EXTRACTA MATHEMATICAE Vol. **27**, Núm. 2, 231 – 244 (2012)

Semicentral Idempotents in the Multiplication Ring of a Centrally Closed Prime Ring

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Presented by Consuelo Martínez — Received July 9, 2012

Abstract: Let R be a ring and let $M(R)$ stand for the multiplication ring of R. An idempotent E in $M(R)$ is called left semicentral if its range $E(R)$ is a right ideal of R. In the case that R is prime and centrally closed we give a description of the left semicentral idempotents in $M(R)$. As an application we prove that, if, in addition, $M(R)$ is Baer (respectively, regular or Rickart), then R is Baer (respectively, regular or Rickart). Similar results for $*$-rings are also proved.

Key words: Prime ring, extended centroid, multiplication ring, semicentral idempotent, Baer ring.

AMS *Subject Class.* (2010): 16N60, 16E50.

Introduction

Let R be a (unital associative) ring and let $\mathrm{End}_{\mathbb{Z}}(R)$ stand for the ring of all endomorphisms of the additive group of R. For each a in R, let L_a and R_a denote the *left* and *right multiplications* by a, respectively. The *multiplication ring* of R is defined as the subring $M(R)$ of $\mathrm{End}_{\mathbb{Z}}(R)$ generated by the set $\{L_a, R_a : a \in R\}$. If for any $a, b \in R$ we define the *two-sided multiplication* $M_{a,b} \in \mathrm{End}_{\mathbb{Z}}(R)$ by $M_{a,b}(x) = axb$, it is clear that $L_a = M_{a,1}$, $R_a = M_{1,a}$, $\mathrm{Id}_R = M_{1,1}$, and

$$M(R) = \Big\{ \sum_{i=1}^{n} M_{a_i,b_i} : n \in \mathbb{N},\ a_i, b_i \in R\ (1 \leq i \leq n) \Big\}.$$

We say that an idempotent E in $M(R)$ is *left* (respectively, *right*, or *two-sided*) *semicentral* if its range $E(R)$ is a right (respectively, left, or two-sided) ideal of R.

This research was partially supported by the Spanish MICINN and Fondos FEDER, jointly, trough proyect MTM2009-12067, and, in addition, by the Junta de Andalucía Grant FQM290.

Our aim is to provide a description of the semicentral idempotents in the multiplication ring of a centrally closed prime ring. While the general theory of rings of quotients is developed in many books, we shall mostly follow [1]. Recall that a ring R is called *prime* if the product of two nonzero ideals of R is always nonzero (equivalently, the condition $aRb = 0$, where $a, b \in R$, implies $a = 0$ or $b = 0$), and R is called *semiprime* if it contains no nonzero nilpotent ideals (equivalently, the condition $aRa = 0$, where $a \in R$, implies $a = 0$). The extended centroid C of a semiprime ring R can be defined as the center of its two-sided symmetric ring of quotients $Q_s(R)$, and R is said to be *centrally closed* whenever C coincides with the center of R. Moreover, R is prime if and only if C is a field. We prove that the left semicentral idempotents in $M(R)$, for R centrally closed prime ring, are just of the form

$$E = L_e + \sum_{i=1}^{n} M_{x_i,y_i}$$

for suitable e idempotent in R, $n \geq 0$, $x_i, y_i \in R$ satisfying $ex_i = x_i$, $x_ie = 0$, and $x_ix_j = 0$ for all $i, j \in \{1, \ldots, n\}$, and such that both sets $\{e, x_1, \ldots, x_n\}$ and $\{1, y_1, \ldots, y_n\}$ are linearly C-independent.

As usual, for a subset S of a ring R, the *left* respectively *right annihilator* of S will be defined by

$$\mathrm{Ann}_\ell(S) := \{a \in R : aS = 0\} \text{ and } \mathrm{Ann}_r(S) := \{a \in R : Sa = 0\}.$$

Clearly $\mathrm{Ann}_\ell(S)$ is a left ideal of R and $\mathrm{Ann}_r(S)$ is a right ideal of R. Recall that a ring R is a *Rickart ring* if for each x in R there are idempotents e and f in R such that $\mathrm{Ann}_r(x) = eR$ and $\mathrm{Ann}_\ell(x) = Rf$. A ring R is a *regular ring* if for each x in R there exists an element y in R such that $x = xyx$ (equivalently, $xR = eR$ for suitable idempotent e in R). A ring R is a *Baer ring* if for each subset S of R there is an idempotent e in R such that $\mathrm{Ann}_r(S) = eR$. As an application of the description of the semicentral idempotents in $M(R)$, for R centrally closed prime ring, we derive that if $M(R)$ is a Rickart, regular, or Baer ring, then R so is. Similar results for centrally closed $*$-prime $*$-rings are also obtained. The classical books here are [2, 3, 6, 7].

