Field and Particle Theory

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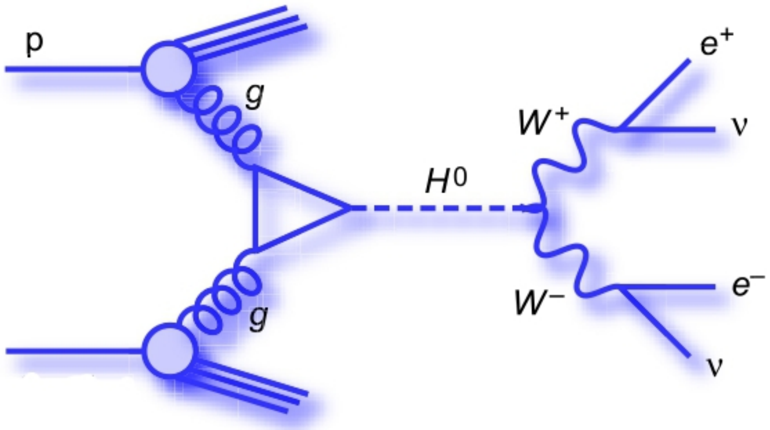

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Outline

1. Introduction: particles *vs* fields.
   Lorentz and Poincaré symmetries
2. Classical field theory
3. Quantization of free fields
4. Field interactions and Feynman diagrams
5. Observables: cross sections and decay widths
6. Elementary processes in quantum electrodynamics
7. The Standard Model of the electroweak and strong interactions.
   The Higgs boson

1. Introduction

Introduction | Why quantum fields?

- The Quantum Field Theory (QFT) is a synthesis of
  - Special Relativity and
  - Quantum Mechanics

  [It is possible to write a relativistic version of Schrödinger equation. In fact, he was the first to come up with what we call today the Klein-Gordon equation, that he later discarded because it couldn't describe properly the fine structure of hydrogen spectrum, so he had to settle for its non-relativistic limit]

- However, wave equations (relativistic or not) cannot explain particle number changing processes

  Moreover, relatistic wave equations suffer from *pathologies*:
  - negative probability densities
  - negative energy solutions
  - violation of the causality principle
    (non vanishing probability of particle propagation beyond the light cone)

Introduction | Why quantum fields?

- The QFT:
  - provides a natural framework for states with arbitrary number of particles (Fock space)
  - makes sense of negative energy solutions (antiparticles)
  - solves the causality problem
    (a particle propagating beyond the light cone is indistinguishable from its antiparticle going in opposite direction; both amplitudes cancel each other)
  - explains the spin-statistics connection
    (as a consequence of field quantization)
  - allows calculation of physical observables with very high accuracy in agreement with experiment (cross sections, lifetimes, magnetic moments, ...)

$$\left.\begin{array}{ll}\text{ex:} & g_e/2 = 1.001\,159\,652\,182\,032\,(720) \\ \text{th:} & \text{QED (5 loops!)}\end{array}\right\} \Rightarrow \alpha^{-1} = 137.035\,999\,150\,(33)$$

- We will use natural units $\hbar = c = 1$. Then, the following physical magnitudes have the same dimensions: [length] = [time] = [energy]$^{-1}$ = [mass]$^{-1}$
- A useful relation is:

$$\begin{aligned}\hbar c &= 197.326\,9631(49)\ \text{MeV fm}\\ \hbar c &\simeq 200\ \text{MeV fm} \Rightarrow 25\ \text{GeV}^{-2} \simeq 10^{-30}\ \text{m}^2 = 10\ \text{mbarn}\end{aligned} \tag{1}$$

1 fm $= 10^{-15}$ m (one Fermi, the order of the proton radius), 1 barn $= 10^{-24}$ cm^2

- Our sign convention for Minkowski metric is

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}, \quad g^{\mu}{}_{\nu} = g^{\mu\alpha}g_{\alpha\nu} = \delta^{\mu}_{\nu} = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix} \tag{2}$$

- We will use Einstein convention implying summation over repeated indices:

$$A_\mu B^\mu = \sum_{\mu=0}^{3} A_\mu B^\mu = g_{\mu\nu}A^\mu B^\nu = A^0B^0 - A^1B^1 - A^2B^2 - A^3B^3\ , \tag{3}$$

where $A^\mu = (A^0, \vec{A}) = (A^0, A^1, A^2, A^3)$ are contravariant vectors and $A_\mu = g_{\mu\nu}A^\nu = (A^0, -\vec{A}) = (A^0, -A^1, -A^2, -A^3) = (A_0, A_1, A_2, A_3)$ are covariant. In particular, $x^\mu = (x^0, \vec{x}) = (t, \vec{x})$ and

$$\partial_\mu = \frac{\partial}{\partial x^\mu}\ , \quad \partial^\mu = \frac{\partial}{\partial x_\mu}\ , \tag{4}$$

$$\Box = \partial_\mu\partial^\mu = \partial_0^2 - \nabla^2\ , \quad \nabla^2 = \partial_x^2 + \partial_y^2 + \partial_z^2\ . \tag{5}$$

Greek indices $(\mu, \nu, \ldots)$ take values 0,1,2,3. Latin indices $(i, j, \ldots)$ are reserved for spatial components. The four-momentum is then

$$p^\mu = \mathrm{i}\partial^\mu = (p^0, \vec{p}) = (E, \vec{p})\ , \quad p_\mu p^\mu = E^2 - \vec{p}^2 = m^2\ , \tag{6}$$

$$p^0 = \mathrm{i}\partial^0 = \mathrm{i}\frac{\partial}{\partial t}\ , \quad p^k = \mathrm{i}\partial^k = \mathrm{i}\frac{\partial}{\partial x_k} = -\mathrm{i}\frac{\partial}{\partial x^k} = -\mathrm{i}\partial_k \equiv -\mathrm{i}\nabla^k \tag{7}$$

Introduction — Notation, units and coventions

- We will use Heaviside-Lorentz units for electromagnetism in which the fine structure constant is

$$\alpha = \frac{e^2}{4\pi\hbar c} = 1/137.035\,999\,150 \,. \tag{8}$$

Then the electric charge unit for $\hbar = c = 1$ is $e = \sqrt{4\pi\alpha}$ (dimensionless), Maxwell equations read

$$\begin{aligned} \nabla \cdot \vec{E} = \rho \,, \quad & \nabla \times \vec{B} - \partial_t \vec{E} = \vec{j} \\ \nabla \cdot \vec{B} = 0 \,, \quad & \nabla \times \vec{E} + \partial_t \vec{B} = 0 \end{aligned} \tag{9}$$

and the Coulomb potential between two charges $Q_1 = eq_1$ and $Q_2 = eq_2$ is

$$V(r) = \frac{Q_1 Q_2}{4\pi r} = q_1 q_2 \frac{\alpha}{r} \,. \tag{10}$$

Lie groups

- A group is a set of elements G, not necessarily countable, with an internal operation satisfying the requirements: asssociativity, existence of a neutral element e and existence of an inverse element a^{-1} for every element a.
- The elements g of a Lie group depend in a continuous and differentiable way on a set of real parameters θ_a, $a = 1, \ldots, N$, namely $g(\theta)$, where:

$$\begin{aligned} g(0) &= e && \text{(the neutral element)} \\ g^{-1}(\theta) &= g(-\theta) && \text{(inverse element).} \end{aligned}$$

The Lie group is also a manifold. N is the group dimension.

- A subgroup is a subset of G which is also a group.

– An invariant (or normal) subgroup H is such that $\forall h \in H$ and $\forall g \in G$, $ghg^{-1} \in H$.

– A simple group is a nontrivial group whose only invariant subgroups are the trivial group and the group itself.

For instance, SU(n) is simple and U(n) is not simple.

Lie groups

- A representation R maps every element g to a linear operator $D_R(g)$ on a vector space, $g \mapsto D_R(g)$, such that:
  (i) $D_R(e) = \mathbb{1}$ (unit operator),
  (ii) $D_R(g_1)D_R(g_2) = D_R(g_1 g_2)$.
  On a vector space of finite dimension, g is represented by a matrix $n \times n$, $[D_R(g)]^i{}_j$, inducing a linear transformation on the vector space which is given by how it acts on the basis vectors $(\phi^1, \ldots, \phi^n)$, $\phi^i \mapsto [D_R(g)]^i{}_j \phi^j$.
- Two representations R and R' are equivalent if $\exists S$ such that $D_R(g) = S^{-1} D_{R'}(g) S$, $\forall g$. Namely, they are related by a change of basis.
- The representation R is reducible if it leaves invariant a nontrivial subspace. Otherwise it is irreducible (irrep). We say that R is completely reducible if $\forall g$, $D_R(g)$ can be written in blocks, i.e., one can choose a basis $\{\phi^i\}$ so that there are vector subspaces not mixing with the others under the action of the group. In this case, R can be written as the direct sum of several irreps: $D_R = D_1 \oplus D_2 \oplus \ldots$

Lie groups

- If an element of a Lie group is infinitely close to the identity then $D_R(\delta\theta) = \mathbb{1} - \mathrm{i}\delta\theta^a T^a_R$. The operators $T^a_R = \mathrm{i}\partial D_R/\partial\theta^a|_{\theta=0}$, with $a = 1, \ldots, N$, are the group generators in the representation R. The number of generators is the group dimension. For an arbitrary transformation: $D_R(\theta) = \exp\{-\mathrm{i}\theta^a T^a_R\}$.
- If D_R is a unitary representation (the inverse of each element is its adjoint) then the generators are Hermitian.
  And every unitary representation is completely reducible.
  Remember that physical observables are Hermitian operators.
- The generators satisfy the Lie algebra: $[T^a, T^b] = \mathrm{i} f^{abc} T^c$, where f^{abc} are the structure constants of the group, independent of the representation. In order to find the group representation it is enough to find the algebra representation.
- If G is Abelian, $[T^a, T^b] = 0$ and $\exp\{-\mathrm{i}\alpha^a T^a\}\exp\{-\mathrm{i}\beta^b T^b\} = \exp\{-\mathrm{i}(\alpha^c + \beta^c)T^c\}$. The irreps of an Abelian group are one-dimensional.

Lie groups

- Casimir operators are those commuting with every generator. They are a multiple of the identity and the proportionality constant λ is used to label the irreps.

– For instance, SU(2) (group of rotations in three dimensions) has three generators, the angular momentum operators J^k with $k = 1, 2, 3$,
satisfying the Lie algebra $[J^k, J^\ell] = \mathrm{i}\epsilon^{k\ell m} J^m$.
SU(2) has one Casimir operator: $\vec{J}^2 = (J^1)^2 + (J^2)^2 + (J^3)^2 = \lambda \mathbb{1}$, with $\lambda = j(j+1)$.
The irreps of SU(2) are labeled by $j = 0, \frac{1}{2}, 1, \ldots$ and have dimension $2j + 1$.
Here ϵ is the totally antisymmetric Levi-Civita tensor,

$$\epsilon^{ijk} = \begin{cases} +1 & \text{si } (ijk) \text{ is an even permutation of } (123), \\ -1 & \text{si } (ijk) \text{ is an odd permutation of } (123), \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

Lie groups

- A Lie group is compact if its parameter space is compact.
Examples: the group of rotations is compact; the group of translations is not.

– If a group is compact the parameter labeling the irreps takes discrete values (e.g. the spin j of the group of rotations) and if it is not compact it takes continuous values (e.g. the momentum p of space translations).

– The finite dimensional reps of a compact group are unitary.
The finite dimensional reps of a not compact, simple group are not unitary.[a]

- The Lorentz group, that we will review in the following, is a simple and non compact Lie group. Its finite-dimensional representations are not unitary and its unitaty representations are infinite-dimensional (Hilbert space of one particle).

[a] However, if the group is not simple the reps may be unitary or not. An example of a non compact, non simple group with unitary finite dimensional representations is the group of translations in one dimension. The group of boosts along one direction has non unitary representations. This latter group is a non invariant and non simple subgroup of the Lorentz group, which is simple.

The Lorentz group

- It is defined as the group of linear coordinate transformations

$$x^\mu \mapsto x'^\mu = \Lambda^\mu{}_\nu x^\nu, \quad \mu,\nu \in \{0,1,2,3\}, \quad x^\mu = (t,x,y,z) \tag{12}$$

preserving the quadratic form

$$x_\mu x^\mu = g_{\mu\nu} x^\mu x^\nu = t^2 - x^2 - y^2 - z^2. \tag{13}$$

Therefore, it is isomorphic to the group $O(1,3)$. Formally,

$$g_{\mu\nu} x'^\mu x'^\nu = g_{\mu\nu}(\Lambda^\mu{}_\rho x^\rho)(\Lambda^\nu{}_\sigma x^\sigma) = g_{\rho\sigma} x^\rho x^\sigma \quad (\forall x) \tag{14}$$

$$\Rightarrow g_{\rho\sigma} = g_{\mu\nu}\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma = (\Lambda^T)_\rho{}^\mu g_{\mu\nu}\Lambda^\nu{}_\sigma \tag{15}$$

$$\Rightarrow g = \Lambda^T g \Lambda. \tag{16}$$

The Lorentz group

- On the other hand, from the 00 component of (15),

$$1 = (\Lambda^0{}_0)^2 - \sum_{i=1}^{3} (\Lambda^i{}_0)^2 \Rightarrow (\Lambda^0{}_0)^2 \geq 1 \Rightarrow \begin{cases} \Lambda^0{}_0 \geq 1 \\ \Lambda^0{}_0 \leq -1 \end{cases} \tag{17}$$

and from (16),

$$(\det \Lambda)^2 = 1 \Rightarrow \det \Lambda = \pm 1. \tag{18}$$

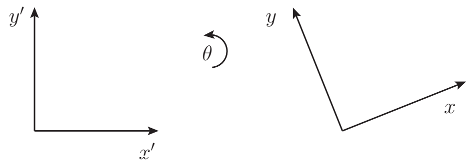
Therefore, we can distinguish four types of Lorentz transformations:

The Lorentz group

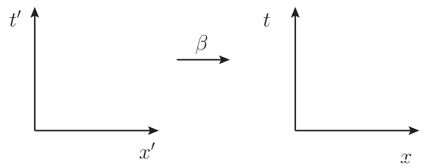
1. Orthochronous ($\Lambda^0{}_0 \geq 1$) proper ($\det \Lambda = +1$)

   They form a subgroup. It is isomorphic to SO(1,3). From now on, we will refer to this as the *Lorentz group*. Its elements are continuous transformations (Lie group) that can be connected with the identity by successive infinitesimal transformations. They are:

   - rotations in three space dimensions:

   - boosts (pure Lorentz transformations):

   The remaining transformations do not form a group and can be written as the product of inversions (discrete transformations) and orthochronous proper Lorentz transformations Λ_P. They are the following

The Lorentz group

2. Non orthochronous ($\Lambda^0{}_0 \leq -1$) proper ($\det \Lambda = +1$)

   Transformations of the kind:
   $\Lambda_P \times \{\mathrm{diag}(-,-,-,-), \mathrm{diag}(-,-,+,+), \mathrm{diag}(-,+,-,+), \mathrm{diag}(-,+,+,-)\}$.
   They include the total inversion, $\mathrm{diag}(-,-,-,-)$.

3. Orthochronous ($\Lambda^0{}_0 \geq 1$) improper ($\det \Lambda = -1$)

   Transformations of the kind:
   $\Lambda_P \times \{\mathrm{diag}(+,+,+,-), \mathrm{diag}(+,+,-,+), \mathrm{diag}(+,-,+,+), \mathrm{diag}(+,-,-,-)\}$.
   They include the space inversion, $\mathrm{diag}(+,-,-,-)$.

4. Non orthochronous ($\Lambda^0{}_0 \leq -1$) improper ($\det \Lambda = -1$)

   Transformations of the kind:
   $\Lambda_P \times \{\mathrm{diag}(-,-,-,+), \mathrm{diag}(-,-,+,-), \mathrm{diag}(-,+,-,-), \mathrm{diag}(-,+,+,+)\}$.
   They include the time inversion, $\mathrm{diag}(-,+,+,+)$.

