilon$ has been obtained. From the description of $\mathcal{L}_{A^2}$ and $\mathcal{I}_U^\varepsilon$, and taking into account that U' was explicitly determined in Proposition 2.7, it can readily be concluded that

$$\mathcal{I}_A^\varepsilon = \{V \oplus W\colon\ V \in \mathcal{I}_U^\varepsilon,\ W \in \mathcal{I}_{U'}^\varepsilon\}. \qquad \square$$

In a complete analogy with Theorem 2.9, we are going to relate now the complementarity for the different closures without any distinction of cases.

Theorem 3.9. *Let A be an algebra. Then the following assertions are equivalent*:

(i) *A is ε-complemented*;
(ii) *$A = U \oplus [U : A]_{\mathrm{ann}}$, for every ε-closed ideal U of A*;
(iii) *$A = \operatorname{Ann}(A) \oplus \widehat{A^2}$ and $\widehat{A^2}$ is a π-complemented m.s.p. algebra*;
(iv) *A is isomorphic to an algebra of the form $B_0 \oplus B$, where B_0 is a null algebra and B is a π-complemented m.s.p. algebra, with B $\sharp$-unital whenever $B_0 \neq 0$.*

In this case, every ε-closed ideal U of A is an ε-complemented algebra, and $\mathcal{I}_A^\varepsilon = \{V \oplus W\colon\ V \in \mathcal{I}_U^\varepsilon,\ W \in \mathcal{I}_{U'}^\varepsilon\}$, where U' is the ε-complement of U in A.

Proof. (i) ⇒ (ii). Given an ε-closed ideal U of A, by Theorem 2.9, $[U : A]_{\mathrm{ann}}$ is the unique ε-quasicomplement of U in A. Therefore, $A = U \oplus [U : A]_{\mathrm{ann}}$.

(ii) ⇒ (iii). Clearly A is ε-complemented. When $\operatorname{Ann}(A) = 0$, by Proposition 3.1, we see that A is π-complemented m.s.p. In view of 1.8(2) we have $A = \widehat{A^2}$, and it can be concluded that $\widehat{A^2}$ is a π-complemented m.s.p. algebra. When $\operatorname{Ann}(A) \neq 0$, by Proposition 3.8, we see that $A = \operatorname{Ann}(A) \oplus A^2$ and A^2 is a $\sharp$-unital π-complemented m.s.p. algebra. Finally, from 1.14, we have $\widehat{A^2} = A^2$.

(iii) ⇒ (iv). When $\operatorname{Ann}(A) = 0$, this implication is clear. Assume that $\operatorname{Ann}(A) \neq 0$. Then, by 1.14, we have $\widehat{A^2} = A^2$, and A^2 is $\sharp$-unital.

(iv) ⇒ (i). When $B_0 = 0$, this implication follows from Proposition 3.1. When $B_0 \neq 0$, since B is $\sharp$-unital, we see that $B^2 = B$. Therefore, $A = \operatorname{Ann}(A) \oplus A^2$ and A^2 is a $\sharp$-unital π-complemented m.s.p. algebra. Now, by applying Proposition 3.8, we conclude that A is ε-complemented.

Finally, in the case in which A satisfies the equivalences in the statement, from Propositions 3.1 and 3.8 we see that the final part in the statement follows. $\square$

Corollary 3.10. *Let A be an algebra. Then the following assertions are equivalent*:

(i) *A is ε-complemented*;
(ii) *A is ε-quasicomplemented and the ε-closure is additive.*

Proof. (i) ⇒ (ii). Given $U, V \in \mathcal{I}_A^\varepsilon$, keeping in mind the description of $\mathcal{I}_A^\varepsilon$ given in Theorem 3.9, we see, firstly, that $V = V_1 \oplus V_2$ for $V_1 \in \mathcal{I}_U^\varepsilon$ and $V_2 \in \mathcal{I}_{U'}^\varepsilon$, and secondly that $U + V = U \oplus V_2 \in \mathcal{I}_A^\varepsilon$. Thus, the ε-closure is additive.

(ii) ⇒ (i). This implication is obvious. $\square$

In view of item (iv) in Theorem 3.9, in order to provide a structure theory for ε-complemented algebras we need to establish those analogous to Theorems 2.15 and 2.16 for π-complemented m.s.p. algebras. First we discuss the case in which a direct sum is $\sharp$-unital.

Proposition 3.11. *Let $\{A_i\}_{i \in I}$ be a nonempty family of nonzero algebras, and set $A = \bigoplus_{i \in I} A_i$. Then the following assertions are equivalent*:

(i) *A is $\sharp$-unital*;
(ii) *I is finite and A_i is $\sharp$-unital, for all $i \in I$.*

Proof. (i) $\Rightarrow$ (ii). For each $i \in I$, A_i is $\sharp$-unital because of Lemma 2.13. On the other hand, we have $M^\sharp(A) = \bigoplus_{i\in I}[A_i : A]^\sharp$ by Proposition 3.3. Since A is $\sharp$-unital, it follows that $Id_A = E_{i_1} + \cdots + E_{i_n}$ for suitable $E_{i_j} \in [A_{i_j} : A]^\sharp$ and $\{i_1, \dots, i_n\} \subseteq I$. Therefore $A = Id_A(A) = (E_{i_1} + \cdots + E_{i_n})(A) \subseteq A_{i_1} + \cdots + A_{i_n}$, and hence I is finite.

(ii) $\Rightarrow$ (i). Assume that $I = \{1, 2, \dots, n\}$. By Proposition 2.4, for each $i \in I$, there exists $E_i \in [A_i : A]^\sharp$ such that $E_i(x) = x$ for all $x \in A_i$. Set $E = \sum_{i=1}^n E_i \in M^\sharp(A)$. For each $a \in A$, by writing $a = \sum_{i=1}^n a_i$ for $a_i \in A_i$, we see that $E(a) = \sum_{i=1}^n E_i(a_i) = \sum_{i=1}^n a_i = a$. Therefore $E = Id_A$, and so A is $\sharp$-unital. $\square$

Corollary 3.12. *Let A be an algebra. Then the following assertions are equivalent*:

(i) *A is π-complemented m.s.p.*;
(ii) *A is isomorphic to $A_0 \oplus A_1$, where A_0 is a π-radical π-complemented m.s.p. algebra and A_1 is a π-decomposable π-complemented m.s.p. algebra.*

In this case, $A_0 \cong \pi\text{-}\mathrm{Rad}(A)$ and $A_1 \cong \overline{\pi\text{-}\mathrm{Soc}(A)}$. Moreover, A is $\sharp$-unital if, and only if, both A_0 and A_1 are $\sharp$-unital.

Proof. Implication (ii) $\Rightarrow$ (i) is a direct consequence of 1.20 and 1.15(3). Implication (i) $\Rightarrow$ (ii) follows from 1.18 by considering the algebras $A_0 = \pi\text{-}\mathrm{Rad}(A)$ and $A_1 = \overline{\pi\text{-}\mathrm{Soc}(A)}$. Note that, by 1.8(4), both algebras are m.s.p. Finally, note that the equivalence adding $\sharp$-unital to both sentences follows from Proposition 3.11. $\square$

Given a nonempty family of nonzero algebras $\{A_i\}_{i\in I}$, and given a subalgebra A of $\prod_{i\in I} A_i$, A is said to be a *$\sharp$-unital subalgebra* of $\prod_{i\in I} A_i$ if there exists $E \in M^\sharp(A)$ such that $p'_i(E) = Id_{A_i}$ for all $i \in I$.

Corollary 3.13. *Let A be a nonzero algebra. Then the following assertions are equivalent*:

(i) *A is ($\sharp$-unital) π-decomposable π-complemented m.s.p.*;
(ii) *There exists a nonempty family of nonzero m.p. algebras $\{A_i\}_{i\in I}$ such that A can be regarded as a ($\sharp$-unital) subalgebra of $\prod_{i\in I} A_i$ containing $\bigoplus_{i\in I} A_i$ and satisfying $p_J(A) \subseteq A$ for all $J \subseteq I$.*

In this case, $\mathcal{L}_A = \{p_J(A)\colon\ J \subseteq I\}$.