1. The main results

We begin by stating some immediate characterizations of semicentral idempotents in the multiplication ring.

PROPOSITION 1.1. *Let R be a ring and let E be an idempotent in $M(R)$. Then the following conditions are equivalent:*

(i) *E is a left (respectively, right) semicentral idempotent in $M(R)$.*

(ii) *$E(E(a)b) = E(a)b$ (respectively, $E(bE(a)) = bE(a)$) for all $a, b \in R$.*

(iii) *$ER_aE = R_aE$ (respectively, $EL_aE = L_aE$) for every $a \in R$.*

COROLLARY 1.2. *Let R be a ring and let E be an idempotent in $M(R)$. Then the following conditions are equivalent:*

(i) *E is a two-sided semicentral idempotent in $M(R)$.*

(ii) *$E(E(a)b) = E(a)b$ and $E(bE(a)) = bE(a)$ for all $a, b \in R$.*

(iii) *$ETE = TE$ for every $T \in M(R)$.*

Note that the two-sided semicentral idempotents in $M(R)$ in our sense are just the left semicentral idempotents in the ring $M(R)$ in the sense of [4]. Clearly every central idempotent in $M(R)$ is two-sided semicentral. The converse is true whenever R is prime.

PROPOSITION 1.3. *Let R be a prime ring. For $E \in M(R)$, the following conditions are equivalent:*

(i) *E is a central idempotent.*

(ii) *E is a two-sided semicentral idempotent.*

(iii) *$E = 0$ or Id_R.*

Proof. The implications (iii) $\Rightarrow$ (i) $\Rightarrow$ (ii) are true in a general context.
(ii) $\Rightarrow$ (iii). If E is a two-sided semicentral idempotent in $M(R)$, then

$$(\mathrm{Id}_R - E)M(R)E = 0.$$

Since $M(R)$ is a prime ring [5, Proposition 4], it follows that $E = 0$ or Id_R. ∎

In order to obtain a description of the one-sided semicentral idempotents in the multiplication ring of a centrally closed prime ring, we will make heavy use of the following well-known fact [1, Corollary 6.1.3]:

Let R be a centrally closed prime ring, and let $a_i, b_i \in R$ $(1 \leq i \leq n)$ be such that $\sum_{i=1}^{n} a_i x b_i = 0$ for every $x \in R$. If $a_1, \ldots, a_n$ are linearly C-independent, then $b_1 = \cdots = b_n = 0$.

Given $T \in M(R)\backslash\{0\}$, we will say that the *length* of T is $n \in \mathbb{N}$ if $T = \sum_{i=1}^{n} M_{a_i,b_i}$ for some $a_i, b_i \in R$ and T cannot be written also as $\sum_{i=1}^{m} M_{c_i,d_i}$ for some $m < n$, $c_i, d_i \in R$.

LEMMA 1.4. *Let R be a centrally closed prime ring and let T be a nonzero element in $M(R)$. Then T has length n if and only if $T = \sum_{i=1}^{n} M_{a_i,b_i}$ for some $a_i, b_i \in R$ with $a_1, \ldots, a_n$ linearly C-independent and $b_1, \ldots, b_n$ linearly C-independent.*

Proof. Assume that T has length n. If $T = \sum_{i=1}^{n} M_{a_i,b_i}$, then it is clear that any linear C-dependence of the a_i's or the b_i's allows us to write T as a sum of two-sided multiplications with less than n summands. Therefore, both $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$ are linearly C-independent sets.

Conversely, assume that $T = \sum_{i=1}^{n} M_{a_i,b_i}$ and that both $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$ are linearly C-independent sets. To obtain a contradiction, we suppose that $T = \sum_{j=1}^{m} M_{c_j,d_j}$ for some $m < n$, $c_1, \ldots, c_m$ linearly C-independent and $d_1, \ldots, d_m$ linearly C-independent. Then, there exists $k, \ell \in \{1, \ldots, n\}$ such that a_k is linearly C-independent of the c_j's and a_ℓ is linearly C-dependent of the c_j's. By the incomplete basis theorem, there exists a subset of $\{a_1, \ldots, a_n\}$, which we will assume $\{a_1, \ldots, a_p\}$, such that $\{a_1, \ldots, a_p, c_1, \ldots, c_m\}$ is a basis of the C-vector subspace generated by $\{a_1, \ldots, a_n, c_1, \ldots, c_m\}$. So for each $k \in \{p+1, \ldots, n\}$ we can write

$$a_k = \sum_{i=1}^{p} \alpha_k^i a_i + \sum_{j=1}^{m} \beta_k^j c_j \quad (\alpha_k^i, \beta_k^j \in C).$$