The Lorentz group

- Let us see the number of parameters of the Lorentz group (of orthochronous proper transformations). Taking an arbitrary infinitesimal transformation $\Lambda^\mu{}_\nu = \delta^\mu_\nu + \omega^\mu{}_\nu$, equation (15) implies:

$$\begin{aligned} g_{\rho\sigma} = g_{\mu\nu}\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma &= g_{\mu\nu}(\delta^\mu_\rho + \omega^\mu{}_\rho)(\delta^\nu_\sigma + \omega^\nu{}_\sigma) \\ &= g_{\rho\sigma} + \omega_{\rho\sigma} + \omega_{\sigma\rho} + \mathcal{O}(\omega^2) \Rightarrow \omega_{\rho\sigma} = -\omega_{\sigma\rho}. \end{aligned} \qquad (19)$$

Therefore, ω is antisymmetric and has 6 independent parameters.

Any Λ can be written as the product of rotations (R), that can be parametrized by 3 angles $\theta \in [0, 2\pi]$ about the axes x, y, z counterclockwise, and boosts (L), that can be parametrized by the 3 components of the velocity $\beta \in (-1, 1)$ along the axes x, y, z:

$$\Lambda = RL$$

The Lorentz group

- In particular,

$$R_x = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c_\theta & -s_\theta \\ 0 & 0 & s_\theta & c_\theta \end{pmatrix}, \; R_y = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_\theta & 0 & s_\theta \\ 0 & 0 & 1 & 0 \\ 0 & -s_\theta & 0 & c_\theta \end{pmatrix}, \; R_z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_\theta & -s_\theta & 0 \\ 0 & s_\theta & c_\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad (20)$$

$$L_x = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; L_y = \begin{pmatrix} \gamma & 0 & \gamma\beta & 0 \\ 0 & 1 & 0 & 0 \\ \gamma\beta & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; L_z = \begin{pmatrix} \gamma & 0 & 0 & \gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\beta & 0 & 0 & \gamma \end{pmatrix}, \qquad (21)$$

with $c_\theta = \cos\theta$, $s_\theta = \sin\theta$ y $\gamma = 1/\sqrt{1-\beta^2}$.

The Lorentz group

- It is covenient to replace the velocity parameter β by the rapidity $\eta \in (-\infty, \infty)$

$$\eta = \frac{1}{2} \ln \frac{1+\beta}{1-\beta} \tag{22}$$

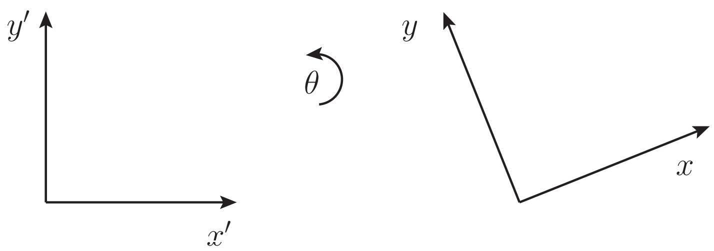
  which is an additive parameter, as is the angle θ.

  Namely, if we perform two boosts of rapidities η_A and η_B along the same direction $\hat{n}$ then $L_{\hat{n}}(\eta_A) L_{\hat{n}}(\eta_B) = L_{\hat{n}}(\eta_A + \eta_B)$.

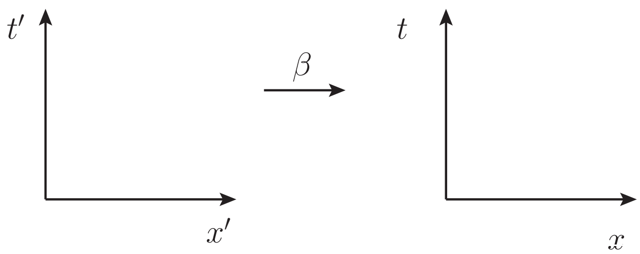
  This is very easy to check from the properties of the hyperbolic functions, since

$$\gamma = \cosh \eta, \quad \gamma\beta = \sinh \eta. \tag{23}$$

The Lorentz group



$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$



$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix}$$

$$\gamma = \cosh \eta \ , \quad \gamma\beta = \sinh \eta$$

The Lorentz group

- Let us find the generators algebra taking infinitesimal transformations:

$$R_x(\delta\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\delta\theta \\ 0 & 0 & \delta\theta & 1 \end{pmatrix} = \mathbb{1} - \mathrm{i}\delta\theta J^1 \Rightarrow J^1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathrm{i} \\ 0 & 0 & \mathrm{i} & 0 \end{pmatrix} \tag{24}$$

$$R_y(\delta\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \delta\theta \\ 0 & 0 & 1 & 0 \\ 0 & -\delta\theta & 0 & 1 \end{pmatrix} = \mathbb{1} - \mathrm{i}\delta\theta J^2 \Rightarrow J^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 & 0 \\ 0 & -\mathrm{i} & 0 & 0 \end{pmatrix} \tag{25}$$

$$R_z(\delta\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\delta\theta & 0 \\ 0 & \delta\theta & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \mathbb{1} - \mathrm{i}\delta\theta J^3 \Rightarrow J^3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} & 0 \\ 0 & \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{26}$$

The Lorentz group

$$L_x(\delta\eta) = \begin{pmatrix} 1 & \delta\eta & 0 & 0 \\ \delta\eta & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \mathbb{1} - \mathrm{i}\delta\eta K^1 \Rightarrow K^1 = \begin{pmatrix} 0 & \mathrm{i} & 0 & 0 \\ \mathrm{i} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{27}$$

$$L_y(\delta\eta) = \begin{pmatrix} 1 & 0 & \delta\eta & 0 \\ 0 & 1 & 0 & 0 \\ \delta\eta & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \mathbb{1} - \mathrm{i}\delta\eta K^2 \Rightarrow K^2 = \begin{pmatrix} 0 & 0 & \mathrm{i} & 0 \\ 0 & 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{28}$$

$$L_z(\delta\eta) = \begin{pmatrix} 1 & 0 & 0 & \delta\eta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \delta\eta & 0 & 0 & 1 \end{pmatrix} = \mathbb{1} - \mathrm{i}\delta\eta K^3 \Rightarrow K^3 = \begin{pmatrix} 0 & 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 & 0 \end{pmatrix} \tag{29}$$

The Lorentz group

- Note that, because the boost parameter space is not compact, the boost generators are not Hermitian ($(K^m)^\dagger = -K^m$).
- The Lie algebra of the Lorentz group is

$$[J^k, J^\ell] = \mathrm{i}\epsilon^{k\ell m} J^m, \quad [K^k, K^\ell] = -\mathrm{i}\epsilon^{k\ell m} J^m, \quad [J^k, K^\ell] = \mathrm{i}\epsilon^{k\ell m} K^m \quad (k, \ell, m \in \{1, 2, 3\}) \tag{30}$$

- We see that rotations close the algebra, since SU(2) is a subgroup of the Lorentz group. However, boosts are not a subgroup.
- It is convenient to rewrite these 6 generators as

$$A^m = \frac{1}{2}(J^m + \mathrm{i}K^m), \quad B^m = \frac{1}{2}(J^m - \mathrm{i}K^m). \tag{31}$$

  A^m y B^m are Hermitian and satisfy the Lie algebra:

$$[A^k, A^\ell] = \mathrm{i}\epsilon^{k\ell m} A^m, \quad [B^k, B^\ell] = \mathrm{i}\epsilon^{k\ell m} B^m, \quad [A^k, B^\ell] = 0. \tag{32}$$

The Lorentz group

- Then we see that the Lorentz group is locally isomorphic to the SU(2)×SU(2) group, because both have the same algebra.

▷ This allows us to label its irreps as (j_1, j_2), of dimension $(2j_1 + 1)(2j_2 + 1)$.

▷ Note that we have found finite-dimensional irreps of the Lorentz group, but they are not unitary, because it is not compact:

$$\Lambda = \exp\{-\mathrm{i}(\theta^m J^m + \eta^m K^m)\} \equiv \exp\{-\mathrm{i}(\vec{\theta} \cdot \vec{J} + \vec{\eta} \cdot \vec{K})\}, \tag{33}$$

$$\Lambda^{-1} = \exp\{\mathrm{i}(\vec{\theta} \cdot \vec{J} + \vec{\eta} \cdot \vec{K})\} \neq \Lambda^\dagger = \exp\{\mathrm{i}(\vec{\theta} \cdot \vec{J} - \vec{\eta} \cdot \vec{K})\}. \tag{34}$$

The Lorentz group

- Another way to write the Lorentz group generators is the following. We take as parameters the 6 independent elements of an antisymmetric matrix $\omega_{\mu\nu} = -\omega_{\nu\mu}$. Then the generators are the 6 independent components of the antisymmetric operator $J^{\mu\nu} = -J^{\nu\mu}$,

$$\Lambda = \exp\left\{-\frac{\mathrm{i}}{2}\omega_{\mu\nu}J^{\mu\nu}\right\} \tag{35}$$

(the factor $\frac{1}{2}$ compensates for the sum $\forall\mu,\nu$ instead of $\forall\mu<\nu$) with

$$J^k = \frac{1}{2}\epsilon^{k\ell m}J^{\ell m} \Rightarrow \begin{cases} J^1 = J^{23} = -J^{32} \\ J^2 = J^{31} = -J^{13} \\ J^3 = J^{12} = -J^{21} \end{cases} \tag{36}$$

$$K^k = J^{0k} = -J^{k0}. \tag{37}$$

The Lorentz group

- These parameters relate to angles and rapidities by

$$\theta^k = \frac{1}{2}\epsilon^{k\ell m}\omega^{\ell m} \Rightarrow \begin{cases} \theta^1 = \omega^{23} = -\omega^{32} = \omega_{23} = -\omega_{32} \\ \theta^2 = \omega^{31} = -\omega^{13} = \omega_{31} = -\omega_{13} \\ \theta^3 = \omega^{12} = -\omega^{21} = \omega_{12} = -\omega_{21} \end{cases} \tag{38}$$

$$\eta^k = \omega^{0k} = -\omega^{k0} = -\omega_{0k} = \omega_{k0}. \tag{39}$$

- The generators can be written in a covariant way as

$$(J^{\mu\nu})^{\rho}{}_{\sigma} = \mathrm{i}(g^{\mu\rho}\delta^{\nu}_{\sigma} - g^{\nu\rho}\delta^{\mu}_{\sigma}). \tag{40}$$

▷ The Lie algebra of these generators is

$$[J^{\mu\nu}, J^{\rho\sigma}] = \mathrm{i}(g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho}). \tag{41}$$

- We have just shown the four-dimensional representation of the Lorentz group, that served to define the group. We may ask whether it is irreducible (it is) or whether it is its smallest nontrivial representation (it is not, as we will see). This is the so called vector representation of the Lorentz group:

$$\mathbf{4}: \qquad \Lambda^\mu{}_\nu = \left[\exp\{-\mathrm{i}(\vec{\theta}\cdot\vec{J}+\vec{\eta}\cdot\vec{K})\}\right]^\mu{}_\nu = \left[\exp\left\{-\frac{\mathrm{i}}{2}\omega_{\alpha\beta}J^{\alpha\beta}\right\}\right]^\mu{}_\nu \tag{42}$$

A four-vector V^μ (or V_μ) is a vector of the invariant irreducible vector space on which Λ acts:

$$V^\mu \mapsto \Lambda^\mu{}_\nu V^\nu, \quad V_\mu \mapsto \Lambda_\mu{}^\nu V_\nu. \tag{43}$$

Note that $\Lambda^\mu{}_\nu$ and $\Lambda_\mu{}^\nu$ are equivalent representations, since they are related by a similarity transformation $S = g_{\mu\nu}$,

$$\Lambda^\mu{}_\nu = g^{\mu\rho}\Lambda_\rho{}^\sigma g_{\sigma\nu}. \tag{44}$$

It is customary to identify the term representation with that of representation space. In this sense, we say that V^μ and V_μ are equivalent irreps.

- V_μ is the vector associated to V^μ in the dual space, so the matrix $\Lambda_\mu{}^\nu$ is the inverse of $\Lambda^\mu{}_\nu$. In fact, using (15):

$$\Lambda_\tau{}^\nu\Lambda^\mu{}_\nu = \Lambda_\tau{}^\nu g^{\mu\rho}\Lambda_\rho{}^\sigma g_{\sigma\nu} = \Lambda_\tau{}^\nu g_{\nu\sigma}\Lambda_\rho{}^\sigma g^{\mu\rho} = g_{\tau\rho}g^{\mu\rho} = \delta^\mu_\tau. \tag{45}$$

- One can build higher dimensional representations by performing the tensor product $\mathbf{4}\otimes\mathbf{4}\otimes\cdots$. They are called tensor representations and their vectors are tensors with several indices (its number is called rank). Then, a tensor of two (contravariant) indices $T^{\mu\nu}$ transforms as:

$$\mathbf{4}\otimes\mathbf{4}: \qquad T^{\mu\nu} \mapsto \Lambda^\mu{}_{\mu'}\Lambda^\nu{}_{\nu'}T^{\mu'\nu'}. \tag{46}$$

The tensor-product representation is reducible. In particular, if $T^{\mu\nu}$ is symmetric (antisymmetric) its transformed tensor is also symmetric (antisymmetric). Moreover, the trace is invariant (scalar).[a]

[a] The trace $T = g_{\mu\nu}T^{\mu\nu} \mapsto g_{\mu\nu}\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma T^{\rho\sigma} = g_{\rho\sigma}T^{\rho\sigma} = T$, where we have used (15).

- In fact, rank-two tensors can be written as the direct sum of invariant irreducible subspaces:

$$\mathbf{4} \otimes \mathbf{4} = \mathbf{1} \oplus \mathbf{6} \oplus \mathbf{9}, \tag{47}$$

so any rank-two tensor can be decomposed in

$$T^{\mu\nu} = \frac{1}{4} g^{\mu\nu} T + A^{\mu\nu} + S^{\mu\nu}, \tag{48}$$

$$T = g_{\mu\nu} T^{\mu\nu} = T^{\mu}_{\ \mu} \quad \text{(trace)}, \tag{49}$$

$$A^{\mu\nu} = \frac{1}{2}(T^{\mu\nu} - T^{\nu\mu}) \quad \text{(antisymmetric part)}, \tag{50}$$

$$S^{\mu\nu} = \frac{1}{2}(T^{\mu\nu} + T^{\nu\mu}) - \frac{1}{4} g^{\mu\nu} T \quad \text{(traceless symmetric part)}. \tag{51}$$

- By the same reasoning as before, $T^{\mu\nu}$, $T^{\mu}_{\ \nu}$, $T_{\mu}^{\ \nu}$ and $T_{\mu\nu}$ are equivalent reducible representations of the Lorentz group.
- An example of a rank-two tensor is the metric tensor $g_{\mu\nu}$, which is also invariant from the definition of Lorentz transformation (15).

- An important irrep of any Lie group is the adjoint representation, whose dimension is equal to the number of generators, that can be built from the structure constants,

$$(T^a_{\text{adj}})^{bc} = -\mathrm{i} f^{abc}. \tag{52}$$

▷ One may show, in general, that the structure constants satisfy the Lie algebra of the group using the Jacobi identity:

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0, \tag{53}$$

replacing $A = T^a$, $B = T^b$, $C = T^c$, which implies

$$f^{abd} f^{cde} + f^{bcd} f^{ade} + f^{cad} f^{bde} = 0. \tag{54}$$

▷ In the case of the Lorentz group, locally isomorphic to SU(2)×SU(2), the adjoint representation is the antisymmetric tensor representation $A^{\mu\nu}$.
(The structure constants of SU(n) are antisymmetric in the three indices.)