Proof. (i) $\Rightarrow$ (ii). By 1.19, there exists a nonempty family of nonzero prime algebras $\{A_i\}_{i\in I}$ such that A can be regarded as a subalgebra of $\prod_{i\in I} A_i$ containing $\bigoplus_{i\in I} A_i$ and satisfying $p_J(A) \subseteq A$ for all $J \subseteq I$. Moreover, $\mathcal{L}_A = \{p_J(A)\colon\ J \subseteq I\}$. Since, for each $i \in I$, $A_i = p_i(A) \in \mathcal{L}_A$, from 1.8(4) it follows that A_i is m.s.p. Now, taking into account 1.21, we can confirm that all the A_i's are m.p. algebras.

(ii) $\Rightarrow$ (i). This implication follows from 1.19 and 1.20.

Finally, note that the equivalence adding $\sharp$-unital to both sentences follows from Lemma 2.13. $\square$

4. Algebras whose multiplication algebra is ε'-complemented

This section discusses an outstanding subclass of the class of all ε-complemented algebras, namely the class consisting of all algebras whose multiplication algebra is ε'-complemented. One of the peculiarities of the algebras in this subclass is the close relation existing between closed ideals in the algebra and the central idempotents in the multiplication algebra.

The next result is elemental and will be used without further mention.

Lemma 4.1. *Let X be a vector space. If E is an idempotent in $L(X)$, then $Id_X - E$ is an idempotent in $L(X)$, $\mathrm{Ker}(E) = (Id_X - E)(X)$, and $X = E(X) \oplus \mathrm{Ker}(E)$.*

For an algebra A, $\Im_A$ denotes the set of all central idempotents in A. Recall that $\Im_A$ is a lattice for the partial order given by $e \leqslant f$ if $ef = e$, and with meet and join operations given by $e \wedge f = ef$ and $e \vee f = e + f - ef$. Moreover, $\Im_A$ is a boolean algebra whenever A has a unit.

Proposition 4.2. *Let A be an algebra. Then*:

(1) *If E is an idempotent in $M(A)$, then $\operatorname{Ker}(E) = (M(A)E)_{\mathrm{ann}}$ and $\operatorname{Ker}(E)^{\mathrm{ann}} = M(A)E$. In particular, $E(A)$ and $\operatorname{Ker}(E)$ are ε-closed subspaces of A.*
(2) *If $E \in \Im_{M(A)}$, then $E(A)$ and $\operatorname{Ker}(E)$ are ε-complemented ideals of A, and $[E(A) : A] = M(A)E$.*
(3) *The map $E \mapsto E(A)$ is a lattice monomorphism from $\Im_{M(A)}$ into $\mathcal{I}^{\varepsilon}_A$.*

Proof. (1) Let E be an idempotent in $M(A)$. If $x \in \operatorname{Ker}(E)$, then $FE(x) = 0$ for all $F \in M(A)$, and hence $x \in (M(A)E)_{\mathrm{ann}}$. Conversely, if $x \in (M(A)E)_{\mathrm{ann}}$, then $FE(x) = 0$ for all $F \in M(A)$, and in particular $E(x) = Id_A E(x) = 0$, and so $x \in \operatorname{Ker}(E)$. Thus the equality $\operatorname{Ker}(E) = (M(A)E)_{\mathrm{ann}}$ is proved.

If $F \in \operatorname{Ker}(E)^{\mathrm{ann}}$, then $0 = F(\operatorname{Ker}(E)) = F(Id_A - E)(A)$, hence $0 = F(Id_A - E)$, and so $F = FE \in M(A)E$. Conversely, if $F \in M(A)E$, and $F = GE$ for some $G \in M(A)$, then $F(\operatorname{Ker}(E)) = GE(\operatorname{Ker}(E)) = 0$, and so $F \in \operatorname{Ker}(E)^{\mathrm{ann}}$. Thus the equality $\operatorname{Ker}(E)^{\mathrm{ann}} = M(A)E$ is proved.

From the equalities $\operatorname{Ker}(E) = (M(A)E)_{\mathrm{ann}}$ and

$$E(A) = \operatorname{Ker}(Id_A - E) = \big(M(A)(Id_A - E)\big)_{\mathrm{ann}}$$

it follows that $\operatorname{Ker}(E)$ and $E(A)$ are ε-closed subspaces of A.

(2) If E is a central idempotent in $M(A)$, then $M(A)E$ is an ideal of $M(A)$, and hence $\operatorname{Ker}(E) = (M(A)E)_{\mathrm{ann}}$ is an ideal of A. Since $E(A) = \operatorname{Ker}(Id_A - E)$, it follows that $E(A)$ is also an ideal of A. Moreover, the decomposition $A = E(A) \oplus \operatorname{Ker}(E)$ yields that both ideals are ε-complemented. On the other hand, it is clear that

$$\big[E(A) : A\big] = E\big[E(A) : A\big] \subseteq EM(A) \subseteq \big[E(A) : A\big],$$

and hence $[E(A) : A] = EM(A) = M(A)E$.

(3) Note that, for E_1, E_2 central idempotents in $M(A)$, we have the following properties:

(a) $E_1(A) = E_2(A)$ implies $E_1 = E_2$,
(b) $E_1E_2(A) = E_1(A) \cap E_2(A)$,
(c) $(E_1 + E_2 - E_1E_2)(A) = E_1(A) + E_2(A)$,

and consequently the map $E \mapsto E(A)$ is a lattice monomorphism from $\Im_{M(A)}$ into $\mathcal{I}^{\varepsilon}_A$. $\square$

Proposition 4.3. *Let A be an algebra. Consider the following conditions*:

(1) *$M(A)$ is π-complemented*;
(2) *For each ε-closed ideal U of A, there exists $E \in \Im_{M(A)}$ such that $[U : A] = M(A)E$*;
(3) *$M(A)$ is semiprime.*

Then the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) *hold.*

Proof. (1) $\Rightarrow$ (2). Given $U \in \mathcal{I}^{\varepsilon}_A$, by 1.9, we see that

$$[U : A] = \operatorname{Ann}\big(U^{\mathrm{ann}}\big) \in \mathcal{I}^{\pi}_{M(A)},$$

and hence $M(A) = [U : A] \oplus \operatorname{Ann}([U : A])$. Then we can write $Id_A = E + F$ with $E \in [U : A]$ and $F \in \operatorname{Ann}([U : A])$. It is fairly evident that E is the unit element of the algebra $[U : A]$, and consequently E is a central idempotent in $M(A)$ such that $[U : A] = M(A)E$.

(2) $\Rightarrow$ (3). Let U be an ε-closed ideal of A and assume that $[U : A] = M(A)E$ for a suitable $E \in \Im_{M(A)}$. Then, we see that $E \in [U : A]$, and hence $E(A) \subseteq U$. From Proposition 4.2(1), we have

$$\operatorname{Ker}(E) = \big(M(A)E\big)_{\text{ann}} = [U : A]_{\text{ann}} \quad \text{and} \quad A = E(A) \oplus \operatorname{Ker}(E).$$

Therefore, $A \subseteq U + [U : A]_{\text{ann}}$, and as a result $A = U + [U : A]_{\text{ann}}$.

Now, assume that U is an arbitrary ideal of A. Since $[U : A] \subseteq [\widehat{U} : A]$, and hence $[\widehat{U} : A]_{\text{ann}} \subseteq [U : A]_{\text{ann}}$, it follows from the above that $A = \widehat{U} + [\widehat{U} : A]_{\text{ann}} \subseteq (U + [U : A]_{\text{ann}})^{\wedge}$, and as a result $A = (U + [U : A]_{\text{ann}})^{\wedge}$. Finally, by 1.9, we conclude that $M(A)$ is semiprime. $\square$

Proposition 4.4. *Let A be an algebra and U be an ideal of A. If there exists an ideal V of A such that $A = U \oplus V$ and $M(A) = [U : A] \oplus [V : A]$, then the evaluation in the elements of U determines an algebra isomorphism from $[U : A]$ onto $M(U)$.*

Proof. By Proposition 2.4, the evaluation in the elements of U determines an algebra monomorphism φ from $[U : A]$ into $M(U)$, which takes $[U : A]^\sharp$ onto $M^\sharp(U)$. By writing $Id_A = E + F$ for $E \in [U : A]$ and $F \in [V : A]$, we see that $E(x) = Id_A(x) = x = Id_U(x)$ for all $x \in U$, and consequently $\varphi(E) = Id_U$. Therefore $\varphi([U : A]) = M(U)$. Thus, the map φ is an algebra isomorphism from $[U : A]$ onto $M(U)$. $\square$