Therefore, the equality $\sum_{i=1}^{n} M_{a_i,b_i} = \sum_{j=1}^{m} M_{c_j,d_j}$ yields to

$$\sum_{i=1}^{p} a_i x \left(b_i + \sum_{k=p+1}^{n} \alpha_k^i b_k \right) = \sum_{j=1}^{m} c_j x \left(d_j - \sum_{k=p+1}^{n} \beta_k^j b_k \right)$$

for every $x \in R$. Hence $b_1 + \sum_{k=p+1}^{n} \alpha_k^i b_k = 0$ -a contradiction. Thus T has length n. ∎

Our main result is the following.

THEOREM 1.5. *Let R be a centrally closed prime ring, let E be in $M(R)\backslash\{0\}$ and let $n \geq 0$. Then E is a left semicentral idempotent in $M(R)$ of length $n+1$ if and only if*

$$E = L_e + \sum_{i=1}^{n} M_{x_i,y_i}$$

for suitable e idempotent in R, $x_i, y_i \in R$ satisfying $ex_i = x_i$, $x_i e = 0$, and $x_i x_j = 0$ for all $i, j \in \{1, \ldots, n\}$, and such that both sets $\{e, x_1, \ldots, x_n\}$ and $\{1, y_1, \ldots, y_n\}$ are linearly C-independent.

Proof. It is easy to see that, if E is of the form just described in the statement, then E is a left semicentral idempotent in $M(R)$. Moreover, by Lemma 1.4, E has length $n+1$.

In order to prove the converse, assume that E is a left semicentral idempotent in $M(R)$ of length $n+1$. Write $E = \sum_{i=0}^{n} M_{a_i,b_i}$ for suitable $a_i, b_i \in R$, and take into account that, by Lemma 1.4, $\{a_0, a_1, \ldots, a_n\}$ and $\{b_0, b_1, \ldots, b_n\}$ are each linearly C-independent sets. Set $a_{i,j} = a_i a_j$. Then the equality $E(E(x)y) = E(x)y$ can be rewritten as follows

$$\sum_{i,j=0}^{n} a_{i,j} x b_j y b_i = \sum_{k=0}^{n} a_k x b_k y. \tag{1.1}$$

First assume that $\{a_0, a_1, \ldots, a_n\}$ is a C-basis of the vector subspace generated by the set $S := \{a_{i,j}, a_k : 0 \leq i, j, k \leq n\}$ and that for each i, j

$$a_{i,j} = \sum_{k=0}^{n} \alpha_k^{i,j} a_k \quad (\alpha_k^{i,j} \in C).$$

Then (1.1) gives that

$$\sum_{k=0}^{n} a_k x \left(b_k y - \sum_{i,j=0}^{n} \alpha_k^{i,j} b_j y b_i \right) = 0,$$

and consequently, for each k we have

$$b_k y - \sum_{i,j=0}^{n} \alpha_k^{i,j} b_j y b_i = 0.$$

Writing this equality in the form

$$b_k y \left(1 - \sum_{i=0}^{n} \alpha_k^{i,k} b_i\right) - \sum_{\substack{j=0 \\ j\neq k}}^{n} b_j y \left(\sum_{i=0}^{n} \alpha_k^{i,j} b_i\right) = 0,$$

we see that

$$1 - \sum_{i=0}^{n} \alpha_k^{i,k} b_i = 0 \quad \text{and} \quad \sum_{i=0}^{n} \alpha_k^{i,j} b_i = 0 \quad (j \neq k).$$

These equalities together with the linear C-independence of $b_0, b_1, \ldots, b_n$ give that $\alpha_k^{i,k} = \alpha_{k'}^{i,k'}$ for all i, k, k' and $\alpha_k^{i,j} = 0$ for all i, j, k with $j \neq k$. Set $\alpha_i = \alpha_k^{i,k}$. Then, we have

$$\sum_{i=0}^{n} \alpha_i b_i = 1 \quad \text{and} \quad a_{i,j} = \alpha_i a_j.$$

By suitable reordering of the summands appearing in E we can assume the existence of m with $0 \leq m \leq n$ such that $\alpha_i \neq 0$ for $i \leq m$ and $\alpha_i = 0$ otherwise. Now consider $e = \alpha_0^{-1} a_0$, $x_i = \alpha_i^{-1} a_i - \alpha_0^{-1} a_0$, $y_i = \alpha_i b_i$ if $1 \leq i \leq m$ and $x_i = a_i$, $y_i = b_i$ otherwise. It is easy to check that $E = L_e + \sum_{i=1}^{n} M_{x_i, y_i}$, e is an idempotent in R, and $x_i, y_i \in R$ satisfy $ex_i = x_i$, $x_i e = 0$, and $x_i x_j = 0$ for all i, j, and both sets $\{e, x_1, \ldots, x_n\}$ and $\{1, y_1, \ldots, y_n\}$ are linearly C-independent.