- It is worth looking how the irreps of the Lorentz group transform under the subgroup of rotations. In general, the reducible representations can be written as the direct sum of several irreps of the rotation group, each one labeled by a value of the spin j (remember that they have dimension $2j+1$). Then,

$$V^\mu = (V^0, \vec{V}) \in \mathbf{4} \text{ under the Lorentz group,} \tag{55}$$

$$V^\mu \in 0 \oplus 1 \text{ labeled by } j = 0, 1 \text{ under the rotation group,} \tag{56}$$

i.e. V^0 is a scalar under rotations (spin 0) and $\vec{V}$ is a 3-vector (spin 1).
On the other hand,

$$T^{\mu\nu} \in \mathbf{4} \otimes \mathbf{4} = \mathbf{1} \oplus \mathbf{6} \oplus \mathbf{9} \text{ under Lorentz} \tag{57}$$

$$= (0 \oplus 1) \otimes (0 \oplus 1) = 0 \oplus (1 \oplus 1) \oplus (0 \oplus 1 \oplus 2) \text{ under rotations,} \tag{58}$$

where we have used that the direct product of irreps of the rotation group is

$$j_1 \otimes j_2 = |j_1 - j_2| \oplus |j_1 - j_2 + 1| \oplus \cdots \oplus |j_1 + j_2|. \tag{59}$$

▷ Therefore,

$$\mathbf{1}: \quad T \in 0 \qquad \text{(also a scalar under rotations)} \tag{60}$$

$$\mathbf{6}: \quad A^{\mu\nu} \in 1 \oplus 1 \qquad \begin{cases} A^{0i} & \text{(two independent 3-vector under} \\ \frac{1}{2}\epsilon^{ijk}A^{jk} & \text{rotations mixing under Lorentz)} \end{cases} \tag{61}$$

For instance, the electromagnetic tensor $F^{\mu\nu}$ contains two 3-vectors:
the electric field $E^i = -F^{0i}$ and the magnetic field $B^i = -\frac{1}{2}\epsilon^{ijk}F^{jk}$.
Another example are the generators themselves (36,37).

$$\mathbf{9}: \qquad S^{\mu\nu} \in 0 \oplus 1 \oplus 2 \qquad \begin{cases} S^{00} \\ S^{0i} \\ S^{ij} \text{ con } \sum_i S^{ii} = -S^{00} \end{cases} \tag{62}$$

▷ In general, a tensor $T^{\mu\nu\rho\cdots}$ with N indices contains spins $j = 0, 1, \ldots, N$.

- We have seen that the vector representation and all the tensor representations of the Lorentz group contain representations of integer spin j $(0, 1, \dots)$ under the rotation group.

- Strictly speaking, the representations of half-integer j $(\frac{1}{2}, \frac{3}{2}, \dots)$ are not valid, because for them $R^j(0) \neq R^j(2\pi) = -\mathbb{1}$. However, since the observables in quantum mechanics are quadratic in the wave function, a global minus sign is acceptable and we can allow them. Then the physically relevant rotation group is not SO(3) but SU(2) (both have the same algebra, and hence, the same irreps). The fundamental representation of SU(2) (group of 2×2 unitary matrices with unit determinant) has $j = \frac{1}{2}$ (dimension 2) and is called spinorial representation or spinor. The SU(2) generators in this representation are one half of the Pauli matrices:

$$J^k = \frac{1}{2}\sigma^k, \qquad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{63}$$

- Any SU(2) representation can be obtained from the tensor product of the spinors. For example,

$$\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1. \tag{64}$$

- Likewise, the representations (j_1, j_2) of the Lorentz group can be built from the tensor product of the spinorial representations $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$, both of dimension $(2j_1 + 1)(2j_2 + 1) = 2$.

  Their vectors are called Weyl spinors $\psi_L \in (\frac{1}{2}, 0)$, $\psi_R \in (0, \frac{1}{2})$ and they have two components.

  They are denoted left-handed and right-handed for reasons that we will see soon.

- Let us find the explicit form of the spinorial representations $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$:

$$\vec{A} = \frac{1}{2}(\vec{J}+\mathrm{i}\vec{K}), \quad \vec{B} = \frac{1}{2}(\vec{J}-\mathrm{i}\vec{K}) \Rightarrow \vec{J} = \vec{A}+\vec{B}, \quad \vec{K} = -\mathrm{i}(\vec{A}-\vec{B}) \tag{65}$$

and recalling (33) we have

$$\psi_L : \quad \vec{A} = \frac{\vec{\sigma}}{2}, \quad \vec{B} = 0 \Rightarrow \vec{J} = \frac{\vec{\sigma}}{2}, \quad \vec{K} = -\mathrm{i}\frac{\vec{\sigma}}{2}$$

$$\Lambda_L = \exp\left\{(-\mathrm{i}\vec{\theta}-\vec{\eta})\cdot\frac{\vec{\sigma}}{2}\right\} \tag{66}$$

$$\psi_R : \quad \vec{A} = 0, \quad \vec{B} = \frac{\vec{\sigma}}{2} \Rightarrow \vec{J} = \frac{\vec{\sigma}}{2}, \quad \vec{K} = \mathrm{i}\frac{\vec{\sigma}}{2}$$

$$\Lambda_R = \exp\left\{(-\mathrm{i}\vec{\theta}+\vec{\eta})\cdot\frac{\vec{\sigma}}{2}\right\} \tag{67}$$

- Note that $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ are conjugate representations:

$$\sigma^2\Lambda_L^*\sigma^2 = \Lambda_R. \tag{68}$$

To check it, use $\sigma^2\sigma^i\sigma^2 = -\sigma^{i*}$.

▷ We can then define the conjugate spinor of ψ_L, that transforms as a ψ_R, as follows:

$$\psi_L^c \equiv \mathrm{i}\sigma^2\psi_L^* \in (0,\tfrac{1}{2}) \quad \text{(the i factor is a convention)} \tag{69}$$

since $\sigma^2\psi_L^* \mapsto \sigma^2(\Lambda_L\psi_L)^* = \sigma^2\Lambda_L^*\sigma^2\sigma^2\psi_L^* = \Lambda_R(\sigma^2\psi_L^*)$.

▷ Then, using $\sigma^{2*} = -\sigma^2$, we must consistently define the conjugate of ψ_R, transforming as a ψ_L, as follows,

$$\psi_R^c \equiv -\mathrm{i}\sigma^2\psi_R^* \in (\tfrac{1}{2},0). \tag{70}$$

- It is worth mentioning that the spinorial representations are complex, because

$$\psi_L \mapsto \exp\left\{(-\mathrm{i}\vec{\theta} - \vec{\eta}) \cdot \frac{\vec{\sigma}}{2}\right\} \psi_L, \tag{71}$$

$$\psi_R \mapsto \exp\left\{(-\mathrm{i}\vec{\theta} + \vec{\eta}) \cdot \frac{\vec{\sigma}}{2}\right\} \psi_R. \tag{72}$$

  Although ψ_L and ψ_R may be real in a given reference frame they will be complex in another.

- However, in the vector representation and the higher rank tensor representations one can impose the condition to be real, $V_\mu^* = V_\mu$, $T_{\mu\nu}^* = T_{\mu\nu}$, etc., which is consistent for every reference frame because $\Lambda^\mu{}_\nu$ is real.

- By the way, the spinorial representation $(\frac{1}{2}, \frac{1}{2})$ has complex dimension 4. Its vectors are composed of two independent Weyl spinors $((\psi_L)_\alpha, (\xi_R)_\beta)$, $\alpha, \beta \in \{1,2\}$.

  One can see that

$$\xi_R^\dagger \sigma^\mu \psi_R \quad \text{and} \quad \xi_L^\dagger \overline{\sigma}^\mu \psi_L$$

  transform as contravariant four-vectors where

$$\xi_L \equiv -\mathrm{i}\sigma^2 \xi_R^*, \quad \psi_R \equiv \mathrm{i}\sigma^2 \psi_L^*, \quad \sigma^\mu \equiv (1, \vec{\sigma}), \quad \overline{\sigma}^\mu \equiv (1, -\vec{\sigma}).$$

- A field is a function of the spacetime coordinates with well defined transformation properties under the Lorentz group.

  In general, if the coordinates transform

$$x^\mu \mapsto x'^\mu = \Lambda^\mu{}_\nu x^\nu \qquad \text{(infinitesimally: } x'^\mu = x^\mu + \delta x^\mu\text{)} \tag{73}$$

  a field $\phi(x)$ (that may have or not Lorentz indices or others) transforms

$$\phi(x) \mapsto \phi'(x'). \tag{74}$$

- Our aim is to build Lorentz invariant field theories. For finding the representations of the Lorentz group in this space of functions we have to compare $\phi(x)$ with its infinitesimal transformation $\phi'(x) = \phi'(x' - \delta x)$:

$$\begin{aligned}\delta\phi(x) \equiv \phi'(x) - \phi(x) &= \phi'(x' - \delta x) - \phi(x) \\ &= \phi'(x') - \delta x^\rho \partial_\rho \phi(x) - \phi(x) \\ &= \phi'(x') - \phi(x) + \frac{\mathrm{i}}{2}\omega_{\mu\nu}(J^{\mu\nu})^\rho{}_\sigma x^\sigma \partial_\rho \phi(x) \\ &\equiv -\frac{\mathrm{i}}{2}\omega_{\mu\nu} J^{\mu\nu}_\phi \phi(x),\end{aligned} \tag{75}$$

  where $J^{\mu\nu}_\phi$ are the generators in the infinite-dimensional representation of the Lorentz group on the field ϕ. In the next-to-last step we have approximated $\partial_\rho\phi'(x)$ by $\partial_\rho\phi(x)$, since they differ at the next order in δx, and in the last step we have written

$$\delta x^\rho = -\frac{\mathrm{i}}{2}\omega_{\mu\nu}(J^{\mu\nu})^\rho{}_\sigma x^\sigma, \quad (J^{\mu\nu})^\rho{}_\sigma = \mathrm{i}(g^{\mu\rho}\delta^\nu_\sigma - g^{\nu\rho}\delta^\mu_\sigma). \tag{76}$$

Field representations — Scalars

- Under Lorentz transformations, scalar fields satisfy

$$\phi'(x') = \phi(x). \tag{77}$$

- Then, from (75),

$$\delta\phi(x) = \frac{\mathrm{i}}{2}\omega_{\mu\nu}(J^{\mu\nu})^{\rho}{}_{\sigma}x^{\sigma}\partial_{\rho}\phi(x) \equiv -\frac{\mathrm{i}}{2}\omega_{\mu\nu}L^{\mu\nu}\phi(x) \tag{78}$$

$$\Rightarrow L^{\mu\nu} = -(J^{\mu\nu})^{\rho}{}_{\sigma}x^{\sigma}\partial_{\rho} = \mathrm{i}(x^{\mu}\partial^{\nu} - x^{\nu}\partial^{\mu}). \tag{79}$$

- Recalling that $p^{\mu} = \mathrm{i}\partial^{\mu}$, we see that the generators read $L^{\mu\nu} = x^{\mu}p^{\nu} - x^{\nu}p^{\mu}$. In particular, the generator of rotations is the orbital angular momentum, as expected:

$$L^{i} = \frac{1}{2}\epsilon^{ijk}L^{jk} = \epsilon^{ijk}x^{j}p^{k}. \tag{80}$$

- Note that the finite-dimensional representations of the Lorentz group may be unitary and indeed this is unitary because the $L^{\mu\nu}$ are Hermitian.

Field representations — Weyl, Dirac and Majorana

- Under Lorentz transformations, Weyl fields satisfy

$$\psi_L(x) \mapsto \psi'_L(x') = \Lambda_L\psi_L(x), \quad \psi_R(x) \mapsto \psi'_R(x') = \Lambda_R\psi_R(x). \tag{81}$$

- Then, from (75) and focussing on $\psi_L(x)$,

$$\begin{aligned}\delta\psi_L(x) &= (\Lambda_L - \mathbb{1})\psi_L(x) + \frac{\mathrm{i}}{2}\omega_{\mu\nu}(J^{\mu\nu})^{\rho}{}_{\sigma}x^{\sigma}\partial_{\rho}\psi_L(x)\\ &= -\frac{\mathrm{i}}{2}\omega_{\mu\nu}S_L^{\mu\nu}\psi_L(x) - \frac{\mathrm{i}}{2}\omega_{\mu\nu}L^{\mu\nu}\psi_L(x) \equiv -\frac{\mathrm{i}}{2}\omega_{\mu\nu}J_L^{\mu\nu}\psi_L(x),\end{aligned} \tag{82}$$

where we have applied (78) to the second term and we have replaced

$$\Lambda_L - \mathbb{1} \equiv -\frac{\mathrm{i}}{2}\omega_{\mu\nu}S_L^{\mu\nu} = -\mathrm{i}(\vec{\theta}\cdot\vec{J} + \vec{\eta}\cdot\vec{K}), \quad \vec{J} = \frac{\vec{\sigma}}{2}, \quad \vec{K} = -\mathrm{i}\frac{\vec{\sigma}}{2}. \tag{83}$$

- Therefore, the generators on this Weyl field representation are $J_L^{\mu\nu} = L^{\mu\nu} + S_L^{\mu\nu}$. In particular, the rotation generators (total angular momentum) are $J_L^i = L^i + S^i$, which have two contributions: the orbital part, $L^i = \epsilon^{ijk}x^j p^k$, and the one due to spin, $S^i = \frac{1}{2}\sigma^i$.

Field representations — Weyl, Dirac and Majorana

- The boost generators are $J_L^{0k} = L^{0k} - \frac{i}{2}\sigma^k$, which are *not* Hermitian, so the infinite-dimensional representation of the Lorentz group on Weyl fields ψ_L is not unitary.

- Likewise, check that for $\psi_R(x)$,

$$\delta\psi_R(x) = -\frac{i}{2}\omega_{\mu\nu}J_R^{\mu\nu}\psi_R(x), \quad J_R^{\mu\nu} = L^{\mu\nu} + S_R^{\mu\nu}, \tag{84}$$

  where

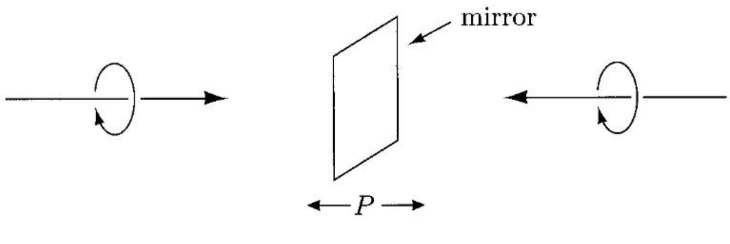
$$\Lambda_R - \mathbb{1} \equiv -\frac{i}{2}\omega_{\mu\nu}S_R^{\mu\nu} = -i(\vec\theta\cdot\vec J + \vec\eta\cdot\vec K), \quad \vec J = \frac{\vec\sigma}{2}, \quad \vec K = i\frac{\vec\sigma}{2}. \tag{85}$$

  The rotation generators, $J_R^i = L^i + S^i$, are the same as for $\psi_L(x)$. The boost generators are $J_R^{0k} = L^{0k} + \frac{i}{2}\sigma^k$, also not Hermitian, so the infinite-dimensional representation of the Lorentz group on Weyl fields ψ_R is not unitary either.

Field representations — Weyl, Dirac and Majorana

- Note that under a space inversion, denoted parity transformation (excluded from our definition of the Lorentz group),

$$(t,\vec x) \mapsto (t,-\vec x) \Rightarrow \vec\beta \mapsto -\vec\beta \Rightarrow \vec J \mapsto \vec J, \quad \vec K \mapsto -\vec K \Rightarrow \vec A \leftrightarrow \vec B \tag{86}$$



- This means that the representation (j_1, j_2) of the Lorentz group is not a valid representation when parity is included, unless $j_1 = j_2$, since the parity-transformed of a vector in (j_1, j_2) is a vector in (j_2, j_1). In particular, Weyl spinors, regardless of whether they are in $(\frac{1}{2}, 0)$ or $(0, \frac{1}{2})$, do not form invariant subspaces under parity.