Lemma 4.5. *Let A be an algebra. If the map $E \mapsto E(A)$ is an order isomorphism from $\Im_{M(A)}$ onto $\mathcal{I}_A^\varepsilon$, then:*

(1) *A is ε-complemented, $M(A)$ is π-complemented, and the lattice isomorphism $U \mapsto [U : A]$ from $\mathcal{I}_A^\varepsilon$ onto $\mathcal{M}_{M(A)}$ takes ε-complements in π-complements;*
(2) *$M(U)$ is a π-complemented algebra, for every ε-closed ideal U of A.*

Proof. (1) Since, by assumption, $\mathcal{I}_A^\varepsilon = \{E(A)\colon\ E \in \Im_{M(A)}\}$, it follows from Proposition 4.2(2) that A is ε-complemented. Therefore, from Theorem 2.9, the map $U \mapsto [U : A]$ is a lattice isomorphism from $\mathcal{I}_A^\varepsilon$ onto $\mathcal{M}_{M(A)}$, and consequently $\mathcal{M}_{M(A)} = \{[E(A) : A]\colon\ E \in \Im_{M(A)}\}$. Moreover, note that, for each $E \in \Im_{M(A)}$, we have the decompositions $A = E(A) \oplus (Id_A - E)(A)$ and $M(A) = M(A)E \oplus M(A)(Id_A - E) = [E(A) : A] \oplus [(Id_A - E)(A) : A]$. From this we conclude that $M(A)$ is π-complemented and the lattice isomorphism $U \mapsto [U : A]$ from $\mathcal{I}_A^\varepsilon$ onto $\mathcal{M}_{M(A)}$ takes ε-complements in π-complements.

(2) Given $U \in \mathcal{I}_A^\varepsilon$, by part (1), there exists $V \in \mathcal{I}_A^\varepsilon$ such that $A = U \oplus V$ and $M(A) = [U : A] \oplus [V : A]$. Therefore, by Proposition 4.4, $M(U) \cong [U : A]$. Finally, since $M(A)$ is π-complemented, by 1.13, we can assert that $[U : A]$ is a π-complemented algebra, and hence so is $M(U)$. $\square$

We are now in a position to state our main result in this section.

Theorem 4.6. *Let A be an algebra. Then the following assertions are equivalent:*

(i) *$M(A)$ is ε'-complemented;*
(ii) *$M(A)$ is π-complemented and $\varepsilon' = \pi$;*
(iii) *$\varepsilon' = \pi$ and for each ε-closed ideal U of A, there exists $E \in \Im_{M(A)}$ such that $[U : A] = M(A)E$;*
(iv) *The map $E \mapsto E(A)$ is an order isomorphism from $\Im_{M(A)}$ onto $\mathcal{I}_A^\varepsilon$;*
(v) *$A = \operatorname{Ann}(A) \oplus \widehat{A^2}$ and $\widehat{A^2}$ is an m.s.p. algebra with $M(\widehat{A^2})$ π-complemented;*
(vi) *A is isomorphic to an algebra of the form $B_0 \oplus B$, where B_0 is a null algebra and B is an m.s.p. algebra such that $M(B)$ is π-complemented, with B $\sharp$-unital whenever $B_0 \neq 0$.*

In this case, A is ε-complemented and the lattice isomorphism $U \mapsto [U : A]$ from $\mathcal{I}_A^\varepsilon$ onto $\mathcal{M}_{M(A)}$ takes ε-complements in π-complements. Moreover, $M(U)$ is an ε'-complemented algebra, for every ε-closed ideal U of A.

Proof. (i) $\Rightarrow$ (ii). By Theorem 2.9, $\varepsilon' = \pi$, and so $M(A)$ is π-complemented.

(ii) $\Rightarrow$ (iii). This implication follows from implication (1) $\Rightarrow$ (2) in Proposition 4.3.

(iii) $\Rightarrow$ (iv). By Proposition 4.2(3), the map $E \mapsto E(A)$ is a lattice monomorphism from $\Im_{M(A)}$ into $\mathcal{I}_A^\varepsilon$. Given an ε-closed ideal U of A, by assumption, there exists $E \in \Im_{M(A)}$ such that $[U : A] = M(A)E$. Therefore $[U : A] = [E(A) : A]$ because of Proposition 4.2(2). On the other hand, by Proposition 4.3, $M(A)$ is semiprime, and hence, in view of Theorem 2.9, we can assert that the map $V \mapsto [V : A]$ is a lattice isomorphism from $\mathcal{I}_A^\varepsilon$ onto $\mathcal{M}_{M(A)}$. Therefore, the equality $[U : A] = [E(A) : A]$ yields that $U = E(A)$. Thus, the map $E \mapsto E(A)$ is a lattice isomorphism from $\Im_{M(A)}$ onto $\mathcal{I}_A^\varepsilon$.

(iv) $\Rightarrow$ (v). By Lemma 4.5(1), A is ε-complemented, and hence, by Theorem 3.9, $A = \mathrm{Ann}(A) \oplus \widehat{A^2}$ and $\widehat{A^2}$ is a π-complemented m.s.p. algebra. Since $\widehat{A^2}$ is an ε-closed ideal of A, again by Lemma 4.5(2), it follows that $M(\widehat{A^2})$ is a π-complemented algebra.

(v) $\Rightarrow$ (vi). If $\mathrm{Ann}(A) \neq 0$, then $\widehat{A^2} = A^2$ is a $\sharp$-unital algebra by 1.14.

(vi) $\Rightarrow$ (i). If $B_0 = 0$, then A is an m.s.p. algebra such that $M(A)$ is π-complemented, and we conclude that $M(A)$ is ε'-complemented because of 1.8(1). Now, assume that $B_0 \neq 0$, and hence B is $\sharp$-unital. From the above case, it follows that $M(B)$ is ε'-complemented. Since $A^2 \cong B^2 = B$, we see that $M(A^2)$ is ε'-complemented. On the other hand, by 1.14, $\mathrm{Ann}(A)$ is ε-quasicomplemented in A, and hence, by Proposition 2.5, we can regard $M(A^2)$ as an ideal of $M(A)$ and

$$\mathcal{I}_{M(A)}^{\varepsilon'} = \left\{\mathcal{P}, \mathbb{K}(Id_A - Id_{A^2}) \oplus \mathcal{P}\colon\ \mathcal{P} \in \mathcal{I}_{M(A^2)}^{\varepsilon'}\right\}.$$

From this it immediately follows that $M(A)$ is also ε'-complemented.

Now, assume that A satisfies the equivalent conditions in the statement. By Lemma 4.5, A is ε-complemented and the lattice isomorphism $U \mapsto [U : A]$ from $\mathcal{I}_A^\varepsilon$ onto $\mathcal{M}_{M(A)}$ takes ε-complements in π-complements. Moreover, for each ε-closed ideal U of A, $M(U)$ is a π-complemented algebra. Since, by Theorem 3.9, U is an ε-complemented algebra, Theorem 2.9 allows us to confirm that $\varepsilon' = \pi$ in $M(U)$. Thus, $M(U)$ is a π-complemented algebra with $\varepsilon' = \pi$, and hence $M(U)$ is an ε'-complemented algebra. $\square$

Remark 4.7. When $\mathrm{Ann}(A) = 0$, the condition $\varepsilon' = \pi$ in clauses (ii) and (iii) in the above theorem can be dropped. Indeed, if A is an algebra with zero annihilator satisfying either of the conditions (1) or (2) in Proposition 4.3, then $M(A)$ is semiprime. Therefore A is m.s.p., and hence $\varepsilon' = \pi$ because of 1.8(1).

However, when $\mathrm{Ann}(A) \neq 0$, the condition $\varepsilon' = \pi$ cannot be dropped: there are algebras with a π-complemented multiplication algebra which are not ε-quasicomplemented, and consequently do not satisfy the condition $\varepsilon' = \pi$ (see Example 2.12).

Let us mention two important consequences of Theorem 4.6.