Finally suppose, towards a contradiction, that $\{a_0, a_1, \ldots, a_n\}$ is not a C-basis of the vector subspace generated by S. If S is a linearly C-independent set, then it follows from (1.1) that $b_0 y = 0$ for every $y \in R$, hence $b_0 = 0$ -a contradiction. Therefore there exists a nonempty proper subset Γ of $\{0, 1, \ldots, n\} \times \{0, 1, \ldots, n\}$ such that

$$\{a_{i,j}, a_k : (i,j) \in \Gamma,\ 0 \leq k \leq n\}$$

is a C-basis of the vector subspace generated by S. Accordingly, for each $(p, q) \notin \Gamma$, we may write

$$a_{p,q} = \sum_{(i,j)\in\Gamma} \alpha_{i,j}^{p,q} a_{i,j} + \sum_{k=0}^{n} \beta_k^{p,q} a_k \quad (\alpha_{i,j}^{p,q}, \beta_k^{p,q} \in C).$$

Now, from (1.1) we see that

$$\sum_{(i,j)\in\Gamma} a_{i,j}x\left(b_j y b_i + \sum_{(p,q)\notin\Gamma} \alpha_{i,j}^{p,q} b_q y b_p\right) = \sum_{k=0}^{n} a_k x\left(b_k y - \sum_{(p,q)\notin\Gamma} \beta_k^{p,q} b_q y b_p\right).$$

As a consequence, for a fixed $(i_0, j_0) \in \Gamma$, we have

$$b_{j_0} y b_{i_0} + \sum_{(p,q)\notin\Gamma} \alpha_{i_0,j_0}^{p,q} b_q y b_p = 0,$$

hence

$$b_{j_0} y\left(b_{i_0} + \sum_{(p,j_0)\notin\Gamma} \alpha_{i_0,j_0}^{p,j_0} b_p\right) + \sum_{j\neq j_0} b_j y\left(\sum_{(p,j)\notin\Gamma} \alpha_{i_0,j_0}^{p,j} b_p\right) = 0,$$

and so

$$b_{i_0} + \sum_{(p,j_0)\notin\Gamma} \alpha_{i_0,j_0}^{p,j_0} b_p = 0,$$

which is a contradiction. ∎

Let R be a ring, and let R^{op} stand for the opposite ring of R. Since the additive groups of R and R^{op} agree, we can identify their endomorphism rings $\mathrm{End}_{\mathbb{Z}}(R) \equiv \mathrm{End}_{\mathbb{Z}}(R^{op})$, as well as their multiplication rings $M(R) \equiv M(R^{op})$. More precisely, if $M_{a,b}^{op}$ denote the two-sided multiplication determined by the elements a and b in the opposite ring R^{op}, then note that $M_{a,b}^{op} = M_{b,a}$.

Corollary 1.6. *Let R be a centrally closed prime ring, let E be in $M(R)\backslash\{0\}$ and let $n \geq 0$. Then E is a right semicentral idempotent in $M(R)$ of length $n+1$ if and only if*

$$E = R_e + \sum_{i=1}^{n} M_{x_i,y_i}$$

for suitable e idempotent in R, $x_i, y_i \in R$ satisfying $y_i e = y_i$, $e y_i = 0$, and $y_i y_j = 0$ for all $i, j \in \{1, \ldots, n\}$, and such that both sets $\{1, x_1, \ldots, x_n\}$ and $\{e, y_1, \ldots, y_n\}$ are linearly C-independent.

Proof. Note that R^{op} is a centrally closed prime ring. It is clear that $E \in M(R)$ is a right semicentral idempotent in $M(R)$ of length $n+1$ if and only if $E \in M(R^{op})$ is a left semicentral idempotent in $M(R^{op})$ of length $n+1$. Now, the result follows straightforwardly from Theorem 1.5 applied to R^{op}. ∎

Corollary 1.7. *Let R be a centrally closed prime ring. We have:*

(1) *If E is a left semicentral idempotent in $M(R)$, then there exists an idempotent e in R such that $EL_e = L_e$ and $L_eE = E$. In particular, $E(R) = eR$.*

(2) *If E is a right semicentral idempotent in $M(R)$, then there exists an idempotent e in R such that $ER_e = R_e$ and $R_eE = E$. In particular, $E(R) = Re$.*

Proof. (1) We may assume that $E \neq 0$. By Theorem 1.5, we have

$$E = L_e + \sum_{i=1}^{n} M_{x_i,y_i}$$

for suitable e idempotent in R, $n \geq 0$, $x_i, y_i \in R$ such that $ex_i = x_i$, $x_ie = 0$, and $x_ix_j = 0$ for all $i, j \in \{1, \ldots, n\}$. Note that these conditions imply that $EL_e = L_e$ and $L_eE = E$, and therefore $E(R) = eR$.