- However, we can define the Dirac field, of four complex components:

$$\psi(x) = \begin{pmatrix} \psi_L(x) \\ \psi_R(x) \end{pmatrix} \tag{87}$$

that under Lorentz transformations (orthochronous, proper), $x^\mu \mapsto x'^\mu = \Lambda^\mu{}_\nu x^\nu$,

$$\psi(x) \mapsto \psi'(x') = \Lambda_D \psi(x), \quad \Lambda_D = \begin{pmatrix} \Lambda_L & 0 \\ 0 & \Lambda_R \end{pmatrix} \tag{88}$$

and under parity, $x^\mu = (t, \vec{x}) \mapsto \tilde{x}^\mu = (t, -\vec{x})$,

$$\psi(x) \mapsto \psi'(\tilde{x}) = \begin{pmatrix} \psi_R(\tilde{x}) \\ \psi_L(\tilde{x}) \end{pmatrix} = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \psi(\tilde{x}). \tag{89}$$

- The charge conjugate of a Dirac spinor is another Dirac spinor,

$$\psi^c = \begin{pmatrix} \psi_R^c \\ \psi_L^c \end{pmatrix} = \begin{pmatrix} -\mathrm{i}\sigma^2 \psi_R^* \\ \mathrm{i}\sigma^2 \psi_L^* \end{pmatrix} = -\mathrm{i} \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix} \psi^* \tag{90}$$

and, of course, $(\psi^c)^c = \psi$. Note that the coordinates x^μ do not change under charge conjugation.

- Dirac fields, not Weyl fields, are the basic objects in field theories which are invariant under parity, like QED (quantum electrodynamics) and QCD (quantum chromodynamics).

Field representations | Weyl, Dirac and Majorana

- Finally, a Majorana spinor is a Dirac spinor with non-independent ψ_L and ψ_R components,

$$\psi_R = \zeta \mathrm{i}\sigma^2 \psi_L^* \Rightarrow \psi_M = \begin{pmatrix} \psi_L \\ \zeta \mathrm{i}\sigma^2 \psi_L^* \end{pmatrix}, \quad |\zeta|^2 = 1. \tag{91}$$

A Majorana spinor has two degrees of freedom, like a Weyl spinor, but it is charge self-conjugate,

$$\psi_M^c = \begin{pmatrix} \zeta^* \psi_L \\ \mathrm{i}\sigma^2 \psi_L^* \end{pmatrix} = \zeta^* \psi_M. \tag{92}$$

Field representations | vector

- Under Lorentz transformations, a vector field satisfies

$$V^\mu(x) \mapsto V'^\mu(x') = \Lambda^\mu{}_\nu V^\nu(x). \tag{93}$$

- Then, from (75) and (78),

$$\delta V^\mu(x) = (\Lambda^\mu{}_\nu - \delta^\mu_\nu) V^\nu(x) - \frac{\mathrm{i}}{2}\omega_{\rho\sigma} L^{\rho\sigma} V^\mu(x) \equiv -\frac{\mathrm{i}}{2}\omega_{\rho\sigma} J_V^{\rho\sigma} V^\mu(x). \tag{94}$$

- Writing $J_V^{\rho\sigma} = L^{\rho\sigma} + S_V^{\rho\sigma}$, as before, we see that

$$\Lambda^\mu{}_\nu - \delta^\mu_\nu \equiv -\frac{\mathrm{i}}{2}\omega_{\rho\sigma}(S_V^{\rho\sigma})^\mu{}_\nu = -\frac{\mathrm{i}}{2}\omega_{\rho\sigma}(J^{\rho\sigma})^\mu{}_\nu \Rightarrow S_V^{\rho\sigma} = J^{\rho\sigma}. \tag{95}$$

Poincaré group

- The Poincaré group includes Lorentz transformations and spacetime translations,

$$x^\mu \mapsto x'^\mu = x^\mu + a^\mu. \tag{96}$$

- Taking an infinitesimal translation $a^\mu = \epsilon^\mu$,

$$\begin{aligned} x'^\mu \equiv (\mathbb{1} - \mathrm{i}\epsilon_\rho P^\rho)x^\mu &\Rightarrow \delta x^\mu = \epsilon^\mu = -\mathrm{i}\epsilon_\rho P^\rho x^\mu \\ &\Rightarrow P^\rho = \mathrm{i}\partial^\rho. \end{aligned} \tag{97}$$

  We see, as expected, that the generators of translations are the 4 components of the four-momentum operator P^μ. Then, a general translation reads $\exp\{-\mathrm{i}a_\mu P^\mu\}$.

Poincaré group

- The Poincaré algebra, written in a covariant fashion, is

$$[P^\mu, P^\nu] = 0, \tag{98}$$

$$[P^\mu, J^{\rho\sigma}] = \mathrm{i}(g^{\mu\rho}P^\sigma - g^{\mu\sigma}P^\rho), \tag{99}$$

$$[J^{\mu\nu}, J^{\rho\sigma}] = \mathrm{i}(g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho}). \tag{100}$$

- The last line corresponds to the algebra of the Lorentz subgroup (41). Translations are also a subgroup. It is convenient to write explicitly the commutation relations of the generators of translations, rotations and boosts:

$$[P^0, J^k] = 0, \tag{101}$$

$$[P^k, J^\ell] = \mathrm{i}\epsilon^{k\ell m}P^m, \tag{102}$$

$$[P^0, K^k] = \mathrm{i}P^k, \tag{103}$$

$$[P^k, K^\ell] = \mathrm{i}P^0\delta^{k\ell}. \tag{104}$$

Poincaré group

- Note that, since the Hamiltonian is $H = P^0$ (generator of time translations), we have that $[H, P^k] = [H, J^k] = 0$ but $[H, K^k] \neq 0$. This does *not* mean that only linear momentum and angular momentum are conserved quatities, because K^i depends explicitly on time, and

$$\frac{\mathrm{d}}{\mathrm{d}t} K^k = \mathrm{i}[H, K^k] + \frac{\partial}{\partial t} K^k = 0. \tag{105}$$

  As a consequence, there are also conserved quantities associated to boosts, as we will see in next chapter when we will study the Noether's theorem.

Poincaré group | Field representations

- We have seen already that fields form infinite-dimensional represenations of the Lorentz group, with generators

$$J^{\mu\nu} = L^{\mu\nu} + S^{\mu\nu}, \tag{106}$$

  where $L^{\mu\nu} = \mathrm{i}(x^\mu \partial^\nu - x^\nu \partial^\mu)$ and $S^{\mu\nu}$ depends on the field type (scalar, spinorial, ...)

- Let us find now the representation of translations. To that end, we impose that every field component, whether it is tensorial or spinorial, must satisfy

$$\phi'(x') = \phi(x), \quad x'^\mu = x^\mu + a^\mu. \tag{107}$$

  Then, performing an infinitesimal translation $a^\mu = \epsilon^\mu$,

$$\delta\phi(x) = \phi'(x' - \epsilon) - \phi(x) = \phi'(x') - \epsilon^\mu \partial_\mu \phi(x) - \phi(x) = -\epsilon^\mu \partial_\mu \phi(x). \tag{108}$$

  Therefore, comparing this expression to

$$\phi'(x' - \epsilon) = \exp\{-\mathrm{i}(-\epsilon_\mu) P^\mu\} \phi'(x') \Rightarrow \delta\phi(x) = \mathrm{i}\epsilon_\mu P^\mu \phi(x) \tag{109}$$

  we have $P^\mu = \mathrm{i}\partial^\mu$.

Poincaré group | Field representations

- To verify that our findings are consistent, we can check the commutation rules (99) using the representation on fields of the Lorentz group generators (106) and the generators of translations (109) taking into account that $S^{\mu\nu}$ is independent of the spacetime coordinates, and hence commutes with ∂^μ,

$$[P^\mu, J^{\rho\sigma}] = [P^\mu, L^{\rho\sigma}] = [\mathrm{i}\partial^\mu, \mathrm{i}(x^\rho\partial^\sigma - x^\sigma\partial^\rho)] \\ = -(g^{\mu\rho}\partial^\sigma - g^{\mu\sigma}\partial^\rho) = \mathrm{i}(g^{\mu\rho}P^\sigma - g^{\mu\sigma}P^\rho), \qquad (110)$$

where we have applied the rule $[A, BC] = [A, B]C + B[A, C]$ and the identity $[\partial^\mu, x^\nu] = g^{\mu\nu}$.

Poincaré group | Representation on one-particle states

- Now we have all we need to build field Lagrangians invariant under Poincaré transformations. We will see that upon quantization the fields create and annihilate particles (and antiparticles).

▷ It is then useful to identify the Poinciaré invariant Hilbert space of one-particle states, namely what are the irreps of the Poincaré group labeled by the Casimir operators whose vectors are identified by quantum number which are the eigenvalues of a set of commuting generators,

$$|\psi\rangle = |\vec{p}, j_3, \dots\rangle$$

- These irreps must be unitary to guarantee that the scalar products of states remain invariant under changes of reference frame,

$$\langle\psi_1|\psi_2\rangle = \langle\psi_1|\,\mathcal{P}^\dagger\mathcal{P}\,|\psi_2\rangle$$

Therefore, the Poincaré transformations $\mathcal{P}$ are represented by unitary operators on this space, and the generators J^i (rotations), K^i (boosts) and P^μ (translations) are given by Hermitian operators.

- The Poincaré group is not compact, so its unitary representations are infinite-dimensional. That is why the Hilbert space of one-particle states has infinite dimension.

▷ However the field representations are not necessarily unitary, as we have seen. Hence, it is important not to confuse representations on fields with representations on the Hilbert space of one particle.

- The Poincaré group has two Casimir operators:

$$m^2 = P_\mu P^\mu \quad \text{and} \quad W_\mu W^\mu, \tag{111}$$

where W^μ is the Pauli-Lubanski vector defined by

$$W^\mu = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma} J_{\nu\rho} P_\sigma. \tag{112}$$

- Both operatos commute, because

$$\begin{aligned}
[W^\mu, P^\alpha] &= -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}[J_{\nu\rho}P_\sigma, P^\alpha] \\
&= -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}(J_{\nu\rho}[P_\sigma, P^\alpha] + [J_{\nu\rho}, P^\alpha]P_\sigma) \\
&= -\frac{\mathrm{i}}{2}\epsilon^{\mu\nu\rho\sigma}(g^\alpha_\rho P_\nu - g^\alpha_\nu P_\rho)P_\sigma \\
&= -\frac{\mathrm{i}}{2}\epsilon^{\mu\nu\alpha\sigma}P_\nu P_\sigma + \frac{\mathrm{i}}{2}\epsilon^{\mu\alpha\rho\sigma}P_\rho P_\sigma = 0.
\end{aligned} \tag{113}$$

and they also commute with every Poincaré generator, so they are Casimir operators.

- Moreover $m^2 = P_\mu P^\mu$ and $W_\mu W^\mu$ are Lorentz invariant, so we can use their eigenvalues to label the irreps, and they are the same in any reference frame. We must distinguish two cases (**Wigner classification**):

Poincaré group | Representation on one-particle states

- $\boxed{\text{Case } m \neq 0}$

  It is then simpler to work in the rest frame, $p^\mu = (m, 0, 0, 0)$. Then,

$$\left.\begin{aligned} W^0 &= 0 \\ W^i &= -\frac{m}{2}\epsilon^{ijk0}J^{jk} = \frac{m}{2}\epsilon^{0ijk}J^{jk} = \frac{m}{2}\epsilon^{ijk}J^{jk} = mJ^i \end{aligned}\right\} \Rightarrow W_\mu W^\mu = -m^2 j(j+1). \tag{114}$$

  Therefore, the $\boxed{\text{irreps are labeled by } m, j}$ (mass and spin) and the vectors are labeled by $|j_3 = -j \dots j; \dots\rangle$.

▷ We see that massive particles of spin j have $\boxed{2j+1 \text{ degrees of freedom}}$
This is because, once we have performed a boost to take the massive particle to the reference frame where it has a four-moment $p^\mu = (m, 0, 0, 0)$, we still have total freedom to rotate the system in three dimensions.

▷ We say that SU(2) is the *little group* (set of Lorentz transformations leaving invariant a given election of p^μ).

Poincaré group | Representation on one-particle states

- $\boxed{\text{Case } m = 0}$

  In this case the rest frame does not exist. We can choose one where $p^\mu = (\omega, 0, 0, \omega)$, describing a massless particle moving along the z axis. Then,

$$\left.\begin{aligned} -W^0 = W^3 &= \omega J^3 \\ W^1 &= \omega(J^1 + K^2) \\ W^2 &= \omega(J^2 - K^1) \end{aligned}\right\} \Rightarrow \quad W_\mu W^\mu = -\omega^2[(J^1+K^2)^2 + (J^2-K^1)^2]. \tag{115}$$

  Now the *little group* is SO(2) (rotations in the plane perpendicular to the motion), whose $\boxed{\text{irreps}}$ are $\boxed{\text{one-dimensional}}$ (Abelian group) and are labeled by a number $h \in \{0, \pm\frac{1}{2}, \pm 1, \dots\}$ called helicity (projection of the angular momentum along the direction of motion):[a]

$$h = \hat{p} \cdot \vec{J}. \tag{116}$$

[a] The elements of SO(2) in the irrep h are given by $R(\theta) = \exp\{-\mathrm{i}h\theta\}$.

- $\boxed{\text{Case } \boldsymbol{m = 0}}$ (cont'd)

▷ Note that $h = j_3$ for our choice of direction of motion, and $j_3 = \pm j$.

▷ The irreps h and $-h$ are distinct (they do not mix under Poincaré transformations), although in theories symmetric under P the corresponding massless particles are given the same name. We just say that they are the same particle in two different helicity states.

▷ Then we speak of 'a photon' ($m = 0, j = 1$) that can be right-handed or left-handed if $h = \pm 1$, respectively. The photon is a massless particle of spin 1, but the state with $j_3 = 0$ does not exist.

▷ Likewise, we will see that the massless Weyl fields ψ_L and ψ_R ($m = 0, j = \frac{1}{2}$) have helicities $h = -\frac{1}{2}$, $h = +\frac{1}{2}$, respectively.
They represent different particles if the theory is not symmetric under parity (e.g. in the SM the neutrino is masslees and it is ν_L; the ν_R could simply not exist).

- Two final comments:

▷ We can always speak of helicity as the projection of the angular momentum along the direction of motion, but it is an invariant quantity under Poincaré transformations only if the particle is massless.

Sometimes we use the term *chirality*, to refer to a massless particle transforming as the left-handed or the right-handed representation of the Poincaré group. The chirality coincides with the helicity for massless particles.

▷ The embedding of unitary representations of the Poincaré group (particles) into a field theory is not trivial. For example, the vector field $V_\mu(x)$ describes both spin 0 and spin 1 (too many degrees of freedom).

  – To construct a spin 1 unitary field theory of massive paticles we will have to choose carefully the Lagrangian so that the spin 0 component of the field is never excited.

  – And for spin 1 massless particles we have to choose a Lagrangian that propagates only transversely polarized states (left and right helicities). This is achieved by introducing the gauge invariance, an invariance under parametrizations related to charge conservation at the origin of interactions.

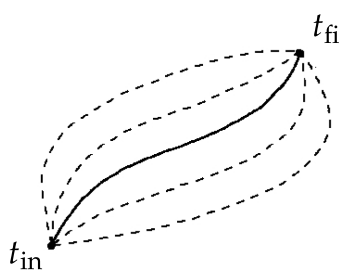
2. Classical field theory

Euler-Lagrange equations | Systems of N particles

- Let us first review the basic principle of classical mechanics for a system of N particles in the Lagrangian formalism. This system has $3N$ degrees of freedom described by a set of coordinates $q_i(t), i = 1, 2, \ldots, 3N$.

- The Lagrangian L is a function of the q_i and and their time derivatives $\dot{q}_i$, $L = L(q, \dot{q})$. In general, $L(q, \dot{q}) = \sum_i \frac{1}{2} m_i \dot{q}_i^2 - V(q)$ (kinetic term minus potential). We will assume that the system is conservative, so the Lagrangian does not depend explicitly on time. The action S is defined as

$$S = \int \mathrm{d}t \; L(q, \dot{q}). \tag{1}$$



Euler-Lagrange equations | Systems of N particles

- The principle of least action states that the trajectory of the system in the configuration space from an initial state $q_{\rm in} = q(t_{\rm in})$ to a final one $q_{\rm fi} = q(t_{\rm fi})$, both fixed, is an extreme of the action (usually a minimum):

$$\delta S = \delta \int_{t_{\rm in}}^{t_{\rm fi}} \mathrm{d}t\; L(q,\dot q) = \int_{t_{\rm in}}^{t_{\rm fi}} \mathrm{d}t\; \delta L(q,\dot q) = 0. \tag{2}$$

We can write

$$\delta L = \sum_i \left[\frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot q_i}\delta \dot q_i \right] = \sum_i \left[\frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot q_i}\frac{\mathrm{d}}{\mathrm{d}t}\delta q_i \right], \tag{3}$$

where we have used

$$\delta \dot q_i = \frac{\partial \dot q_i}{\partial \alpha}\delta\alpha = \frac{\partial}{\partial \alpha}\left(\frac{\mathrm{d}q_i}{\mathrm{d}t}\right)\delta\alpha = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial q_i}{\partial \alpha}\right)\delta\alpha = \frac{\mathrm{d}}{\mathrm{d}t}\delta q_i \tag{4}$$

with α being a set of parameters such that $q_i = q_i(\alpha, t)$ is sufficiently smooth, in such a way that the derivatives with respecto to α and t commute, because we can discretize both variations.