Corollary 4.8. *For an algebra A, the following assertions are equivalent*:

(i) *$M(A)$ is ε'-complemented*;
(ii) *$M(A)$ is ε'-quasicomplemented and the ε'-closure is additive.*

Proof. Since implication (ii) $\Rightarrow$ (i) is obvious, all that remains is to show that (i) $\Rightarrow$ (ii). By Theorem 4.6, $M(A)$ is π-complemented and $\varepsilon' = \pi$. Since, by 1.13, the π-closure is additive for π-complemented algebras, we deduce that the ε'-closure in $M(A)$ is additive. $\square$

Corollary 4.9. *If A is an ε-complemented algebra with either nonzero annihilator or $\sharp$-unital, then $M(A)$ is ε'-complemented.*

Proof. From Theorem 3.9 we see that A is isomorphic to an algebra of the form $B_0 \oplus B$, where B_0 is a null algebra and B is a $\sharp$-unital π-complemented m.s.p. algebra. Note that, by Proposition 3.1

and Corollary 3.6, $M(B)$ is π-complemented. Now, from Theorem 4.6 it follows that $M(A)$ is ε'-complemented. $\square$

Example 4.10. π-Decomposable ε-complemented algebras may have a multiplication algebra that is not ε'-complemented. Let c_{00} be the algebra of all quasi-null sequences on $\mathbb{K}$ endowed with the coordinatewise algebra operations. By Corollary 3.13, c_{00} is a π-decomposable ε-complemented algebra. Moreover, it is immediately apparent that the algebra c_{qc} can be regarded as the unitization of c_{00}. For each $\{\alpha_n\}$ in c_{qc} consider the mapping $\varphi(\{\alpha_n\})$ from c_{00} into c_{00} given by

$$\varphi\big(\{\alpha_n\}\big)\big(\{a_n\}\big) = \{\alpha_n a_n\}.$$

It is easy to verify that $\varphi(\{\alpha_n\})$ is a linear mapping on c_{00}, and that φ is an algebra monomorphism from c_{qc} into $L(c_{00})$ with rank equal to $M(c_{00})$. Thus, via φ, we may regard c_{qc} as the multiplication algebra of c_{00}. Note that $\varepsilon' = \pi$ because c_{00} is an m.s.p. algebra. However, c_{qc} is not π-complemented, as noted in Example 3.2.

Let us now discuss the structure theory.

Proposition 4.11. *Let A be an algebra and U, V be ideals of A. Assume that $A = U \oplus V$. Then the following assertions are equivalent*:

(i) $M(A) = [U : A] \oplus [V : A]$;
(ii) *The evaluation in the elements of U determines an algebra isomorphism from $[U : A]$ onto $M(U)$, and the evaluation in the elements of V determines an algebra isomorphism from $[V : A]$ onto $M(V)$*;
(iii) *At least one of the algebras U and V is $\sharp$-unital.*

Proof. (i) $\Rightarrow$ (ii). This implication follows directly from Proposition 4.4.

(ii) $\Rightarrow$ (iii). Note that $[U : A] \cap [V : A] = [U \cap V : A] = [0 : A] = 0$. By Proposition 3.3, we have

$$M^\sharp(A) = [U : A]^\sharp \oplus [V : A]^\sharp \subseteq [U : A] \oplus [V : A] \subseteq M(A).$$

Since $M^\sharp(A)$ has a codimension less than or equal to 1 in $M(A)$, at least one of the following equalities $[U : A]^\sharp = [U : A]$ and $[V : A]^\sharp = [V : A]$ holds. Now, keeping in mind Proposition 2.4, we can conclude that at least one of the algebras U and V is $\sharp$-unital.

(iii) $\Rightarrow$ (i). Assume that U is a $\sharp$-unital algebra. By Proposition 2.4, $[U : A]^\sharp$ has a unit element, say E, and $E(x) = x$ for every $x \in U$. Note that $E(V) \subseteq U \cap V$, and hence $E(V) = 0$. Given $a \in A$, by writing $a = x + y$ for $x \in U$ and $y \in V$, we see that $(Id_A - E)(a) = a - E(x) = y$. Therefore $Id_A - E \in [V : A]$. Thus $[U : A] + [V : A]$ is an ideal of $M(A)$ containing Id_A, and as a result $M(A) = [U : A] + [V : A]$. Finally, since $U \cap V = 0$ yields that $[U : A] \cap [V : A] = 0$, we conclude that $M(A) = [U : A] \oplus [V : A]$. $\square$

We can now state the analogue of Theorems 2.15 and 2.16 for m.s.p. algebras with a π-complemented multiplication algebra.

Corollary 4.12. *Let A be an algebra. Then the following assertions are equivalent*:

(i) *A is m.s.p. with $M(A)$ π-complemented*;
(ii) *A is isomorphic to $A_0 \oplus A_1$, where A_0 is π-radical m.s.p. with $M(A_0)$ π-complemented, A_1 is π-decomposable m.s.p. with $M(A_1)$ π-complemented, and at least one of the algebras A_0 and A_1 is $\sharp$-unital.*

In this case, $A_0 \cong \pi\text{-}\mathrm{Rad}(A)$ and $A_1 \cong \overline{\pi\text{-}\mathrm{Soc}(A)}$. Moreover, A is $\sharp$-unital if, and only if, both A_0 and A_1 are $\sharp$-unital.

Proof. (i) $\Rightarrow$ (ii). By Proposition 3.5, A is π-complemented. Now, by Corollary 3.12, $A = A_0 \oplus A_1$, where A_0 is a π-radical m.s.p. algebra and A_1 is a π-decomposable m.s.p. algebra. Since A_0 and A_1 are closed ideals of A, from Theorem 4.6 it follows that $M(A_0)$ and $M(A_1)$ are π-complemented algebras, and $M(A) = [A_0 : A] \oplus [A_1 : A]$. Finally, by Proposition 4.11, we conclude that at least one of the algebras A_0 and A_1 is $\sharp$-unital.

(ii) $\Rightarrow$ (i). By Proposition 3.5, A_0 and A_1 are π-complemented. Now, by Corollary 3.12, A is an m.s.p. algebra. From Proposition 4.11, $M(A) = [A_0 : A] \oplus [A_1 : A]$, $M(A_0) \cong [A_0 : A]$, and $M(A_1) \cong [A_1 : A]$. Thus, $M(A)$ is the direct sum of two π-complemented algebras, and we conclude that $M(A)$ is π-complemented because of 1.15(3).

The two clauses in the final part of the statement were proved in Corollary 3.12. $\square$

Corollary 4.13. *Let A be a nonzero algebra. Then the following assertions are equivalent*:

(i) *A is ($\sharp$-unital) π-decomposable m.s.p. with $M(A)$ π-complemented*;
(ii) *There exists a nonempty family of nonzero m.p. algebras $\{A_i\}_{i\in I}$ such that A can be regarded as a ($\sharp$-unital) subalgebra of $\prod_{i\in I} A_i$ containing $\bigoplus_{i\in I} A_i$ and satisfying that, for each $J \subseteq I$, there exists $E_J \in \Im_{M(A)}$ such that $p_J(A) = E_J(A)$.*

In this case, $\mathcal{L}_A = \{p_J(A)\colon\ J \subseteq I\}$.

Proof. (i) $\Rightarrow$ (ii). By Proposition 3.5, A is π-complemented. Now, by Corollary 3.13, there exists a nonempty family of ($\sharp$-unital) nonzero m.p. algebras $\{A_i\}_{i\in I}$ such that A can be regarded as a ($\sharp$-unital) subalgebra of $\prod_{i\in I} A_i$ containing $\bigoplus_{i\in I} A_i$, and $\mathcal{L}_A = \{p_J(A)\colon\ J \subseteq I\}$. Since $\varepsilon' = \pi$ (by 1.8(1)) and $M(A)$ is π-complemented (by assumption), from Theorem 4.6 it follows that, for each $J \subseteq I$, there exists $E_J \in \Im_{M(A)}$ such that $p_J(A) = E_J(A)$.

(ii) $\Rightarrow$ (i). Since, for each $J \subseteq I$, there exists $E_J \in \Im_{M(A)}$ such that $p_J(A) = E_J(A)$, it follows that $p_J(A) \subseteq A$. Now, by Corollary 3.13, A is ($\sharp$-unital) π-decomposable m.s.p. and $\mathcal{L}_A = \{p_J(A)\colon\ J \subseteq I\}$. From this equality, and keeping in mind Proposition 4.2(3), we can assert that the map $E \mapsto E(A)$ is a lattice isomorphism from $\Im_{M(A)}$ into $\mathcal{L}_A$. Finally, from Theorem 4.6 it follows that $M(A)$ is π-complemented. $\square$

5. Complemented algebras. Finite dimensional algebras

In this last section, complementarity with respect to the above discussed closures is related to classical complementarity, with special attention being paid to the finite dimensional case.