(2) This assertion can be proved similarly, taking into account Corollary 1.6. ∎

A $*$-*ring* is a ring R endowed with an *involution*, that is a map $* : R \to R$ satisfying

$$(a + b)^* = a^* + b^*, \quad (ab)^* = b^*a^*, \quad \text{and} \quad (a^*)^* = a.$$

Lemma 1.8. *Let R be a centrally closed prime ring. Then $M(R)$ is a $*$-ring for the involution $\circ$ defined by*

$$T = \sum_{i=1}^{n} M_{a_i,b_i} \mapsto T^\circ := \sum_{i=1}^{n} M_{b_i,a_i}.$$

Proof. In order to prove the map $T \mapsto T^\circ$ is well-defined, we show that $\sum_{i=1}^{n} M_{b_i,a_i} = 0$ whenever $\sum_{i=1}^{n} M_{a_i,b_i} = 0$. This is clear whenever $a_1 = \cdots = a_n = 0$. Assume that some a_i is nonzero. By suitable reordering of the summands we may assume the existence of m with $1 < m \leq n$ such that $\{a_1, \ldots, a_m\}$ is a C-basis of the vector subspace generated by the set $\{a_1, \ldots, a_n\}$. For each j with $m < j \leq n$, write $a_j = \sum_{i=1}^{m} \lambda_i^j a_i$ ($\lambda_i^j \in C$). Then, we have

$$0 = \sum_{i=1}^{n} M_{a_i,b_i} = \sum_{i=1}^{m} M_{a_i,b_i+\sum_{j=m+1}^{n} \lambda_i^j b_j},$$

hence, for every i with $1 \leq i \leq m$, we obtain that $b_i + \sum_{j=m+1}^{n} \lambda_i^j b_j = 0$, and so

$$0 = \sum_{i=1}^{m} M_{b_i + \sum_{j=m+1}^{n} \lambda_i^j b_j, a_i} = \sum_{i=1}^{n} M_{b_i, a_i},$$

as required. The proofs of the remaining assertions are straightforward. ∎

Note that the involution $\circ$ on $M(R)$ given by Lemma 1.8 is not linked to any involution on R. Therefore, when R is actually a $*$-ring, the involution $*$ on $M(R)$ given by Proposition 1.9 below becomes more useful in order to relate R and $M(R)$ as $*$-rings.

Let R be a $*$-ring with involution $*$. For each $T \in \mathrm{End}_{\mathbb{Z}}(R)$, let T' stand for the endomorphism of the additive group of R defined by $T'(x) := T(x^*)^*$ for every $x \in R$. It is clear that the map $T \mapsto T'$ becomes an involutive automorphism of the ring $\mathrm{End}_{\mathbb{Z}}(R)$.

Proposition 1.9. *Let R be a centrally closed prime $*$-ring. Then $M(R)$ is a $*$-ring for the involution defined by*

$$T = \sum_{i=1}^{n} M_{a_i, b_i} \mapsto T^* = \sum_{i=1}^{n} M_{a_i^*, b_i^*}.$$

Proof. Note that if $T \in M(R)$ and $T = \sum_{i=1}^{n} M_{a_i, b_i}$, then $T' = \sum_{i=1}^{n} M_{b_i^*, a_i^*}$ belongs also to $M(R)$. Therefore, we can regard the map $T \mapsto T'$ as an involutive automorphism of $M(R)$. By considering the involution $\circ$ on $M(R)$ provided by Lemma 1.8, and noticing that $'$ and $\circ$ commute, we find that the map $T \mapsto T^* := (T^\circ)'$ becomes an involution on $M(R)$, and the proof is complete. ∎

If R is a centrally closed prime $*$-ring, then the involution $*$ on $M(R)$ given by the above proposition will hereafter be referred to as the *involution associated to the involution $*$ on R.*

The self-adjoint idempotents in a $*$-ring are called *projections.*

Corollary 1.10. *Let R be a centrally closed prime $*$-ring and let E be in $M(R)$. Consider $M(R)$ as a $*$-ring for the involution associated to the involution $*$ on R. Then:*