Euler-Lagrange equations | Systems of N particles

- On the other hand, integrating by parts:[a]

$$\int_{t_{\rm in}}^{t_{\rm fi}} \mathrm{d}t\; \frac{\partial L}{\partial \dot q_i}\frac{\mathrm{d}}{\mathrm{d}t}\delta q_i = \cancel{\left[\frac{\partial L}{\partial \dot q_i}\delta q_i\right]_{t_{\rm in}}^{t_{\rm fi}}} - \int_{t_{\rm in}}^{t_{\rm fi}} \mathrm{d}t\; \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot q_i}\delta q_i. \tag{5}$$

Therefore,

$$\delta S = \int_{t_{\rm in}}^{t_{\rm fi}} \mathrm{d}t\; \sum_i \left[\frac{\partial L}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot q_i} \right]\delta q_i = 0, \quad \forall \delta q_i \tag{6}$$

$$\Rightarrow \boxed{\frac{\partial L}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot q_i} = 0} \quad \text{(Euler-Lagrange equations)} \tag{7}$$

[a] In fact: $\displaystyle\int_a^b \mathrm{d}t\; u\frac{\mathrm{d}v}{\mathrm{d}t} = [uv]_a^b - \int_a^b \mathrm{d}t\; v\frac{\mathrm{d}u}{\mathrm{d}t}$.

Euler-Lagrange equations | Systems of N particles

- Remember that in the Hamiltonian formalism the basic object is

$$H(p,q) = \sum_i p_i \dot{q}_i - L, \quad p_i = \frac{\partial L}{\partial \dot{q}_i}. \tag{8}$$

- Taking the differential of this expression we get

$$\begin{aligned} \mathrm{d}H &= \sum_i \left\{ \dot{q}_i \mathrm{d}p_i + p_i \mathrm{d}\dot{q}_i - \left(\frac{\partial L}{\partial q_i} \mathrm{d}q_i + \frac{\partial L}{\partial \dot{q}_i} \mathrm{d}\dot{q}_i \right) \right\} &(9) \\ &= \sum_i \{ \dot{q}_i \mathrm{d}p_i - \dot{p}_i \mathrm{d}q_i \} &(10) \end{aligned}$$

where we have used the Euler-Lagrange equations (7) and the momentum definition in (8). This shows that the Hamiltonian H is a function of p and q. The previous expression leads to:

$$\boxed{\dot{q}_i = \frac{\partial H}{\partial p_i}\,, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}} \quad \text{(Hamilton equations)} \tag{11}$$

Euler-Lagrange equations | Systems of N particles

- Defining the Poisson bracket of whatever two dynamical variables f_1 and f_2

$$[f_1, f_2]_P = \sum_i \left\{ \frac{\partial f_1}{\partial q_i} \frac{\partial f_2}{\partial p_i} - \frac{\partial f_1}{\partial p_i} \frac{\partial f_2}{\partial q_i} \right\} \tag{12}$$

it is easy to check that

$$[q_r, p_s]_P = \delta_{rs} \tag{13}$$

and the Hamilton equations can be rewritten as

$$\dot{q}_r = [q_r, H]_P\,, \quad \dot{p}_r = [p_r, H]_P \tag{14}$$

and in general for any dynamical variable f one has

$$\dot{f} \equiv \frac{\mathrm{d}f}{\mathrm{d}t} = [f, H]_P + \frac{\partial f}{\partial t} \tag{15}$$

where the term $\partial f / \partial t$ is non vanishing if f depends explicitly on time.

Euler-Lagrange equations | Continuous medium

- Assume now that, instead of a system with a finite number of degrees of freedom, we have a continuous medium. Then the system is described by a field $\phi(x)$,

$$q_i(t) \longrightarrow \phi(t,\vec{x}) = \phi(x) \tag{16}$$

and its dymamics is described by a Lagrangian,

$$L = \int \mathrm{d}^3x \; \mathcal{L}(\phi, \partial_\mu\phi). \tag{17}$$

- From now on, we will call Lagrangian to the Lagrangian density $\mathcal{L}$.
Then the action is

$$S = \int \mathrm{d}t \; L = \int \mathrm{d}^4x \; \mathcal{L}(\phi, \partial_\mu\phi). \tag{18}$$

Euler-Lagrange equations | Continuous medium

- The principle of least action reads:

$$\delta S = \int \mathrm{d}^4x \left[\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi)\right] = \int \mathrm{d}^4x \left[\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right]\delta\phi = 0, \tag{19}$$

where the boundary condition now is not $q_i(t_{\mathrm{in}})$ and $q_i(t_{\mathrm{fi}})$ fixed but fields remain constant in the infinity, since

$$\int \mathrm{d}^4x \, \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi) = \cancel{\int \mathrm{d}^4x \, \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right]} - \int \mathrm{d}^4x \, \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi \tag{20}$$

and we have used the Stokes theorem,

$$\int_V \mathrm{d}^4x \, \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right] = \int_\Sigma \mathrm{d}A \; n_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right] \tag{21}$$

(n^μ is the vector normal to the surface) and the boundary condition

$$\delta\phi|_\Sigma = 0. \tag{22}$$

Euler-Lagrange equations | Continuous medium

- Then we have:

$$\boxed{\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = 0} \quad \text{(Euler-Lagrange equation for field } \phi\text{)}. \tag{23}$$

▷ Note that if one adds to the Lagrangian a term of the form (total derivative):

$$\mathcal{L} \longrightarrow \mathcal{L} + \partial_\mu K^\mu(\phi) \tag{24}$$

the equations of motion do not change due to the boundary condition imposing that fields are constant in the infinity, because using again the Stokes theorem,

$$\int_V \mathrm{d}^4x\, \partial_\mu K^\mu = \int_\Sigma \mathrm{d}A\, n_\mu K^\mu, \tag{25}$$

a constant term is added to the action and equation $\delta S = 0$ remains untouched.

Euler-Lagrange equations | Continuous medium

- In the Hamiltonian formalism we define the conjugate momentum of field ϕ,

$$\Pi(x) = \frac{\partial\mathcal{L}(x)}{\partial(\partial_0\phi)} \tag{26}$$

and the Hamiltonian density (or Hamiltonian in short),

$$\mathcal{H}(x) = \Pi(x)\partial_0\phi(x) - \mathcal{L}(x) \tag{27}$$

with

$$H = \int \mathrm{d}^3x\, \mathcal{H}(x). \tag{28}$$

Noether's theorem

- We will discuss the relation between coninuous symmetries and conservation laws in classical field theory.

  A global infinitesimal transformation of fields ϕ_i, i.e. with $|\epsilon^a| \ll 1$ independent of the coordinates, reads generically

$$\boxed{\phi_i(x) \mapsto \phi_i'(x') \equiv \phi_i(x) + \epsilon^a F_{i,a}(\phi, \partial\phi)} \tag{29}$$

  and the transformation of coordinates,

$$\boxed{x^\mu \mapsto x'^\mu = x^\mu + \delta x^\mu \equiv x^\mu + \epsilon^a A_a^\mu(x)} \tag{30}$$

  where "i" can be one index, two, ... or none.

- We say that this transformation is a symmetry if it leaves the equations of motion invariant, i.e. the action does not change:

$$S(\phi) \mapsto S(\phi') = S(\phi). \tag{31}$$

Noether's theorem

- Then, to first order in δx,

$$0 = S(\phi') - S(\phi) = \int \mathrm{d}^4x'\ \mathcal{L}'(x') - \int \mathrm{d}^4x\ \mathcal{L}(x) = \int \mathrm{d}^4x \left[\mathcal{L}'(x') - \mathcal{L}(x) + \partial_\mu \delta x^\mu \mathcal{L}(x)\right] \tag{32}$$

  where we have used

$$\mathrm{d}^4x' = \left|\frac{\partial x'^\mu}{\partial x^\nu}\right| \mathrm{d}^4x, \qquad \left|\frac{\partial x'^\mu}{\partial x^\nu}\right| = \begin{vmatrix} 1 + \dfrac{\partial \delta x^0}{\partial x^0} & \dfrac{\partial \delta x^0}{\partial x^1} & \dots \\ \dfrac{\partial \delta x^1}{\partial x^0} & 1 + \dfrac{\partial \delta x^1}{\partial x^1} & \dots \\ \dots & \dots & \dots \end{vmatrix} = 1 + \partial_\mu \delta x^\mu + \mathcal{O}(\delta x)^2. \tag{33}$$

  Then, since

$$\mathcal{L}'(x') = \mathcal{L}'(x) + \delta x^\mu \partial_\mu \mathcal{L}(x) + \mathcal{O}(\delta x)^2 \tag{34}$$

  and $\delta\mathcal{L}(x) = \mathcal{L}'(x) - \mathcal{L}(x)$, equation (32) reads

$$0 = \int \mathrm{d}^4x \left\{\delta\mathcal{L}(x) + \partial_\mu \left[\delta x^\mu \mathcal{L}(x)\right]\right\}. \tag{35}$$

Noether's theorem

- On the other hand,

$$\delta\mathcal{L}(x) = \sum_i \left[\frac{\partial\mathcal{L}}{\partial\phi_i}\delta\phi_i + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\delta(\partial_\mu\phi_i)\right]$$
$$= \sum_i \left\{\left[\frac{\partial\mathcal{L}}{\partial\phi_i} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\right]\delta\phi_i + \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\delta\phi_i\right]\right\} \tag{36}$$

and from

$$\phi_i'(x') = \phi_i(x) + \epsilon^a F_{i,a} = \phi_i(x'^\mu - \epsilon^a A_a^\mu) + \epsilon^a F_{i,a} \tag{37}$$

we have

$$\phi_i'(x) = \phi_i(x^\mu - \epsilon^a A_a^\mu) + \epsilon^a F_{i,a} = \phi_i(x) - \epsilon^a A_a^\mu \partial_\mu\phi_i(x) + \epsilon^a F_{i,a} \tag{38}$$

so

$$\delta\phi_i(x) = \phi_i'(x) - \phi_i(x) = -\epsilon^a[A_a^\mu\partial_\mu\phi_i(x) - F_{i,a}] \tag{39}$$

Noether's theorem

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- Then if $\phi = \phi_{\text{cl}}$ is a solution of the Euler-Lagrangian equations equation (35) reads

$$0 = \int d^4x\, \partial_\mu\left[\sum_i \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\delta\phi_i + \delta x^\mu\mathcal{L}(x)\right] \tag{40}$$

and substituting δx^μ from (30) and $\delta\phi$ from (39) we have

$$0 = \epsilon^a \int d^4x\, \partial_\mu j_a^\mu(\phi_{\text{cl}}), \tag{41}$$

where

$$\boxed{j_a^\mu(\phi) \equiv \sum_i \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}[A_a^\nu(x)\partial_\nu\phi_i(x) - F_{i,a}(\phi,\partial\phi)] - A_a^\mu(x)\mathcal{L}(x)} \tag{42}$$

Noether's theorem

- Suppose we perform a local transformation, $\epsilon^a = \epsilon^a(x)$, on this action which is invariant under global transformations. Then it will not remain invariant but now

$$S(\phi') = S(\phi) + \int \mathrm{d}^4x[\epsilon^a(x)K_a(\phi) - (\partial_\mu\epsilon^a)j_a^\mu(\phi)] + \mathcal{O}(\partial\partial\epsilon) + \mathcal{O}(\epsilon^2), \tag{43}$$

where the coefficient $K_a(\phi)$ vanishes, because in the particular case of ϵ^a constant the global invariance implies $\int \mathrm{d}^4x\ K_a(\phi) = 0$ for any ϕ.

▷ Let us see why we have denoted by $-j_a^\mu(\phi)$ the other coefficient. If every $\epsilon^a(x)$ goes to zero fast enough at infinity, from the Stokes theorem we have

$$\int \mathrm{d}^4x\ \partial_\mu\left(\epsilon^a j_a^\mu(\phi)\right) = 0 \Rightarrow -\int \mathrm{d}^4x\ (\partial_\mu\epsilon^a)j_a^\mu(\phi) = \int \mathrm{d}^4x\ \epsilon^a(x)\partial_\mu j_a^\mu(\phi) \tag{44}$$

and then

$$S(\phi') - S(\phi) = \int \mathrm{d}^4x\ \epsilon^a(x)\partial_\mu j_a^\mu(\phi). \tag{45}$$

Noether's theorem

- Now, taking in particular $\phi = \phi_{\rm cl}$, a solution the the Euler-Lagrange equations, which is an extreme of the action, the previous equation expresses a linear variation of the action around this extreme, and hence it is zero. Therefore

$$0 = \int \mathrm{d}^4x\ \epsilon^a(x)\partial_\mu j_a^\mu(\phi_{\rm cl}) \tag{46}$$

for any $\epsilon^a(x)$ and then

$$\boxed{\partial_\mu j_a^\mu(\phi_{\rm cl}) = 0} \tag{47}$$

This means that (41) does not only implies a vanishing integral but also a vanishing integrand. As a result, $j_a^\mu(\phi_{\rm cl})$ are conserved currents.

Noether's theorem

- Defining the charge

$$Q_a \equiv \int \mathrm{d}^3x\, j_a^0(t,\vec{x}) \tag{48}$$

we see that the conservation of the current (density) $j_a^\mu(x)$ implies that the (total) charge Q_a is conserved, i.e. it is time independent, since

$$\boxed{\partial_t Q_a = \int \mathrm{d}^3x\, \partial_0 j_a^0(t,\vec{x}) = -\int \mathrm{d}^3x\, \partial_i j_a^i(t,\vec{x}) = 0} \tag{49}$$

This is because we assume the fields decrease sufficiently fast at infinity and we have applied the Stokes theorem again.

▷ Two types: internal symmetries, when coordinates do not change, i.e. $A_a^\mu(x) = 0$, and spacetime symmetries if $A_a^\mu(x) \neq 0$.

The conservation of electric charge, isospin, baryon number, etc., are consequences of internal symmetries. Next we will focus on the second type: the invariance under spacetime translations, rotations and boosts.

Noether's theorem | Spacetime translations

- They are given by the following transformations of coordinates and fields (every component if any):

$$x^\mu \mapsto x'^\mu = x^\mu + \epsilon^\mu \Rightarrow \epsilon^a = \epsilon^\nu, \quad A_a^\mu(x) = \delta_\nu^\mu \tag{50}$$

$$\phi_i(x) \mapsto \phi_i'(x') = \phi_i(x) \Rightarrow F_{i,a}(\phi, \partial\phi) = 0. \tag{51}$$

Hence, there are 4 conserved currents forming the energy-momentum tensor,

$$\theta^\mu{}_\nu = \sum_i \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\partial_\nu\phi_i - \delta_\nu^\mu\mathcal{L}, \quad \partial_\mu\theta^\mu{}_\nu = 0 \tag{52}$$

and 4 conserved "charges": the energy and the 3 components of momentum,

$$P_\nu = \int \mathrm{d}^3x\, \theta^0{}_\nu = \int \mathrm{d}^3x \left[\sum_i \frac{\partial\mathcal{L}}{\partial(\partial_0\phi_i)}\partial_\nu\phi_i - \delta_\nu^0\mathcal{L}\right]. \tag{53}$$

Therefore, the invariance under spacetime translations implies four-momentum conservation,

$$\partial_t P^\nu = 0\,, \quad \nu = 0,1,2,3\,. \tag{54}$$

Note that P_0 defined in (54) coincides with the Hamiltonian, defined in (28).