Let us begin with some consequences of Lemma 2.1.

Lemma 5.1. *If A is an algebra and U is a complemented ideal of A such that $U \cap \mathrm{Ann}(A) = 0$, then $[U : A]_{\mathrm{ann}} = \mathrm{Ann}(U)$ is the unique complement of U in A.*

Proof. Let V be a complement of U in A. By Lemma 2.1, we have

$$V \subseteq [U : A]_{\mathrm{ann}} \subseteq \mathrm{Ann}(U) \quad \text{and} \quad U \cap \mathrm{Ann}(U) = 0.$$

Therefore $A = U \oplus V = U \oplus [U : A]_{\mathrm{ann}} = U \oplus \mathrm{Ann}(U)$, and as a consequence $V = [U : A]_{\mathrm{ann}} = \mathrm{Ann}(U)$. $\square$

Proposition 5.2. *If A is a complemented algebra, then $M(A)$ is semiprime, $\mathcal{I}_A^{\pi} = \{U \in \mathcal{I}_A\colon\ \mathrm{Ann}(A) \subseteq U\}$, and every ideal of A is a complemented algebra.*

Proof. By Lemma 2.1(2), we have $A = U + [U : A]_{\mathrm{ann}}$, for every ideal U of A. Thus it can be asserted that $M(A)$ is semiprime, from 1.9.

It is clear that every π-closed ideal of A contains $\mathrm{Ann}(A)$. On the other hand, if U is an ideal of A containing $\mathrm{Ann}(A)$, then, by applying the above lemma to a complement V of U in A, we see that $U = \mathrm{Ann}(V)$, and so U is π-closed. Thus, $\mathcal{I}_A^\pi = \{U \in \mathcal{I}_A\colon \mathrm{Ann}(A) \subseteq U\}$.

Finally, if U is an ideal of A and I is an ideal of U, then clearly I is an ideal of A, and hence $A = I \oplus V$ for a suitable ideal V of A. From this it is immediate that $U = I \oplus (V \cap U)$. Thus U is a complemented algebra. $\square$

Proposition 5.3. *For every nonzero algebra A the following assertions are equivalent*:

(i) *A is complemented with zero annihilator*;
(ii) *$A = U \oplus \mathrm{Ann}(U)$, for every ideal U of A*;
(iii) *A is π-complemented and $\mathcal{I}_A^\pi = \mathcal{I}_A$*;
(iv) *A is m.s.p. and $\mathcal{L}_A = \mathcal{I}_A$*;
(v) *A is decomposable.*

In this case, A is π-decomposable ε-complemented.

Proof. (i) $\Rightarrow$ (ii). For each ideal U of A, by Lemma 5.1, $\mathrm{Ann}(U)$ is a complement of U in A.

(ii) $\Rightarrow$ (iii). Clearly $\mathrm{Ann}(A) = 0$ and A is complemented. Now, by Proposition 5.2, $\mathcal{I}_A^\pi = \mathcal{I}_A$, and hence A is π-complemented.

(iii) $\Rightarrow$ (iv). By 1.13, A is semiprime. Moreover, from 1.6(1)(i) we see that $\mathcal{I}_A^\pi = \mathcal{I}_A^\varepsilon = \mathcal{I}_A$. Finally, by 1.7, we conclude that A is m.s.p. and $\mathcal{L}_A = \mathcal{I}_A$.

(iv) $\Rightarrow$ (v). By 1.8(5), ε-$\mathrm{Rad}(A)$ agrees with the Baer radical of A, and hence ε-$\mathrm{Rad}(A) = 0$. Now, by 1.22, it can be concluded that A is decomposable.

(v) $\Rightarrow$ (i). If $\{B_i\}_{i\in I}$ is a family of simple algebras, then it is clear that $\mathcal{I}_{\bigoplus_{i\in I} B_i} = \{\bigoplus_{i\in J} B_i\colon\ J \subseteq I\}$, and hence $\bigoplus_{i\in I} B_i$ is a complemented algebra with zero annihilator.

When A satisfies the above equivalent conditions, A is π-decomposable ε-complemented because A is complemented and decomposable and ε and π agree with the discrete closure on $\mathcal{I}_A$. $\square$

Corollary 5.4. *For every non-null algebra A the following assertions are equivalent*:

(i) *A is complemented*;
(ii) *A is isomorphic to an algebra of the form $B_0 \oplus B$, where B_0 is a null algebra and B is a decomposable algebra.*

Examples 5.5.

(1) Unital π-decomposable ε-complemented algebras may not be complemented. An example of such an algebra is the algebra ℓ_∞ of all bounded sequences on $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ endowed with the coordinatewise algebra operations. By Corollary 3.13, ℓ_∞ is a unital π-decomposable ε-complemented algebra. Note that π-$\mathrm{Soc}(\ell_\infty) = c_{00}$. Since $\mathrm{Ann}_{\ell_\infty}(c_{00}) = 0$, it follows that $c_{00} \notin \mathcal{I}_{\ell_\infty}^\pi$. Therefore, from the above proposition it follows that ℓ_∞ is not complemented.
(2) Complemented algebras, even with zero annihilator, may have a multiplication algebra that is not ε'-complemented. Note that the algebra c_{00} is decomposable, and hence complemented, but its multiplication algebra is not ε'-complemented (see Example 4.10).
(3) Complemented algebras may not be ε-quasicomplemented. Consider the algebra A consisting of the vector space $\mathbb{K} \times c_{00}$ and the product defined by

$$\big(\lambda, \{a_n\}\big)\big(\mu, \{b_n\}\big) = \big(0, \{a_n b_n\}\big).$$

It is clear that A is a complemented associative commutative algebra with $\mathrm{Ann}(A) = \mathbb{K} \times \{0\}$. For each $\{\alpha_n\}$ in c_{00} consider the mapping $\varphi(\{\alpha_n\})$ from A into A given by

$$\varphi(\{\alpha_n\})(\lambda, \{a_n\}) = (0, \{\alpha_n a_n\}).$$

It is easy to verify that $\varphi(\{\alpha_n\})$ is a linear mapping on A, and that φ is an algebra monomorphism from c_{00} into $L(A)$ with rank equal to $M^\sharp(A)$. Thus, via φ, we may regard c_{00} as the multiplication ideal of A. Since c_{00} does not have a unit element, by 1.14, $\mathrm{Ann}(A)$ is not ε-quasicomplemented in A.

(4) Algebras with a prime multiplication algebra may not be complemented. Algebra A in Example 2.12 has a prime multiplication algebra and satisfies $\mathrm{Ann}(A) \cap A^2 \neq 0$. This fact prevents A from being a complemented algebra. Indeed, since $B^2 = B$ for every decomposable algebra B, it follows that A cannot be isomorphic to an algebra of the form $B_0 \oplus B$, where B_0 is null and B is decomposable. Therefore, by Corollary 5.4, A is not complemented.

Examples (1) and (3) above show that there is no direct relationship between complementarity and ε-quasicomplementarity.

Lemma 5.6. *Let A, B be algebras. Assume that $\mathrm{Ann}(B) = 0$. If U is an ideal of $A \oplus B$ such that $U \cap B$ is a complemented ideal in B, then $U = (U \cap A) \oplus (U \cap B)$.*

Proof. Let U be an ideal of $A \oplus B$ such that $U \cap B$ is a complemented ideal in B, and I be a complement of $U \cap B$ in B. We claim that

$$\{b \in B\colon\ bB + Bb \subseteq U\} \subseteq U.$$

If $b \in B$ satisfies $bB + Bb \subseteq U$, then by writing $b = x + y$ for $x \in U \cap B$ and $y \in I$ we see that

$$yB = (b - x)B \subseteq (U \cap B) \cap I = 0 \quad \text{and} \quad By = B(b - x) \subseteq (U \cap B) \cap I = 0.$$

Therefore $y \in \mathrm{Ann}(B)$, and hence $y = 0$. Thus $b = x \in U$, and the claim is proved.