(1) *E is a left semicentral projection of $M(R)$ if and only if $E = L_e$ for some projection e of R.*

(2) *E is a right semicentral projection of $M(R)$ if and only if $E = R_e$ for some projection e of R.*

Proof. (1) For a projection e of R, it is clear that L_e is a left semicentral projection of $M(R)$. Let E be a left semicentral projection in $M(R)$. We may assume that $E \neq 0$. If E has length 1, then, by Theorem 1.5, $E = L_e$ for suitable idempotent e in R. Therefore

$$e = L_e(1) = E(1) = E^*(1) = L_{e^*}(1) = e^*,$$

hence e is a projection in R, and so the proof is concluded in this case. Suppose, to derive a contradiction, that E has length $n+1$ for $n \in \mathbb{N}$. Then, by Theorem 1.5, $E = L_e + \sum_{i=1}^n M_{x_i,y_i}$ for suitable e idempotent in R, $x_i, y_i \in R$ satisfying $ex_i = x_i$, $x_i e = 0$, and $x_i x_j = 0$ for all $i, j \in \{1, \dots, n\}$, and such that the sets $\{e, x_1, \dots, x_n\}$ and $\{1, y_1, \dots, y_n\}$ are both linearly C-independent. Therefore

$$L_{e^*e} + \sum_{i=1}^n M_{e^*x_i,y_i} = L_{e^*}E = L_e^*E = (EL_e)^* = L_e^* = L_{e^*},$$

and hence

$$L_{e^*(e-1)} + \sum_{i=1}^n M_{e^*x_i,y_i} = 0.$$

Since $1, y_1, \dots, y_n$ are linearly C-independent, we see that $e^* = e^*e$ and $e^*x_i = 0$ for all i. Thus $e^* = e$ and $x_i = ex_i = 0$ for all i, which is a contradiction.

(2) This assertion can be deduced from (1) in the standard way. ∎

2. Prime rings with Baer multiplication ring.

Let R be a ring. Note that, for each left ideal I of R,

$$M_{I,R} := \Big\{ \sum_{i=1}^n M_{x_i,a_i} : n \in \mathbb{N},\ x_i \in I,\ a_i \in R \Big\}$$

is the left ideal of $M(R)$ generated by the set $\{L_x : x \in I\}$. Analogously, for each right ideal I of R,

$$M_{R,I} := \Big\{ \sum_{i=1}^n M_{a_i,x_i} : n \in \mathbb{N},\ a_i \in R,\ x_i \in I \Big\}$$

is the left ideal of $M(R)$ generated by the set $\{R_x : x \in I\}$.

LEMMA 2.1. *Let R be a ring. We have:*

(1) *If I is a left ideal of R such that $\mathrm{Ann}_r(M_{I,R}) = EM(R)$ for suitable idempotent E of $M(R)$, then $\mathrm{Ann}_r(I) = E(R)$.*

(2) *If I is a right ideal of R such that $\mathrm{Ann}_r(M_{R,I}) = EM(R)$ for suitable idempotent E of $M(R)$, then $\mathrm{Ann}_\ell(I) = E(R)$.*

Proof. Assume that I is a left ideal of R such that $\mathrm{Ann}_r(M_{I,R}) = EM(R)$ for suitable idempotent E in $M(R)$. If $a \in \mathrm{Ann}_r(I)$, then $L_a \in \mathrm{Ann}_r(M_{I,R})$, hence $L_a = ET$ for suitable $T \in M(R)$, and so

$$a = L_a(1) = E\big(T(1)\big) \in E(R).$$

Therefore $\mathrm{Ann}_r(I) \subseteq E(R)$. Conversely, since $L_xE = 0$ for every $x \in I$, it follows that $IE(R) = 0$, and so $E(R) \subseteq \mathrm{Ann}_r(I)$. Thus $\mathrm{Ann}_r(I) = E(R)$, and the proof of assertion (1) is complete. The proof of assertion (2) is similar. ∎