- For simplicity, consider a scalar field. The Lorentz transformations read

$$x^\mu \mapsto x'^\mu = x^\mu + \frac{1}{2}\omega^{\rho\sigma}(\delta^\mu_\rho \delta^\nu_\sigma - \delta^\mu_\sigma \delta^\nu_\rho)x_\nu \;\Rightarrow \begin{cases} \epsilon^a = \omega^{\rho\sigma} & = -\omega^{\sigma\rho}, \\ A^\mu_a(x) & = \frac{1}{2}(\delta^\mu_\rho \delta^\nu_\sigma - \delta^\mu_\sigma \delta^\nu_\rho)x_\nu \end{cases} \tag{55}$$

$$\phi(x) \mapsto \phi'(x') = \phi(x) \Rightarrow \quad F_a(\phi) = 0. \tag{56}$$

- Applying Noether's theorem,

$$\begin{aligned} j^\mu{}_{\rho\sigma} &= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \frac{1}{2}(\delta^\nu_\rho x_\sigma - \delta^\nu_\sigma x_\rho)\partial_\nu \phi - \frac{1}{2}(\delta^\mu_\rho x_\sigma - \delta^\mu_\sigma x_\rho)\mathcal{L} \\ &= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \frac{1}{2}(\partial_\rho \phi x_\sigma - \partial_\sigma \phi x_\rho) - \frac{1}{2}(\delta^\mu_\rho x_\sigma - \delta^\mu_\sigma x_\rho)\mathcal{L} \\ &= \frac{1}{2}\left(\theta^\mu{}_\rho x_\sigma - \theta^\mu{}_\sigma x_\rho\right), \quad \partial_\mu j^\mu{}_{\rho\sigma} = 0. \end{aligned} \tag{57}$$

- Therefore, the following tensor contains 6 conserved currents:

$$T^{\mu\rho\sigma} \equiv -(\theta^{\mu\rho}x^\sigma - \theta^{\mu\sigma}x^\rho), \quad \partial_\mu T^{\mu\nu\rho} = 0 \tag{58}$$

and there are 6 charges which are constants of motion,

$$M^{\rho\sigma} = \int \mathrm{d}^3x \; T^{0\rho\sigma} = \int \mathrm{d}^3x \; (x^\rho \theta^{0\sigma} - x^\sigma \theta^{0\rho}), \quad \partial_t M^{\rho\sigma} = 0. \tag{59}$$

Here M^{ij} (angular momentum) come from the invariance under rotations and M^{0i} from the invariance under boosts.

Noether's theorem | Rotations and boosts

- Two remarks are now in order:

  1. Note that equation (58) implies that the energy-momentum tensor must be symmetric, because $\partial_\mu \theta^{\mu\nu} = 0$ and

$$0 = \partial_\mu (x^\rho \theta^{\mu\sigma} - x^\sigma \theta^{\mu\rho}) = x^\rho \partial_\mu \theta^{\mu\sigma} - x^\sigma \partial_\mu \theta^{\mu\rho} + \theta^{\mu\sigma}\delta^\rho_\mu - \theta^{\mu\rho}\delta^\sigma_\mu = \theta^{\rho\sigma} - \theta^{\sigma\rho}. \quad (60)$$

Since the $\theta^{\mu\nu}$ defined in (52) is not necessarily symmetric, one has to add a total derivative of the form $\partial_\lambda f^{\lambda\mu\nu}$, with $f^{\lambda\mu\nu} = -f^{\mu\lambda\nu}$, so that

$$\widetilde{\theta}^{\mu\nu} = \theta^{\mu\nu} + \partial_\lambda f^{\lambda\mu\nu}, \quad \partial_\mu \widetilde{\theta}^{\mu\nu} = \partial_\mu \theta^{\mu\nu} + \partial_\mu \partial_\lambda f^{\lambda\mu\nu} = \partial_\mu \theta^{\mu\nu} = 0 \quad (61)$$

with $\widetilde{\theta}^{\mu\nu} = \widetilde{\theta}^{\nu\mu}$ and, from

$$\int \mathrm{d}^3x\, \partial_\lambda f^{\lambda 0\nu} = \int \mathrm{d}^3x\, \partial_i f^{i0\nu} = 0 \Rightarrow P^\nu = \int \mathrm{d}^3x\, \widetilde{\theta}^{0\nu} = \int \mathrm{d}^3x\, \theta^{0\nu}, \quad (62)$$

the conserved charges are the same as long as the fields, on which f depends, decrease sufficiently fast at infinity.

▷ The symmetrization of the tensor $\theta^{\mu\nu}$ derived from (52) will be needed in the case of the electromagnetic field, as we will see later.

Noether's theorem | Rotations and boosts

- And the second remark:

  2. We are used to the angular momentum conservation ($\partial_t M^{ij} = 0$) but not to the conservation of quantities associated to boosts ($\partial_t M^{0i} = 0$).
  Actually, in quantum mechanics, we have

$$K^k = M^{0k} = P^k t - \int \mathrm{d}^3x\, x^k \theta^{00} = M^{0k}(t) \quad \text{(Heisenberg picture)} \quad (63)$$

and remember that $\partial_t M^{0k} \equiv \mathrm{d}M^{0k}/\mathrm{d}t = \mathrm{i}[H, K^k] + \partial K^k/\partial t = \mathrm{i}^2 P^k + P^k = 0$.
However, unlike energy, momentum and angular momentum, the quantities associated to boosts cannot be used to label states, since the operators representing the boost generators are not always Hermitian and moreover they do not commute with the Hamiltonian.

▷ Note that these K^k are the representation of the generations on the infinite-dimensional representation of scalar fields and, unlike the 4×4 matrix operators introduced in the first chapter, they do depend on the spacetime coordinates.

Scalar fields | The Klein-Gordon equation

- Consider first a real scalar field, $\phi(x) = \phi^*(x)$. An action describing the nontrivial dynamics of the field must contain derivatives, $\partial_\mu\phi$. The Lorentz indices must be contracted, because the action is a scalar. The simplest action is[a]

$$S = \frac{1}{2}\int d^4x\ (\partial_\mu\phi\partial^\mu\phi - m^2\phi^2) = \int d^4x\ \mathcal{L}(x). \tag{64}$$

- The Euler-Lagrange equation for ϕ is then the Klein-Gordon equation,

$$\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = 0 \Rightarrow (\Box + m^2)\phi(x) = 0, \quad \Box \equiv \partial_\mu\partial^\mu. \tag{65}$$

The solutions are plane waves,

$$\phi \propto e^{\pm ipx}, \text{ with } px \equiv p_\mu x^\mu \text{ and } p^2 \equiv p_\mu p^\mu = (p^0)^2 - \vec{p}^2 = m^2.$$

The parameter m is the mass, and by definition $m > 0$.

[a] A term $\phi\Box\phi = \phi\partial_\mu\partial^\mu\phi$ is equivalent to $\partial_\mu\phi\partial^\mu\phi$ up to a total derivative $\partial_\mu(\phi\partial^\mu\phi)$. A linear term $c^3\phi$ is equivalent to the reparametrization $\phi \to \phi - c^3/m^2$, leading to the same dynamics.

Scalar fields | The Klein-Gordon equation

- The most general solution of the Klein-Gordon equation is then,

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_{\vec{p}}}}\left(a_{\vec{p}}e^{-ipx} + a^*_{\vec{p}}e^{ipx}\right)\Big|_{p^0 = E_{\vec{p}} \equiv +\sqrt{m^2+\vec{p}^2}} \tag{66}$$

The field normalization has been chosen for later convenience. Among the solutions, there are positive energy modes (e^{-ipx}) and negative energy modes (e^{+ipx}),[a] whose interpretation will emerge when the field is quantized.

- The sign of the action has been chosen to yield a positive definite Hamiltonian:

$$\Pi_\phi = \frac{\partial\mathcal{L}}{\partial(\partial_0\phi)} = \partial_0\phi \Rightarrow \mathcal{H} = \Pi_\phi\partial_0\phi - \mathcal{L} = \frac{1}{2}\left[(\partial_0\phi)^2 + (\nabla\phi)^2 + m^2\phi^2\right] > 0. \tag{67}$$

The energy-momentum tensor is directly symmetric,

$$\theta^{\mu\nu} = \partial^\mu\phi\partial^\nu\phi - g^{\mu\nu}\mathcal{L} \quad \text{and, in fact,} \quad \mathcal{H} = \theta^{00}. \tag{68}$$

[a] This is because interpreted as a wave function, the field $\phi \sim e^{\mp ip^0t}$ with $p^0 > 0$ satisfies the Schrödinger equation $i\partial_t\phi = \pm E\phi$ with energy $\pm E$ and $E = p^0 > 0$.

- As for the conserved charges associated to rotations,

$$M^{ij} = \int d^3x \; (x^i\theta^{0j} - x^j\theta^{0i}) = \frac{i}{2}\int d^3x \; (\phi L^{ij}\partial_0\phi - \partial_0\phi L^{ij}\phi), \tag{69}$$

where we have used the definition of $L^{ij} = i(x^i\partial^j - x^j\partial^i)$ and integration by parts for $i \neq j$,

$$\int d^3x \; \partial^j[\phi x^i\partial_0\phi] = 0 \Rightarrow \int d^3x \; \partial^j\phi x^i\partial_0\phi = -\int d^3x \; \phi x^i\partial^j\partial_0\phi, \tag{70}$$

$$\int d^3x \; \partial^j[\partial_0\phi x^i\phi] = 0 \Rightarrow \int d^3x \; \partial^j\partial_0\phi x^i\phi = -\int d^3x \; \partial_0\phi x^i\partial^j\phi. \tag{71}$$

- If we define the scalar product of two real scalar fields as

$$\langle\phi_1|\phi_2\rangle \equiv \frac{i}{2}\int d^3x \; \phi_1\overleftrightarrow{\partial_0}\phi_2, \quad f\overleftrightarrow{\partial}g \equiv f\partial g - \partial f g, \tag{72}$$

we have

$$\boxed{M^{ij} = \langle\phi|\, L^{ij}\, |\phi\rangle} \tag{73}$$

which is as expected, because L^{ij} is the representation of J^{ij} on the field space.

- Let use see that $\langle\phi_1|\phi_2\rangle$ is time independent if ϕ_1 and ϕ_2 are solutions of the Klein-Gordon equation, that is in agreement with M^{ij} being a conserved quantity. In fact,

$$\begin{aligned}\partial_0\langle\phi_1|\phi_2\rangle &= \frac{i}{2}\int d^3x \; \partial_0[\phi_1\partial_0\phi_2 - \partial_0\phi_1\phi_2] \\ &= \frac{i}{2}\int d^3x \; \left\{\partial_0\phi_1\partial_0\phi_2 + \phi_1\partial_0^2\phi_2 - \partial_0^2\phi_1\phi_2 - \partial_0\phi_1\partial_0\phi_2\right\} \\ &= \frac{i}{2}\int d^3x \; \left\{\phi_1\nabla^2\phi_2 - \nabla^2\phi_1\phi_2 - m^2\phi_1\phi_2 + m^2\phi_1\phi_2\right\} \\ &= \frac{i}{2}\int d^3x \; \{-\nabla\phi_1\cdot\nabla\phi_2 + \nabla\phi_1\cdot\nabla\phi_2\} = 0,\end{aligned} \tag{74}$$

where we have used

$$(\Box + m^2)\phi_{1,2} = 0 \Rightarrow \partial_0^2\phi_{1,2} = \nabla^2\phi_{1,2} - m^2\phi_{1,2}$$

$$\int \mathrm{d}^3x\ \nabla\cdot(\phi_{1,2}\nabla\phi_{2,1}) = 0 \Rightarrow \int \mathrm{d}^3x\ \phi_{1,2}\nabla^2\phi_{2,1} = -\int \mathrm{d}^3x\ \nabla\phi_{1,2}\cdot\nabla\phi_{2,1}. \qquad (75)$$

- Likewise, we can write

$$\boxed{P^\mu = \int \mathrm{d}^3x\ \theta^{0\mu} = \langle\phi|\, \mathrm{i}\partial^\mu\, |\phi\rangle} \qquad (76)$$

▷ In fact, using again the Klein-Gordon equation and integrating by parts,

$$\begin{aligned} P^0 = \langle\phi|\, \mathrm{i}\partial^0\, |\phi\rangle = \langle\phi|\, \mathrm{i}\partial_0\, |\phi\rangle &= -\frac{1}{2}\int \mathrm{d}^3x\ [\phi\partial_0^2\phi - (\partial_0\phi)^2] \\ &= -\frac{1}{2}\int \mathrm{d}^3x\ [\phi\nabla^2\phi - m^2\phi^2 - (\partial_0\phi)^2] \\ &= \ \frac{1}{2}\int \mathrm{d}^3x\ [(\nabla\phi)^2 + m^2\phi^2 + (\partial_0\phi)^2] = \int \mathrm{d}^3x\ \theta^{00}, \end{aligned} \qquad (77)$$

$$\begin{aligned} P^i = \langle\phi|\, \mathrm{i}\partial^i\, |\phi\rangle &= -\frac{1}{2}\int \mathrm{d}^3x\ [\phi\partial^i\partial_0\phi - \partial_0\phi\partial^i\phi] \\ &= \quad \int \mathrm{d}^3x\ \partial^i\phi\partial_0\phi = \int \mathrm{d}^3x\ \theta^{i0} = \int \mathrm{d}^3x\ \theta^{0i}. \end{aligned} \qquad (78)$$

- And also,

$$\boxed{M^{0i} = \langle\phi|\, L^{0i}\, |\phi\rangle = \int \mathrm{d}^3x\ (x^0\theta^{0i} - x^i\theta^{00})} \qquad (79)$$

▷ In fact,

$$\begin{aligned} M^{0i} = \langle\phi|\, L^{0i}\, |\phi\rangle &= -\frac{1}{2}\int \mathrm{d}^3x\ [\phi(x^0\partial^i - x^i\partial^0)\partial_0\phi - \partial_0\phi(x^0\partial^i - x^i\partial^0)\phi] \\ &= \int \mathrm{d}^3x\ \left\{x^0\partial_0\phi\partial^i\phi + \frac{x^i}{2}[\phi\partial_0^2\phi - (\partial_0\phi)^2]\right\} \\ &= \int \mathrm{d}^3x\ \left\{x^0\partial_0\phi\partial^i\phi - \frac{x^i}{2}[(\partial_0\phi)^2 - \phi\nabla^2\phi + m^2\phi^2]\right\}. \end{aligned} \qquad (80)$$

Scalar fields | The Klein-Gordon equation

- Note that, as anticipated, $L^{\mu\nu}$ and $\mathrm{i}\partial^\mu$ are Hermitian operators, and hence we have an infinite-dimensional unitary representation of the Poincaré group.
  Then $M^{\mu\nu}$ are P^μ are real quatities.