Given $u \in U$, by writing $u = a + b$ for $a \in A$ and $b \in B$ we see that

$$bB = (u - a)B = uB \subseteq U \quad \text{and} \quad Bb = B(u - a) = Bu \subseteq U.$$

Therefore $b \in U \cap B$, hence $a = u - b \in U \cap A$, and so $u = a + b \in (U \cap A) \oplus (U \cap B)$. □

Proposition 5.7. *For every non-null algebra A the following assertions are equivalent*:

(i) *A is complemented and ε-quasicomplemented*;
(ii) *A is complemented and $\varepsilon' = \pi$*;
(iii) *A is isomorphic to an algebra of the form $B_0 \oplus B$, where B_0 is a null algebra and B is a decomposable algebra, with B $\sharp$-unital whenever $B_0 \neq 0$.*

In this case, A is ε-complemented.

Proof. (i) ⇒ (ii). This implication follows from Theorem 2.9.

(ii) ⇒ (iii). By Proposition 5.2, we see that $M(A)$ is semiprime. Now, by Theorem 2.9, A is isomorphic to an algebra of the form $B_0 \oplus B$, where B_0 is a null algebra and B is an m.s.p. algebra, with B $\sharp$-unital whenever $B_0 \neq 0$. Since B is isomorphic to an ideal of A, from Proposition 5.2 it follows that B is a complemented algebra. Finally, by Proposition 5.3, we conclude that B is decomposable.

(iii) ⇒ (i). By Proposition 5.3, B is a complemented m.s.p. algebra. Therefore, by Lemma 5.6, $B_0 \oplus B$ is a complemented algebra, and so is A. Moreover, A is ε-quasicomplemented because of Theorem 2.9.

Assume that A satisfies the equivalent assertions in the statement, and write $A = B_0 \oplus B$, where B_0 is a null algebra and B is a decomposable algebra, with B $\sharp$-unital whenever $B_0 \neq 0$. Then, by Proposition 5.3, B is a π-complemented m.s.p. algebra, and finally, by Theorem 3.9, A is ε-complemented. □

Corollary 5.8. *For every non-null algebra A with* $\mathrm{Ann}(A) \neq 0$, *the following assertions are equivalent*:

(i) *A is complemented and ε-quasicomplemented*;
(ii) *A is isomorphic to an algebra of the form $\bigoplus_{i=0}^{n} B_i$, where B_0 is a null algebra and B_i $(1 \leqslant i \leqslant n)$ is a $\sharp$-unital simple algebra.*

Proof. (i) $\Rightarrow$ (ii). By Proposition 5.7, A is isomorphic to an algebra of the form $B_0 \oplus B$, where B_0 is a nonzero null algebra and B is a $\sharp$-unital decomposable algebra. Now, by Proposition 3.11, $B = \bigoplus_{i=1}^{n} B_i$, where B_i $(1 \leqslant i \leqslant n)$ is a $\sharp$-unital simple algebra.

(ii) $\Rightarrow$ (i). By Proposition 3.11, $B := \bigoplus_{i=1}^{n} B_i$ is $\sharp$-unital. Now, by Proposition 5.7, A is complemented and ε-quasicomplemented. □

The two conditions in the clause (ii) in Proposition 5.7 are independent of each other.

Examples 5.9.

(1) Complemented algebras may not satisfy the $\varepsilon' = \pi$ condition. Algebra A in Example 5.5(3) is complemented, but not ε-quasicomplemented. Therefore, by Proposition 5.7, we can confirm that $\varepsilon' \neq \pi$.
(2) The $\varepsilon' = \pi$ condition, even in a finite dimensional context, does not imply complementarity. Note that, for the algebra A given in Example 2.11, $\mathrm{Ann}(A)$ does not have a complement in A, and so A is not complemented.

Let us now consider complementarity for the multiplication algebra. It is well known that a non-null algebra A is simple whenever $M(A)$ is simple; however, it may occur that A is a simple algebra and $M(A)$ is not simple [14, Theorem 2.5].

Lemma 5.10. *If A is a nonzero algebra with zero annihilator such that $M(A)$ is a complemented algebra, then A is isomorphic to an algebra of the form $\bigoplus_{i=1}^{n} B_i$, where each B_i is a non-null algebra with $M(B_i)$ being a simple algebra.*

Proof. We begin by showing that A is a complemented algebra. By Proposition 5.3 applied to $M(A)$, we can assert that $M(A)$ is semiprime and $\mathcal{I}^{\pi}_{M(A)} = \mathcal{I}_{M(A)}$. Note that, by 1.7, A is m.s.p., and hence, by 1.8(1), $\varepsilon' = \pi$. From Theorem 2.9, we can confirm that $\mathcal{I}_{M(A)} = \{[U : A]\colon\ U \in \mathcal{L}_A\}$. Thus, given $U \in \mathcal{I}_A$, we can write $M(A) = [U : A] \oplus [V : A]$ for a suitable ideal V of A. Therefore, $A = M(A)(A) = ([U : A] \oplus [V : A])(A)$, and we deduce that $A = U + V$. But, $L_{U\cap V}, R_{U\cap V} \subseteq [U \cap V : A] = [U : A] \cap [V : A] = 0$, and hence $U \cap V \subseteq \mathrm{Ann}(A) = 0$. Therefore, $A = U \oplus V$. Thus, A is a complemented algebra.

Now, by Proposition 5.3, we see that A and $M(A)$ are decomposable. Therefore we can regard $A = \bigoplus_{i\in I} B_i$, where B_i $(i \in I)$ is a simple algebra, and, taking into account that $M(A)$ is unital, we can also regard $M(A) = \bigoplus_{k=1}^{n} \mathcal{M}_k$, where $\mathcal{M}_k$ $(1 \leqslant k \leqslant n)$ is a unital simple algebra. For each nonempty subset J of $\{1,\dots,n\}$, set $\mathcal{M}_J = \bigoplus_{k\in J} \mathcal{M}_k$. Note that, for each i, $\mathcal{M}_J(B_i)$ is an ideal contained in B_i, and consequently $\mathcal{M}_J(B_i) = 0$ or B_i. Fix j with $1 \leqslant j \leqslant n$. Since $\mathcal{M}_j(A) \neq 0$, there exists $i_j \in I$ such that $\mathcal{M}_j(B_{i_j}) \neq 0$, and hence $\mathcal{M}_j(B_{i_j}) = B_{i_j}$. Let us show that $\mathcal{M}_j(B_i) = 0$ for all $i \neq i_j$ and $\mathcal{M}_j = [B_{i_j} : A]$. Since $[B_{i_j} : A]$ is a nonzero ideal of $M(A)$, there exists a nonempty subset J of $\{1,\dots,n\}$ such that $[B_{i_j} : A] = \mathcal{M}_J$, and hence $\mathcal{M}_J(A) = B_{i_j}$. From the chain of equalities

$$B_{i_j} = \mathcal{M}_j(B_{i_j}) = \mathcal{M}_j\big(\mathcal{M}_J(A)\big) = (\mathcal{M}_j\mathcal{M}_J)(A),$$

it follows that $j \in J$. Therefore $\mathcal{M}_j\mathcal{M}_J = \mathcal{M}_j$, and hence $\mathcal{M}_j(A) = B_{i_j}$. Thus, for each $i \neq i_j$, we have $\mathcal{M}_j(B_i) \subseteq B_i \cap B_{i_j} = 0$, and so $\mathcal{M}_j(B_i) = 0$. Finally, note that

$$B_{i_j} = \mathcal{M}_J(A) = \left(\bigoplus_{k \in J} \mathcal{M}_k\right)\left(\bigoplus_{i \in I} B_i\right) = \bigoplus_{k \in J} B_{i_k},$$

and hence $J = \{j\}$. Thus $[B_{i_j} : A] = \mathcal{M}_j$.

Now, since $M(A) = \bigoplus_{j=1}^n \mathcal{M}_j$, we see that $M(A) = \bigoplus_{j=1}^n [B_{i_j} : A]$. Therefore we have $A = M(A)(A) = (\bigoplus_{j=1}^n [B_{i_j} : A])(A) = \bigoplus_{j=1}^n B_{i_j}$. Finally, by Proposition 4.11, for each j, the evaluation in the elements of B_{i_j} determines an algebra isomorphism from $[B_{i_j} : A]$ onto $M(B_{i_j})$, hence $M(B_{i_j}) \cong \mathcal{M}_j$, and so $M(B_{i_j})$ is a simple algebra. $\square$

Proposition 5.11. *For every algebra A the following assertions are equivalent*:

(i) *$M(A)$ is complemented*;
(ii) *$M(A)$ is semiprime, $\varepsilon' = \pi$, and $\mathcal{M}_{M(A)} = \mathcal{I}_{M(A)}$*;
(iii) *A is isomorphic to an algebra of the form $\bigoplus_{i=0}^n B_i$, where B_0 is a null algebra and B_i $(1 \leqslant i \leqslant n)$ is a non-null algebra with $M(B_i)$ being a simple algebra.*

In this case, A is a complemented algebra, each B_i $(1 \leqslant i \leqslant n)$ is a simple algebra, $M^\sharp(A) \cong \bigoplus_{i=1}^n M(B_i)$, $M(A) \cong \bigoplus_{i=0}^n M(B_i)$, and $M(A)$ is an ε'-complemented algebra. Moreover, $M(U)$ is a complemented algebra, for every ideal U of A.