THEOREM 2.2. *Let R be a centrally closed prime ring. We have:*

(1) *If $M(R)$ is Rickart, then R is Rickart.*

(2) *If $M(R)$ is regular, then R is regular.*

(3) *If $M(R)$ is Baer, then R is Baer.*

Proof. (1) Assume that $M(R)$ is Rickart. For a given $x \in R$, there exist idempotents E and F in $M(R)$ such that $\mathrm{Ann}_r(L_x) = EM(R)$ and $\mathrm{Ann}_r(R_x) = FM(R)$. Since $M(R)L_x = M_{Rx,R}$ and $M(R)R_x = M_{R,xR}$, and hence $\mathrm{Ann}_r(L_x) = \mathrm{Ann}_r(M_{Rx,R})$ and $\mathrm{Ann}_r(R_x) = \mathrm{Ann}_r(M_{R,xR})$, it follows from Lemma 2.1 that $\mathrm{Ann}_r(Rx) = E(R)$ and $\mathrm{Ann}_\ell(xR) = F(R)$. Therefore E and F are left (resp. right) semicentral idempotents in $M(R)$. Now, by Corollary 1.7, we can confirm the existence of idempotents e and f in R such that $\mathrm{Ann}_r(Rx) = eR$ and $\mathrm{Ann}_\ell(xR) = Rf$. Thus R is Rickart.

(2) Assume that $M(R)$ is regular. For a given $x \in R$, there exists an idempotent E in $M(R)$ such that $L_xM(R) = EM(R)$, hence $xR = E(R)$, and so E is left semicentral. Now, by Corollary 1.7.(1), we conclude that $xR = eR$ for suitable idempotent e in R. Thus R is regular.

(3) Assume that $M(R)$ is Baer. Let I be a left ideal of R. Then, there exists an idempotent E of $M(R)$ such that $\mathrm{Ann}_r(M_{I,R}) = EM(R)$. Arguing as in the proof of assertion (1) we can assert that $\mathrm{Ann}_r(I) = eR$ for suitable idempotent e in R. Thus R is a Baer ring. ∎

We recall that a $*$-ring R is said to be *$*$-prime* if $UV \neq 0$ whenever U and V are nonzero $*$-ideals of R. Every $*$-prime $*$-ring R is semiprime, and hence its involution can be extended uniquely to an involution on $Q_s(R)$ [1, Proposition 2.5.4]. Clearly every prime $*$-ring is $*$-prime. However, there exist nonprime $*$-prime $*$-rings. Indeed, if R is a prime ring, then $R \oplus R^{op}$ endowed with the exchange involution is a nonprime $*$-prime $*$-ring. The next result shows that every centrally closed nonprime $*$-prime $*$-ring is of this type.

Proposition 2.3. *For every $*$-ring R, the following assertions are equivalent:*

(i) *R is a centrally closed nonprime $*$-prime $*$-ring.*

(ii) *There exists an ideal I of R, which is a centrally closed prime ring, such that $R = I \oplus I^*$.*

Proof. (i) $\Rightarrow$ (ii). By the nonprimeness of R there are nonzero ideals J, K of R such that $JK = 0$, hence $(J \cap J^*)(K \cap K^*) = 0$, and so either $J \cap J^* = 0$ or $K \cap K^* = 0$. Assume, for example, that $J \cap J^* = 0$, so that $JJ^* = 0$. Let $\mathrm{Ann}_C(J)$ denote the annihilator of J in C, and let e be the idempotent in C associated to J; that is, e is the unique idempotent in C such that $\mathrm{Ann}_C(J) = (1-e)C$ (cf. [1, Theorem 2.3.9.(ii)]). Since

$$\mathrm{Ann}_C(J^*) = \mathrm{Ann}_C(J)^* = \big((1-e)C\big)^* = (1-e^*)C,$$

it follows that e^* is the idempotent in C associated to J^*. Moreover, the condition $JJ^* = 0$ implies that $ee^* = 0$ (by [1, Lemma 2.3.10]). On the other hand, the $*$-primeness of R implies that $J \oplus J^*$ is an essential ideal of R, hence $J \oplus J^*$ has zero annihilator in R, and in particular $\mathrm{Ann}_C(J \oplus J^*) = 0$. Since $(1-e)(1-e^*) \in \mathrm{Ann}_C(J) \cap \mathrm{Ann}_C(J^*) \subseteq \mathrm{Ann}_C(J \oplus J^*)$, it follows that $(1-e)(1-e^*) = 0$. Therefore $e^* = 1-e$, and hence $R = eR \oplus e^*R$. It is easy to verify that eR is a prime ring. Moreover, since $eQ_s(R) \cap R = eR$, it follows from [1, Proposition 2.3.14] that $Q_s(eR) = eQ_s(R)$, hence the extended centroid of eR is eC, and so eR is centrally closed. Summarizing, $I := eR$ is an ideal of R, which is a centrally closed prime ring, and $R = I \oplus I^*$.

(ii) $\Rightarrow$ (i). It is clear that R is a nonprime $*$-prime $*$-ring. The fact that R is centrally closed follows from the obvious equality

$$Q_s(R) = Q_s(I) \oplus Q_s(I)^*.$$

∎

The involution of a $*$-ring R is called *proper* whenever the condition $a^*a = 0$, for $a \in R$, implies that $a = 0$.