- Finally, we could generalize the Klein-Gordon action to include interactions of the scalar field introducing a potential $V(\phi)$,

$$S = \int \mathrm{d}^4x \, \left\{ \frac{1}{2} \left[\partial_\mu\phi\partial^\mu\phi - m^2\phi^2 \right] - V(\phi) \right\}. \tag{81}$$

  Terms proportional to ϕ^3, ϕ^4, ... in the potential give rise to non linear contributions in the equations of motion, corresponding to field self-interactions:

$$(\Box + m^2)\phi = -\frac{\partial V}{\partial\phi}. \tag{82}$$

Scalar fields | Complex fields

- Suppose now a complex scalar field,

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + \mathrm{i}\phi_2) \tag{83}$$

  where ϕ_1 and ϕ_2 are two real fields of the same mass m. Then

$$\begin{aligned} S &= \int \mathrm{d}^4x \, (\partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi) \\ &= \frac{1}{2}\int \mathrm{d}^4x \, (\partial_\mu\phi_1\partial^\mu\phi_1 - m^2\phi_1^2) + \frac{1}{2}\int \mathrm{d}^4x \, (\partial_\mu\phi_2\partial^\mu\phi_2 - m^2\phi_2^2) \\ &= \int \mathrm{d}^4x \, \mathcal{L}(x). \end{aligned} \tag{84}$$

  It is clear that the Klein-Gordon equation for the complex ϕ is the same as for (65), since both real and imaginary parts satisfy it. The most general solution is

$$\phi(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_{\vec{p}}}} \left(a_{\vec{p}} \mathrm{e}^{-\mathrm{i}px} + b^*_{\vec{p}} \mathrm{e}^{\mathrm{i}px} \right) \Bigg|_{p^0 = E_{\vec{p}} \equiv +\sqrt{m^2+\vec{p}^2}} \tag{85}$$

- The action of this complex field is invariant under global transformations of the group U(1),

$$\phi(x) \mapsto \phi'(x) = \mathrm{e}^{-\mathrm{i}\theta}\phi(x), \quad \phi^*(x) \mapsto \phi'^*(x) = \mathrm{e}^{\mathrm{i}\theta}\phi^*(x) \tag{86}$$

meaning that there is a conserved current associated (take $\phi_i = (\phi, \phi^*)$):

$$x^\mu \mapsto x'^\mu = x^\mu \Rightarrow A^\mu_a(x) = 0 \tag{87}$$

$$\left.\begin{array}{r} \phi(x) \mapsto \phi'(x) = \phi(x) - \mathrm{i}\theta\phi(x) \\ \phi^*(x) \mapsto \phi'^*(x) = \phi^*(x) + \mathrm{i}\theta\phi^*(x) \end{array}\right. \Rightarrow \epsilon^a = \theta, \quad \begin{array}{rl} F_{\phi,a} &= -\mathrm{i}\phi \\ F_{\phi^*,a} &= \mathrm{i}\phi^* \end{array} \tag{88}$$

$$j^\mu = -\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}F_{\phi,a} - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}F_{\phi^*,a} = \mathrm{i}(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*) = \mathrm{i}\phi^*\overleftrightarrow{\partial^\mu}\phi. \tag{89}$$

- The conserved charge is

$$Q = \int \mathrm{d}^3x\, j^0 = \mathrm{i}\int \mathrm{d}^3x\, \phi^*\overleftrightarrow{\partial_0}\phi = \langle\phi|\phi\rangle\,, \quad \partial_t Q = 0, \tag{90}$$

consistent with the generator of the symmetries $\mathrm{e}^{-\mathrm{i}\theta}$ being the identity operator, defining the scalar product of two complex scalar fields ϕ_A and ϕ_B as

$$\langle\phi_A|\phi_B\rangle \equiv \mathrm{i}\int \mathrm{d}^3x\, \phi_A^*\overleftrightarrow{\partial_0}\phi_B. \tag{91}$$

- Consider the Weyl spinors ψ_R and ψ_L. Then

$$\psi_R^\dagger \sigma^\mu \psi_R, \quad \psi_L^\dagger \overline{\sigma}^\mu \psi_L \tag{92}$$

with $\sigma^\mu \equiv (1, \vec{\sigma})$, $\overline{\sigma}^\mu \equiv (1, -\vec{\sigma})$, are Lorentz four-vectors.

▷ To show this, remember that

$$\psi_R \mapsto \exp\left\{(-\mathrm{i}\vec{\theta} + \vec{\eta}) \cdot \frac{\vec{\sigma}}{2}\right\} \psi_R. \tag{93}$$

Consider, for example, an infinitesimal boost of rapidity η along direction x,

$$\begin{aligned} \psi_R^\dagger \sigma^\mu \psi_R \mapsto \psi_R^\dagger \sigma^\mu \psi_R + \eta \psi_R^\dagger \frac{\sigma^1}{2} \sigma^\mu \psi_R + \eta \psi_R^\dagger \sigma^\mu \frac{\sigma^1}{2} \psi_R \\ \Rightarrow \begin{array}{ccl} \psi_R^\dagger \psi_R & \mapsto & \psi_R^\dagger \psi_R + \eta \psi_R^\dagger \sigma^1 \psi_R \\ \psi_R^\dagger \sigma^i \psi_R & \mapsto & \psi_R^\dagger \sigma^i \psi_R + \eta \delta^{i1} \psi_R^\dagger \psi_R, \end{array} \end{aligned} \tag{94}$$

where we have used $\sigma^i \sigma^j + \sigma^j \sigma^i = 2\delta^{ij}$.

We see that $\psi_R^\dagger \sigma^\mu \psi_R$ transforms under this boost like a four-vector v^μ,

$$\begin{pmatrix} v^0 \\ v^1 \\ v^2 \\ v^3 \end{pmatrix} \mapsto \begin{pmatrix} 1 & \eta & 0 & 0 \\ \eta & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v^0 \\ v^1 \\ v^2 \\ v^3 \end{pmatrix}. \tag{95}$$

Consider now an infinitesimal rotation θ about the axis z,

$$\begin{aligned} \psi_R^\dagger \sigma^\mu \psi_R \mapsto \psi_R^\dagger \sigma^\mu \psi_R + \mathrm{i}\theta \psi_R^\dagger \frac{\sigma^3}{2} \sigma^\mu \psi_R - \mathrm{i}\theta \psi_R^\dagger \sigma^\mu \frac{\sigma^3}{2} \psi_R \\ \Rightarrow \begin{array}{ccl} \psi_R^\dagger \psi_R & \mapsto & \psi_R^\dagger \psi_R \\ \psi_R^\dagger \sigma^1 \psi_R & \mapsto & \psi_R^\dagger \sigma^1 \psi_R - \theta \psi_R^\dagger \sigma^2 \psi_R \\ \psi_R^\dagger \sigma^2 \psi_R & \mapsto & \psi_R^\dagger \sigma^2 \psi_R + \theta \psi_R^\dagger \sigma^1 \psi_R \\ \psi_R^\dagger \sigma^3 \psi_R & \mapsto & \psi_R^\dagger \sigma^3 \psi_R, \end{array} \end{aligned} \tag{96}$$

where we have used $\sigma^i \sigma^j - \sigma^j \sigma^i = 2\mathrm{i}\epsilon^{ijk}\sigma^k$.

We see that $\psi_R^\dagger \sigma^\mu \psi_R$ ransforms under this rotation like a four-vector v^μ,

$$\begin{pmatrix} v^0 \\ v^1 \\ v^2 \\ v^3 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\theta & 0 \\ 0 & \theta & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v^0 \\ v^1 \\ v^2 \\ v^3 \end{pmatrix}. \tag{97}$$

And likewise for $\psi_L^\dagger \overline{\sigma}^\mu \psi_L$.

- Now focus on ψ_L. The simplest action involving this field,[a]

$$S = \mathrm{i} \int \mathrm{d}^4x \; \psi_L^\dagger \overline{\sigma}^\mu \partial_\mu \psi_L = \int \mathrm{d}^4x \; \mathcal{L}(x). \tag{98}$$

The factor i is introduced for the Lagrangian to be Hermitian. Let us find the Euler-Lagrange equations, considering that ψ_L and ψ_L^* are independent fields:

$$\begin{array}{rr} [\psi_L^*]: & \mathrm{i}\overline{\sigma}^\mu \partial_\mu \psi_L = 0 \\ [\psi_L]: & -\mathrm{i}\partial_\mu \psi_L^\dagger \overline{\sigma}^\mu = 0 \end{array} \Rightarrow \overline{\sigma}^\mu \partial_\mu \psi_L = 0 \Rightarrow (\partial_0 - \sigma^i \partial_i)\psi_L = 0. \tag{99}$$

(Weyl equation for ψ_L)

[a] We have just shown that $\psi_L^\dagger \overline{\sigma}^\mu \psi_L \mapsto \Lambda^\mu{}_\rho \psi_L^\dagger \overline{\sigma}^\rho \psi_L = \psi_L^\dagger \Lambda^\mu{}_\rho \overline{\sigma}^\rho \psi_L$, because Λ and $\overline{\sigma}$ act on different spaces. On the other hand, $\partial_\mu \mapsto \Lambda_\mu{}^\sigma \partial_\sigma$. Then, the following term is a Lorentz scalar because

$$\psi_L^\dagger \overline{\sigma}^\mu \partial_\mu \psi_L \mapsto \psi_L^\dagger \overline{\sigma}^\rho \Lambda^\mu{}_\rho \Lambda_\mu{}^\sigma \partial_\sigma \psi_L = \psi_L^\dagger \overline{\sigma}^\rho g_\rho^\sigma \partial_\sigma \psi_L = \psi_L^\dagger \overline{\sigma}^\mu \partial_\mu \psi_L.$$

- The Weyl equation for ψ_L is equivalent to a (massless) Klein-Gordon equation for its two components,

$$\partial_0\psi_L = \sigma^i\partial_i\psi_L \Rightarrow \partial_0^2\psi_L = \nabla^2\psi_L \Rightarrow \Box\psi_L = 0 \tag{100}$$

and furthermore provides information about the helicity of the field modes.

▷ Taking a positive (negative) helicity mode of ψ_L,

$$\psi_L(x) = u_L e^{-ipx} \ (u_L e^{ipx}) \tag{101}$$

with $u_L(\vec{p})$ a constant spinor and $p^\mu = (E, \vec{p})$ where $E = |\vec{p}|$ (zero mass), and remembering that

$$\vec{J} = \frac{\vec{\sigma}}{2} \Rightarrow (\hat{p}\cdot\vec{J})\, u_L = \frac{1}{2}\hat{p}\cdot\vec{\sigma}\, u_L \equiv h\, u_L \tag{102}$$

we see that

$$\overline{\sigma}^\mu\partial_\mu\psi_L = (\partial_0 - \sigma^i\partial_i)u_L e^{\mp ipx} = \mp i(E + \vec{\sigma}\cdot\vec{p})u_L e^{\mp ipx} = 0 \Rightarrow \vec{\sigma}\cdot\hat{p}\, u_L = -u_L, \tag{103}$$

meaning that all the modes of ψ_L have negative helicity $h = -\frac{1}{2}$.

- On the other hand, the energy-momentum tensor is

$$\theta^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_L)}\partial^\nu\psi_L - g^{\mu\nu}\mathcal{L} = i\psi_L^\dagger\overline{\sigma}^\mu\partial^\nu\psi_L, \tag{104}$$

where we have used that for fields satisfying the Euler-Lagrange equation (99) the Lagrangian $\mathcal{L} = 0$. The Hamiltonian is $\mathcal{H} = \theta^{00} = i\psi_L^\dagger\partial^0\psi_L$.

- Moreover, the action is invariant under global symmetry transformations of the group U(1),

$$\psi_L \mapsto e^{-i\theta}\psi_L, \tag{105}$$

so there is a conserved current

$$j^\mu = -\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_L)} i\psi_L = \psi_L^\dagger\overline{\sigma}^\mu\psi_L, \quad \partial_\mu j^\mu = 0 \tag{106}$$

and a conserved charge

$$Q = \int d^3x\, j^0 = \int d^3x\, \psi_L^\dagger\psi_L, \quad \partial_t Q = 0. \tag{107}$$

Spinorial fields | Weyl equation

- Likewise one can see that the Weyl equation for ψ_R is

$$\sigma^\mu \partial_\mu \psi_R = 0 \Rightarrow (\partial_0 + \sigma^i \partial_i)\psi_R = 0, \tag{108}$$

which is equivalent to a massless Klein-Gordon equation for its two components,

$$\partial_0 \psi_R = -\sigma^i \partial_i \psi_R \Rightarrow \partial_0^2 \psi_R = \nabla^2 \psi_R \Rightarrow \Box \psi_R = 0. \tag{109}$$

The modes of ψ_R have positive helicity $h = \frac{1}{2}$. The corresponding energy-momentum tensor, current and conserved charge are, respectively,

$$\theta^{\mu\nu} = \mathrm{i}\psi_R^\dagger \sigma^\mu \partial^\nu \psi_R, \quad j^\mu = \psi_R^\dagger \sigma^\mu \psi_R, \quad Q = \int \mathrm{d}^3x \; \psi_R^\dagger \psi_R. \tag{110}$$

Spinorial fields | Dirac equation

- Note that, under a Lorentz transformation,

$$\psi_L \mapsto \Lambda_L \psi_L, \quad \psi_R \mapsto \Lambda_R \psi_R, \quad \text{y } \Lambda_L^\dagger \Lambda_R = \Lambda_R^\dagger \Lambda_L = \mathbb{1}. \tag{111}$$

Therefore, $\psi_L^\dagger \psi_R$ and $\psi_R^\dagger \psi_L$ are Lorentz scalars.

▷ Under parity ($\psi_L \leftrightarrow \psi_R$) the following Hermitian combinations transform as:

$$\begin{aligned} (\psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L) &\mapsto \phantom{-\mathrm{i}}(\psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L) \quad \text{(escalar)} \\ \mathrm{i}(\psi_L^\dagger \psi_R - \psi_R^\dagger \psi_L) &\mapsto -\mathrm{i}(\psi_L^\dagger \psi_R - \psi_R^\dagger \psi_L) \quad \text{(pseudoescalar)} \end{aligned} \tag{112}$$

As a consequence, the Dirac Lagrangian,

$$\mathcal{L}_D = \mathrm{i}\psi_L^\dagger \overline{\sigma}^\mu \partial_\mu \psi_L + \mathrm{i}\psi_R^\dagger \sigma^\mu \partial_\mu \psi_R - m(\psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L) \tag{113}$$

is invariant under parity. In contrast, the Weyl Lagrangian is not.

Spinorial fields | Dirac equation

- Let us find the Euler-Lagrange equations:

$$\begin{array}{rl} [\psi_L^*]: & \mathrm{i}\overline{\sigma}^\mu\partial_\mu\psi_L - m\psi_R = 0 \\ [\psi_L]: & -\mathrm{i}\partial_\mu\psi_L^\dagger\overline{\sigma}^\mu - m\psi_R^\dagger = 0 \\ [\psi_R^*]: & \mathrm{i}\sigma^\mu\partial_\mu\psi_R - m\psi_L = 0 \\ [\psi_R]: & -\mathrm{i}\partial_\mu\psi_R^\dagger\sigma^\mu - m\psi_L^\dagger = 0 \end{array} \Rightarrow \begin{array}{l} \mathrm{i}\overline{\sigma}^\mu\partial_\mu\psi_L = m\psi_R \\ \mathrm{i}\sigma^\mu\partial_\mu\psi_R = m\psi_L. \end{array} \tag{114}$$

This is the Dirac equation in terms of Weyl spinors.

▷ Note that ψ_L and ψ_R are no longer helicity eigenstates and the 2 components of ψ_L and of ψ_R satisfy a Klein-Gordon equation of mass m, because

$$\begin{aligned} &\mathrm{i}\overline{\sigma}^\mu\partial_\mu\psi_L = m\psi_R \Rightarrow (\mathrm{i}\sigma^\nu\partial_\nu)\mathrm{i}\overline{\sigma}^\mu\partial_\mu\psi_L = m\mathrm{i}\sigma^\nu\partial_\nu\psi_R \\ &\qquad\Rightarrow -\frac{1}{2}(\sigma^\mu\overline{\sigma}^\nu + \sigma^\nu\overline{\sigma}^\mu)\partial_\mu\partial_\nu\psi_L = m^2\psi_L \Rightarrow (\Box + m^2)\psi_L = 0, \end{aligned} \tag{115}$$

where we have used (108) and the identity $\sigma^\mu\overline{\sigma}^\nu + \sigma^\nu\overline{\sigma}^\mu = 2g^{\mu\nu}$. The same for ψ_R,

$$(\Box + m^2)\psi_R = 0. \tag{116}$$

Spinorial fields | Dirac equation

- It is convenient to introduce the Dirac field, of 4 components,

$$\psi(x) = \begin{pmatrix} \psi_L(x) \\ \psi_R(x) \end{pmatrix} \quad \text{(chiral representation)} \tag{117}$$

and define the Dirac gamma matrices,

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \Rightarrow \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{pmatrix} \quad \text{(chiral representation)}, \tag{118}$$

satisfying the Clifford algebra,

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}. \tag{119}$$

▷ The Dirac equation is then

$$(\mathrm{i}\not\partial - m)\psi = 0, \quad \not A \equiv \gamma^\mu A_\mu. \tag{120}$$

- We can write the Dirac Lagrangian in a compact form introducing

$$\overline{\psi} \equiv \psi^\dagger \gamma^0 \quad \text{(Dirac adjoint spinor)}. \tag{121}$$

In the chiral representation, $\overline{\psi} = (\psi_R^\dagger, \psi_L^\dagger)$ and

$$\mathcal{L}_D = \overline{\psi}(\mathrm{i}\partial\!\!\!/ - m)\psi. \tag{122}$$

- Another definition is the matrix $\gamma_5 \equiv \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3$, that reads

$$\gamma_5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{(chiral representation)} \tag{123}$$

Therefore, the operators $P_L = \frac{1}{2}(1-\gamma_5)$ and $P_R = \frac{1}{2}(1+\gamma_5)$ are projectors on the Weyl spinors ψ_L and ψ_R, respectively,

$$P_L\psi = \begin{pmatrix} \psi_L \\ 0 \end{pmatrix}, \quad P_R\psi = \begin{pmatrix} 0 \\ \psi_R \end{pmatrix}. \tag{124}$$

- One can chose other representations,

$$\psi'(x) = U\psi(x), \quad \gamma'^\mu = U\gamma^\mu U^\dagger, \quad \overline{\psi}'(x) = \psi'^\dagger(x)\gamma'^0, \tag{125}$$

where U is a constant unitary matrix.