Proof. (i) ⇒ (ii). Assume that A is nonzero, and therefore $M(A)$ is non-null. By Proposition 5.3 applied to $M(A)$, we see that $M(A)$ is semiprime and $\mathcal{I}^\pi_{M(A)} = \mathcal{I}_{M(A)}$. From this, taking into account 1.6(2)(i), we can conclude that $\varepsilon' = \pi$.

(ii) ⇒ (iii). By 1.12 applied to $M(A)$, we see that $M(A)$ is a complemented algebra. On the other hand, by Theorem 2.9, A is isomorphic to an algebra of the form $B_0 \oplus B$, where B_0 is a null algebra and B is an m.s.p. algebra, with B $\sharp$-unital whenever $B_0 \neq 0$. When $B = 0$, the proof is complete. Assume that $B \neq 0$. We claim that $M(B)$ is a complemented algebra. When $B_0 = 0$, we have $A \cong B$, and hence $M(B)$ is complemented. Assume that $B_0 \neq 0$. By 1.14, $\mathrm{Ann}(A)$ is ε-quasicomplemented in A. Note that $A^2 \cong B$ because B is $\sharp$-unital. Now, by Proposition 2.5, $M(B)$ can be regarded as an ideal of $M(A)$ and we can confirm that $M(B)$ is a complemented algebra because of Proposition 5.2. Finally, by Lemma 5.10, $B = \bigoplus_{i=1}^n B_i$, where B_i $(1 \leqslant i \leqslant n)$ is a non-null algebra with $M(B_i)$ being a simple algebra.

(iii) ⇒ (i). If A is a null algebra, then $M(A) = \mathbb{K} Id_A$ is a complemented algebra. Assume that A is non-null and consider $A = \bigoplus_{i=0}^n B_i$, where B_0 is a null algebra and B_i $(1 \leqslant i \leqslant n)$ is a non-null algebra with $M(B_i)$ being a simple algebra. For each i with $1 \leqslant i \leqslant n$, the fact that B_i is non-null yields that $M^\sharp(B_i)$ is a nonzero ideal of $M(B_i)$, and hence the fact that $M(B_i)$ is simple allows us to conclude that B_i is $\sharp$-unital. Therefore, by Proposition 3.11, the algebra $B := \bigoplus_{i=1}^n B_i$ is $\sharp$-unital. Now, by Proposition 3.3, we obtain that $M(B) \cong \bigoplus_{i=1}^n M(B_i)$, and hence $M(B)$ is decomposable. Thus, $M(A)$ is decomposable when $B_0 = 0$. Otherwise, by 1.14, $B_0 = \mathrm{Ann}(A)$ is ε-quasicomplemented in A, and, by Proposition 2.5, we can regard $M(B)$ as an ideal of $M(A)$ and we have $M(A) = \mathbb{K}(Id_A - Id_B) \oplus M(B)$. Note that $\mathbb{K}(Id_A - Id_B) \cong \mathbb{K} Id_{B_0} = M(B_0)$. Summarizing, we have proved that, in any case, $M(A)$ is a decomposable algebra, and hence $M(A)$ is a complemented algebra, from Proposition 5.3.

Now, assume that A satisfies the equivalent assertions in the statement. If A is null, then it is clear that A is complemented, $M^\sharp(A) = 0$, $M(A) = \mathbb{K} Id_A$ is an ε'-complemented algebra, and, for each ideal U of A, $M(U) = \mathbb{K} Id_U$ is a complemented algebra. Assume that A is non-null. Then, A is isomorphic to an algebra of the form $B_0 \oplus B$, where B_0 is a null algebra, $B = \bigoplus_{i=1}^n B_i$, and B_i $(1 \leqslant i \leqslant n)$ is a non-null algebra with $M(B_i)$ being a simple algebra. Therefore, by [14, Theorem 2.5], each B_i $(1 \leqslant i \leqslant n)$ is a simple algebra, and hence B is decomposable. Now, by Proposition 5.7, we conclude that A is complemented.

Note that the algebra isomorphisms $M^\sharp(A) \cong \bigoplus_{i=1}^n M(B_i)$ and $M(A) \cong \bigoplus_{i=0}^n M(B_i)$ are proved above in the proof of implication (iii) ⇒ (i).

On the other hand, since $M(A)$ is complemented and $\mathcal{M}_{M(A)} = \mathcal{I}_{M(A)}$, it is fairly evident that $M(A)$ is ε'-complemented.

Finally, fix $U \in \mathcal{I}_A$, and write $U = S \oplus \bigoplus_{i\in J} B_i$ for a suitable S subspace of B_0 and J subset of $\{1, 2, \ldots, n\}$. Note that the algebra U fulfills condition (iii) in the statement, and as a result $M(U)$ is a complemented algebra. □

Examples 5.12.

(1) Complemented algebras may not have a complemented multiplication algebra. An example of such an algebra is the algebra c_{00} of all quasi-null sequences (see Example 5.5(2)).
(2) There exist unital π-decomposable algebras whose multiplication algebra is ε'-complemented, but which are not complemented. (See Example 5.5(1), and take into account Corollary 4.9.)

From the Wedderburn Theorem, we obtain the Jacobson Theorem, characterizing finite dimensional algebras whose multiplication algebra is semiprime [11] (see also [10] and [3, Proposition 4.8]).

Theorem 5.13 *(Jacobson's Theorem). For an algebra A, the following assertions are equivalent*:

(i) *A is finite dimensional and $M(A)$ is semiprime*;
(ii) *$A = \bigoplus_{i=0}^n B_i$ is a direct sum of ideals, one of them, say B_0, is a finite dimensional null algebra and the others are finite dimensional simple algebras.*

In this case, $M(A) \cong \bigoplus_{i=0}^n M(B_i)$ and $M^\sharp(A) \cong \bigoplus_{i=1}^n M(B_i)$, with $M(B_i)$ being isomorphic to a full matrix algebra over a finite dimensional division algebra Δ_i, for each $1 \leqslant i \leqslant n$.

Proof. Clearly the statement is true for null algebras. Assume that A is non-null.

(i) ⇒ (ii). By the Wedderburn Theorem, $M(A)$ is a decomposable algebra. Therefore, by Proposition 5.3, $M(A)$ is a complemented algebra. Now, assertion (ii) follows from Proposition 5.11.

(ii) ⇒ (i). Assume that $A = \bigoplus_{i=0}^n B_i$, where B_0 is a finite dimensional null algebra and B_i $(1 \leqslant i \leqslant n)$ is a finite dimensional simple algebra. Clearly A is finite dimensional. Moreover, for each i, the fact that B_i is simple implies that $\mathcal{I}_{B_i}^{\varepsilon} = \{0, B_i\}$, therefore B_i is an m.p. algebra because of 1.21, and hence $M^\sharp(B_i)$ is a prime algebra. Note that $M^\sharp(B_0) = 0$. Since, by Proposition 3.3, $M^\sharp(A) \cong \bigoplus_{i=1}^n M^\sharp(B_i)$, it follows from 1.15(2) that $M^\sharp(A)$ is semiprime. Finally, by 1.9, we conclude that $M(A)$ is semiprime.

Now, assume that A satisfies the equivalent assertions in the statement. For each i with $1 \leqslant i \leqslant n$, from Jacobson's density theorem it follows that $M(B_i)$ is isomorphic to a full matrix algebra over a finite dimensional division algebra Δ_i. Finally, the isomorphisms $M(A) \cong \bigoplus_{i=0}^n M(B_i)$ and $M^\sharp(A) \cong \bigoplus_{i=1}^n M(B_i)$ follow from Proposition 5.11. □

For finite dimensional algebras, the different types of (quasi)complementarity agree with the complementarity.