PROPOSITION 2.4. *Let R be a centrally closed nonprime $*$-prime $*$-ring. Then $M(R)$ is a $*$-ring for the involution defined by*

$$T = \sum_{i=1}^{n} M_{a_i,b_i} \mapsto T^* = \sum_{i=1}^{n} M_{a_i^*,b_i^*},$$

which is not proper.

Proof. By Proposition 2.3, there exists an ideal I of R, which is a centrally closed prime ring, such that $R = I \oplus I^*$. Suppose that $a_1, \ldots, a_n, b_1, \ldots, b_n$ are elements in R satisfying $\sum_{i=1}^{n} M_{a_i,b_i} = 0$. By writing $a_i = x_i \oplus y_i^*$ and $b_i = z_i \oplus t_i^*$ for $x_i, y_i, z_i, t_i \in I$, we see that

$$0 = \sum_{i=1}^{n} M_{a_i,b_i} = \sum_{i=1}^{n} M_{x_i \oplus y_i^*, z_i \oplus t_i^*} = \sum_{i=1}^{n} M_{x_i,z_i} + \sum_{i=1}^{n} M_{y_i^*,t_i^*},$$

and consequently $\sum_{i=1}^{n} M_{x_i,z_i} = \sum_{i=1}^{n} M_{y_i^*,t_i^*} = 0$. For each x, y in I, let us denote by $M_{x,y}^I$ the two-sided multiplication determined by x and y in the ring I. It follows from the above that $\sum_{i=1}^{n} M_{x_i,z_i}^I = \sum_{i=1}^{n} M_{t_i,y_i}^I = 0$. Hence, by Lemma 1.8, we have also $\sum_{i=1}^{n} M_{z_i,x_i}^I = \sum_{i=1}^{n} M_{y_i,t_i}^I = 0$, and so $\sum_{i=1}^{n} M_{x_i^*,z_i^*} = \sum_{i=1}^{n} M_{y_i,t_i} = 0$. Therefore

$$\sum_{i=1}^{n} M_{a_i^*,b_i^*} = \sum_{i=1}^{n} M_{x_i^* \oplus y_i, z_i^* \oplus t_i} = \sum_{i=1}^{n} M_{x_i^*,z_i^*} + \sum_{i=1}^{n} M_{y_i,t_i} = 0.$$

Thus the correspondence $T \mapsto T^*$ is a well-defined map. It is routine to verify that this map is an involution on $M(R)$. Finally, note that for $x, y \in I \setminus \{0\}$ we have $M_{x,y} \neq 0$, but $M_{x,y}^* M_{x,y} = 0$, and hence $*$ is not proper. ∎

Putting together Propositions 1.9 and 2.4 we have the following result: *If R is a centrally closed $*$-prime $*$-ring, then $M(R)$ is a $*$-ring for the involution defined by*

$$T = \sum_{i=1}^{n} M_{a_i,b_i} \mapsto T^* = \sum_{i=1}^{n} M_{a_i^*,b_i^*}.$$

This involution will be referred to as the *involution on $M(R)$ associated to the involution $*$ on R.*

Recall that a $*$-ring R is a *Rickart $*$-ring* if for each x in R there is a projection e in R such that $\mathrm{Ann}_r(x) = eR$. A $*$-ring R is a *$*$-regular ring* if for each x in R there is a projection e in R such that $xR = eR$. A $*$-ring R is a *Baer $*$-ring* if for each left ideal I of R there is a projection e in R such that $\mathrm{Ann}_r(I) = eR$.

Theorem 2.5. *Let R be a centrally closed $*$-prime $*$-ring. Consider $M(R)$ endowed with the involution associated to the involution of R. We have:*

(1) *If $M(R)$ is a Rickart $*$-ring, then R is a Rickart $*$-ring.*

(2) *If $M(R)$ is a $*$-regular ring, then R is a $*$-regular ring.*

(3) *If $M(R)$ is a Baer $*$-ring, then R is a Baer $*$-ring.*

Proof. If R is nonprime, then the involution on $M(R)$ associated to the involution on R is not proper (cf. Proposition 2.4), and hence $M(R)$ is not a Rickart $*$-ring [3, 1.10]. Since $*$-regular rings and Baer $*$-rings are Rickart $*$-rings [3, Propositions 1.13 and 1.24], in order to prove the statement we may assume that R is prime. Now, we can argue as in the proof of Theorem 2.2 with Corollary 1.10 instead of Corollary 1.7. ∎

Acknowledgements

The authors are most grateful to the referee for several valuable remarks.

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