Theorem 5.14. *For a finite dimensional algebra A the following are equivalent*:

(i) *A is complemented*;
(ii) *A is ε-complemented*;
(iii) *A is ε-quasicomplemented*;
(iv) *$M(A)$ is semiprime*;
(v) *$M(A)$ is π-complemented*;
(vi) *$M(A)$ is ε'-complemented*;
(vii) *$M(A)$ is complemented.*

Proof. We begin by noting that (iv) ⇒ (i) and (vii) follows from Jacobson's Theorem 5.13.

(i) ⇒ (ii). By Proposition 5.2, $M(A)$ is semiprime. Therefore, by Jacobson's Theorem 5.13, A is isomorphic to an algebra of the form $\bigoplus_{i=0}^{n} B_i$, where B_0 is a null algebra and B_i $(1 \leqslant i \leqslant n)$ is a non-null algebra with $M(B_i)$ being a simple algebra. Now, by Proposition 5.11, we see that $\varepsilon' = \pi$, and finally, by Proposition 5.7, we conclude that A is ε-complemented.

(ii) ⇒ (iii). This implication is obvious.

(iii) ⇒ (iv). This implication follows from Theorem 2.9.

(vii) ⇒ (vi). This implication is a particular case of Proposition 5.11.

(vi) ⇒ (v). This implication is a consequence of Theorem 4.6.

(v) ⇒ (iv). This implication follows from 1.13. □

Let us now discuss the complementarity of finite dimensional ideals in m.s.p. algebras. We begin with the following lemma.

Lemma 5.15. *Let A be an m.s.p. algebra. Then every finite dimensional ideal of A is closed in A.*

Proof. Assume that $A \neq 0$ and U is a nonzero finite dimensional ideal of A. Consider the linear map $\Phi : M(\widehat{U}) \to L(U)$ given by

$$\Phi(F)(x) = F(x), \quad \text{for all } F \in M(\widehat{U}),\ x \in U.$$

Given $F \in M(\widehat{U})$, by the extension property for multiplication operators, we can choose $T \in M(A)$ satisfying $T(x) = F(x)$ for every $x \in \widehat{U}$. Note that the condition $\Phi(F) = 0$ implies $T \in U^{\mathrm{ann}} = \widehat{U}^{\mathrm{ann}}$, and hence $F = 0$. Thus Φ is injective, and consequently $M(\widehat{U})$ is finite dimensional. Now, consider the linear map $\varphi : \widehat{U} \to M(\widehat{U}) \times M(\widehat{U})$ given by

$$\varphi(x) = \left(L_x^{\widehat{U}}, R_x^{\widehat{U}}\right), \quad \text{for all } x \in \widehat{U}.$$

Note that $\mathrm{Ker}(\varphi) = \mathrm{Ann}_{\widehat{U}}(\widehat{U})$. Since, by 1.8(4), $\widehat{U}$ is also an m.s.p. algebra, it follows that $\mathrm{Ker}(\varphi) = 0$. Hence φ is also injective, and consequently $\widehat{U}$ is finite dimensional. Now, by Theorem 5.14, we can assert that $\widehat{U}$ is a complemented algebra. Therefore, by Proposition 5.3, we can confirm that $\mathcal{L}_{\widehat{U}} = \mathcal{I}_{\widehat{U}}$, and hence $U \in \mathcal{L}_{\widehat{U}}$. Keeping in mind that, by 1.8(4), $\mathcal{L}_{\widehat{U}} \subseteq \mathcal{L}_A$, we conclude that $U \in \mathcal{L}_A$. □

Theorem 5.16. *Let A be an m.s.p. algebra. Then every finite dimensional ideal of A is a complemented ideal in A.*

Proof. Assume that $A \neq 0$ and U is a nonzero finite dimensional ideal of A. Since $\mathrm{Ann}(U)$ is closed in A, by 1.8(4), we see that $A/\mathrm{Ann}(U)$ is an m.s.p. algebra. Let $q : A \to A/\mathrm{Ann}(U)$ denote the quotient map. Since, by 1.7, $U \oplus \mathrm{Ann}(U)$ is a dense ideal of A, it follows from 1.5(2) that $q(U) = q(U \oplus \mathrm{Ann}(U))$ is a dense ideal of $A/\mathrm{Ann}(U)$. On the other hand, since $q(U)$ is finite dimensional, by Lemma 5.15, we can assert that $q(U)$ is a closed ideal of $A/\mathrm{Ann}(U)$. Therefore $q(U) = A/\mathrm{Ann}(U)$, and consequently $A = U \oplus \mathrm{Ann}(U)$. □

Corollary 5.17. *Every finite dimensional ideal of an ε-quasicomplemented algebra is a complemented ideal.*

Proof. Let A be an ε-quasicomplemented algebra. By Theorem 2.9, $A = \mathrm{Ann}(A) \oplus \widehat{A^2}$ and $\widehat{A^2}$ is an m.s.p. algebra. If U is a finite dimensional ideal of A, then $U \cap \widehat{A^2}$ is a finite dimensional ideal of $\widehat{A^2}$, and from the above theorem $U \cap \widehat{A^2}$ is a complemented ideal in $\widehat{A^2}$. Since, by Lemma 5.6, $U = (U \cap \mathrm{Ann}(A)) \oplus (U \cap \widehat{A^2})$, it follows that U is a complemented ideal in A. □

Lee and Wong [13, Theorem 1.7] proved that a prime associative algebra A possessing a finite dimensional nonzero right ideal is finite dimensional and simple. The precedent for A primitive was established by Yood [15, Theorem 3.8]. We will conclude this paper with a nonassociative version of this result, which relies on the above theorem.

Corollary 5.18. *Let A be a nonzero m.p. algebra. If A has a nonzero finite dimensional ideal, then A is simple and finite dimensional.*

Proof. By 1.21, $\mathcal{L}_A = \{0, A\}$. Assume that U is a nonzero finite dimensional ideal of A. Since, by the above theorem, every finite dimensional ideal of A is closed, it follows that $U = A$, and hence A is finite dimensional. In particular, all the ideals of A are finite dimensional, and so $\mathcal{I}_A = \{0, A\}$. Thus, A is a simple algebra. $\square$

References

[1] A.A. Albert, The radical of a non-associative algebra, Bull. Amer. Math. Soc. 48 (1942) 891–897.
[2] J.C. Cabello, M. Cabrera, Structure theory for multiplicatively semiprime algebras, J. Algebra 282 (2004) 386–421.
[3] J.C. Cabello, M. Cabrera, Algebras whose multiplication algebra is semiprime. A decomposition theorem, J. Algebra 319 (2008) 911–937.
[4] J.C. Cabello, M. Cabrera, A. Fernández, π-Complemented algebras through pseudocomplemented lattices, Order, in press, doi:10.1007/s11083-011-9214-4.
[5] J.C. Cabello, M. Cabrera, E. Nieto, Closed prime ideals in algebras with semiprime multiplication algebra, Comm. Algebra 35 (2007) 4245–4276.
[6] M. Cabrera, A.A. Mohammed, Extended centroid and central closure of the multiplication algebra, Comm. Algebra 27 (1999) 5723–5736.
[7] M. Cabrera, A.A. Mohammed, Extended centroid and central closure of multiplicatively semiprime algebras, Comm. Algebra 29 (2001) 1215–1233.
[8] M. Cabrera, J. Sánchez, Lie quotients for skew Lie algebras, Algebra Colloq. 16 (2009) 267–274.
[9] M. Cabrera, A.R. Villena, Multiplicative-semiprimeness of nondegenerate Jordan algebras, Comm. Algebra 32 (2004) 3995–4003.
[10] D.R. Finston, On multiplication algebras, Trans. Amer. Math. Soc. 293 (1986) 807–818.
[11] N. Jacobson, A note on nonassociative algebras, Duke Math. J. 3 (1937) 544–548.
[12] N. Jacobson, Abraham Adrian Albert 1905–1972, Bull. Amer. Math. Soc. 80 (1974) 1075–1100.
[13] T.-K. Lee, T.-L. Wong, Semiprime algebras with finiteness conditions, Comm. Algebra 31 (2003) 1823–1835.
[14] F.L. Pritchrad, Ideals in the multiplication algebra of a non-associative K-algebra, Comm. Algebra 21 (1993) 4541–4559.
[15] B. Yood, The strong radical and finite-dimensional ideals, Proc. Amer. Math. Soc. 130 (2001) 139–143.