▷ In this way,

$$\mathcal{L}_D = \psi'^\dagger U\gamma^0(\mathrm{i}\gamma^\mu\partial_\mu - m)U^\dagger\psi' = \overline{\psi}'(\mathrm{i}\gamma'^\mu\partial_\mu - m)\psi', \tag{126}$$

has the same form as the original Lagrangian.

▷ The Clifford algebra remains invariant, $\gamma'^\mu\gamma'^\nu + \gamma'^\nu\gamma'^\mu = 2g^{\mu\nu}$.

- A representation used very frequently is the standard representation or Dirac representation, obtained from the chiral one by

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} . \tag{127}$$

▷ The Dirac field and gamma matrices in the standard representation read

$$\psi = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi_R + \psi_L \\ \psi_R - \psi_L \end{pmatrix} , \tag{128}$$

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} , \quad \gamma_5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} . \tag{129}$$

▷ The Dirac representation is convenient in the non relativistic limit, while the chiral representation is convenient in the ultrarelativistic limit.

- The general solution of the Dirac equation is a superposition of plane waves,

$$\psi(x) \equiv u(\vec{p})\mathrm{e}^{-\mathrm{i}px} \text{ (positive energy modes } E > 0), \tag{130}$$

$$\psi(x) \equiv v(\vec{p})\mathrm{e}^{\mathrm{i}px} \text{ (negative energy modes } -E < 0), \quad E = +\sqrt{m^2 + \vec{p}^2}. \tag{131}$$

Applying (120) to these solutions we have

$$(\not{p} - m)u(\vec{p}) = 0, \quad (\not{p} + m)v(\vec{p}) = 0. \tag{132}$$

- We will find now the explicit form of these solutions in the chiral representation:

$$u(\vec{p}) = \begin{pmatrix} u_L(\vec{p}) \\ u_R(\vec{p}) \end{pmatrix}, \quad v(\vec{p}) = \begin{pmatrix} v_L(\vec{p}) \\ v_R(\vec{p}) \end{pmatrix}. \tag{133}$$

▷ Take first the case $m \neq 0$. Then, in the rest frame system,

$$p^\mu = (m,0,0,0)$$

$$(\not{p} - m)u(0) = 0 \Rightarrow (\gamma^0 - 1)u(0) = 0 \Rightarrow u_L(0) = u_R(0), \tag{134}$$

$$(\not{p} + m)v(0) = 0 \Rightarrow (\gamma^0 + 1)v(0) = 0 \Rightarrow v_L(0) = -v_R(0). \tag{135}$$

Then, focussing on the positive energy spinor $u(\vec{p})$, we can choose

$$u_L^{(s)}(0) = u_R^{(s)}(0) = \sqrt{m}\,\xi^{(s)}, \quad s \in \{1,2\}, \quad \xi^{(r)\dagger}\xi^{(s)} = \delta_{rs},$$

$$\xi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \xi^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{136}$$

- The solutions for an arbitrary $\vec{p}$ can be found performing a boost along $\hat{p} = \vec{p}/|\vec{p}|$,

$$u^{(s)}(\vec{p}) = \begin{pmatrix} e^{-\frac{1}{2}\eta\hat{p}\cdot\vec{\sigma}} u_L^{(s)}(0) \\ e^{+\frac{1}{2}\eta\hat{p}\cdot\vec{\sigma}} u_R^{(s)}(0) \end{pmatrix}. \tag{137}$$

Expanding the exponentials,

$$\begin{aligned} e^{\pm\eta\hat{p}\cdot\vec{\sigma}} &= \sum_{k=0}^{\infty} \frac{1}{(2k)!}\eta^{2k} \pm \hat{p}\cdot\vec{\sigma} \sum_{k=0}^{\infty} \frac{1}{(2k+1)!}\eta^{2k+1} \\ &= \cosh\eta \pm \hat{p}\cdot\vec{\sigma}\sinh\eta = \frac{1}{m}(E \pm \vec{p}\cdot\vec{\sigma}), \end{aligned} \tag{138}$$

with

$$\cosh\eta = \gamma = \frac{E}{m}, \quad \sinh\eta = \gamma\beta = \frac{|\vec{p}|}{m} \tag{139}$$

$$(p\sigma) = p_\mu\sigma^\mu = E - \vec{p}\cdot\vec{\sigma}, \quad (p\bar{\sigma}) = p_\mu\bar{\sigma}^\mu = E + \vec{p}\cdot\vec{\sigma}, \tag{140}$$

$$\Rightarrow u^{(s)}(\vec{p}) = \frac{1}{\sqrt{m}} \begin{pmatrix} \sqrt{(p\sigma)}\; u_L^{(s)}(0) \\ \sqrt{(p\bar{\sigma})}\; u_R^{(s)}(0) \end{pmatrix} = \begin{pmatrix} \sqrt{(p\sigma)}\; \xi^{(s)} \\ \sqrt{(p\bar{\sigma})}\; \xi^{(s)} \end{pmatrix}. \tag{141}$$

- One can write these solutions differently:

$$u^{(s)}(\vec{p}) = \begin{pmatrix} \left[\sqrt{E+|\vec{p}|}\left(\frac{1-\hat{p}\cdot\vec{\sigma}}{2}\right) + \sqrt{E-|\vec{p}|}\left(\frac{1+\hat{p}\cdot\vec{\sigma}}{2}\right)\right]\xi^{(s)} \\ \left[\sqrt{E+|\vec{p}|}\left(\frac{1+\hat{p}\cdot\vec{\sigma}}{2}\right) + \sqrt{E-|\vec{p}|}\left(\frac{1-\hat{p}\cdot\vec{\sigma}}{2}\right)\right]\xi^{(s)} \end{pmatrix}, \tag{142}$$

where we have used

$$\mathrm{e}^{\pm\frac{\eta}{2}\hat{p}\cdot\vec{\sigma}} = \cosh\frac{\eta}{2} \pm \hat{p}\cdot\vec{\sigma}\sinh\frac{\eta}{2} = \mathrm{e}^{\frac{\eta}{2}}\left(\frac{1\pm\hat{p}\cdot\vec{\sigma}}{2}\right) + \mathrm{e}^{-\frac{\eta}{2}}\left(\frac{1\mp\hat{p}\cdot\vec{\sigma}}{2}\right) \tag{143}$$

$$\mathrm{e}^{\pm\frac{\eta}{2}} = \sqrt{\cosh\eta \pm \sinh\eta} = \sqrt{\gamma \pm \gamma\beta} = \sqrt{\frac{E\pm|\vec{p}|}{m}}. \tag{144}$$

- Taking the ultrarelativistic limit ($E \gg m$), $p^\mu \to (E, 0, 0, E)$,

$$\begin{aligned} u^{(1)}(\vec{p}) &\to \sqrt{\frac{E}{2}}\begin{pmatrix}(1-\sigma^3)\xi^{(1)} \\ (1+\sigma^3)\xi^{(1)}\end{pmatrix} = \sqrt{2E}\begin{pmatrix}0 \\ \xi^{(1)}\end{pmatrix} \\ u^{(2)}(\vec{p}) &\to \sqrt{\frac{E}{2}}\begin{pmatrix}(1-\sigma^3)\xi^{(2)} \\ (1+\sigma^3)\xi^{(2)}\end{pmatrix} = \sqrt{2E}\begin{pmatrix}\xi^{(2)} \\ 0\end{pmatrix} \end{aligned} \tag{145}$$

we see that $u^{(1)}$ has right-handed components only and $u^{(2)}$ has left-handed components only, so they are Dirac fields of well defined helicity (chirality), as expected for massless fields.

- Repeating the procedure above for the negative energy spinor $v(\vec{p})$ we obtain

$$v^{(s)}(\vec{p}) = \frac{1}{\sqrt{m}} \begin{pmatrix} \sqrt{(p\sigma)}\, v_L^{(s)}(0) \\ -\sqrt{(p\bar{\sigma})}\, v_R^{(s)}(0) \end{pmatrix} = \begin{pmatrix} \sqrt{(p\sigma)}\, \eta^{(s)} \\ -\sqrt{(p\bar{\sigma})}\, \eta^{(s)} \end{pmatrix}, \quad \eta^{(r)\dagger}\eta^{(s)} = \delta_{rs}, \tag{146}$$

or

$$v^{(s)}(\vec{p}) = \begin{pmatrix} \left[\sqrt{E+|\vec{p}|}\left(\frac{1-\hat{p}\cdot\vec{\sigma}}{2}\right) + \sqrt{E-|\vec{p}|}\left(\frac{1+\hat{p}\cdot\vec{\sigma}}{2}\right)\right]\eta^{(s)} \\ -\left[\sqrt{E+|\vec{p}|}\left(\frac{1+\hat{p}\cdot\vec{\sigma}}{2}\right) + \sqrt{E-|\vec{p}|}\left(\frac{1-\hat{p}\cdot\vec{\sigma}}{2}\right)\right]\eta^{(s)} \end{pmatrix}. \tag{147}$$

- We will see that it turns out convenient to choose

$$\eta^{(s)} = -\mathrm{i}\sigma^2\xi^{(s)*} \Rightarrow \eta^{(1)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \eta^{(2)} = -\begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{148}$$

Then, in the ultrarelativistic limit ($E \gg m$), $p^\mu \to (E, 0, 0, E)$,

$$\begin{aligned} v^{(1)}(\vec{p}) &\to \sqrt{\frac{E}{2}} \begin{pmatrix} (1-\sigma^3)\eta^{(1)} \\ -(1+\sigma^3)\eta^{(1)} \end{pmatrix} = \sqrt{2E} \begin{pmatrix} \eta^{(1)} \\ 0 \end{pmatrix} \\ v^{(2)}(\vec{p}) &\to \sqrt{\frac{E}{2}} \begin{pmatrix} (1-\sigma^3)\eta^{(2)} \\ -(1+\sigma^3)\eta^{(2)} \end{pmatrix} = -\sqrt{2E} \begin{pmatrix} 0 \\ \eta^{(2)} \end{pmatrix} \end{aligned} \tag{149}$$

meaning that $v^{(1)}$ has left-handed components only and $v^{(2)}$ has right-handed components only, so they are again massless Dirac fields with well defined helicity.

Spinorial fields | Dirac equation

- Introducing now the corresponding Dirac adjoint spinors,[a]

$$\overline{u} = u^\dagger \gamma^0, \quad \overline{v} = v^\dagger \gamma^0, \tag{150}$$

satisfying the Dirac equations,

$$\overline{u}(\vec{p})(\not{p} - m) = 0, \quad \overline{v}(\vec{p})(\not{p} + m) = 0, \tag{151}$$

one can show the following orthonormality relations

$$\overline{u}^{(r)}(\vec{p}) u^{(s)}(\vec{p}) = 2m\delta_{rs}, \quad \overline{v}^{(r)}(\vec{p}) v^{(s)}(\vec{p}) = -2m\delta_{rs} \tag{152}$$

$$u^{(r)\dagger}(\vec{p}) u^{(s)}(\vec{p}) = 2E_{\vec{p}}\delta_{rs}, \quad v^{(r)\dagger}(\vec{p}) v^{(s)}(\vec{p}) = 2E_{\vec{p}}\delta_{rs}, \tag{153}$$

$$\overline{u}^{(r)}(\vec{p}) v^{(s)}(\vec{p}) = \overline{v}^{(r)}(\vec{p}) u^{(s)}(\vec{p}) = 0 \tag{154}$$

and the completeness relations,

$$\sum_{s=1,2} u^{(s)}(\vec{p}) \overline{u}^{(s)}(\vec{p}) = \not{p} + m, \quad \sum_{s=1,2} v^{(s)}(\vec{p}) \overline{v}^{(s)}(\vec{p}) = \not{p} - m. \tag{155}$$

[a] Use the identity $\gamma^{\mu\dagger} = \gamma^0 \gamma^\mu \gamma^0$.

Spinorial fields | Dirac equation

- It is important to note that the 16 matrices,

$$\mathbb{1}, \ \gamma_5, \ \gamma^\mu, \ \gamma^\mu\gamma_5, \ \sigma^{\mu\nu} \equiv \frac{\mathrm{i}}{2}[\gamma^\mu, \gamma^\nu] \tag{156}$$

are linearly independent and form a basis for the 4×4 matrices. Then one can define the following fermion bilinears with well defined properties (covariant) under Lorentz transformations,

$$\overline{\psi}\psi, \ \overline{\psi}\gamma_5\psi, \ \overline{\psi}\gamma^\mu\psi, \ \overline{\psi}\gamma^\mu\gamma_5\psi, \ \overline{\psi}\sigma^{\mu\nu}\psi. \tag{157}$$

- One can also check that the Lorentz transformations of the Dirac spinor ψ can be written, in any representation of the gamma matrices, as

$$\psi \mapsto \exp\left\{-\frac{\mathrm{i}}{4}\omega_{\mu\nu}\sigma^{\mu\nu}\right\}\psi, \quad \text{i.e. } J^{\mu\nu} = \frac{1}{2}\sigma^{\mu\nu}. \tag{158}$$

(it is sufficient to check that $J^{\mu\nu} = \frac{1}{2}\sigma^{\mu\nu}$ satisfies the Lorentz algebra.)

Spinorial fields | Dirac equation

- An interesting global symmetry that the massless Dirac Lagrangian has is the chiral symmetry,

$$\psi_L \mapsto e^{-i\theta_L}\psi_L, \quad \psi_R \mapsto e^{-i\theta_R}\psi_R \quad (\theta_L \text{ and } \theta_R \text{ independent}), \tag{159}$$

that, in terms of the Dirac spinor, can be written

$$\psi \mapsto e^{-i\alpha}\psi, \quad \psi \mapsto e^{-i\beta\gamma_5}\psi \quad (\alpha \text{ and } \beta \text{ independent}). \tag{160}$$

In fact, performing infinitesimal transformations,

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \mapsto e^{-i\alpha}\psi = \begin{pmatrix} (1-i\alpha)\psi_L \\ (1-i\alpha)\psi_R \end{pmatrix} \Rightarrow \theta_R = \theta_L \equiv \alpha \tag{161}$$

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \mapsto e^{-i\beta\gamma_5}\psi = \begin{pmatrix} (1+i\beta)\psi_L \\ (1-i\beta)\psi_R \end{pmatrix} \Rightarrow \theta_R = -\theta_L \equiv \beta. \tag{162}$$

Spinorial fields | Dirac equation

- From

$$\psi \mapsto e^{-i\alpha}\psi \Rightarrow \overline{\psi} \mapsto \overline{\psi}e^{i\alpha} \tag{163}$$

$$\psi \mapsto e^{-i\beta\gamma_5}\psi \Rightarrow \overline{\psi} \mapsto \psi^\dagger e^{i\beta\gamma_5}\gamma^0 = \psi^\dagger\gamma^0 e^{-i\beta\gamma_5} = \overline{\psi}e^{-i\beta\gamma_5}, \tag{164}$$

$$\text{using } \gamma_5^\dagger = \gamma_5, \quad \{\gamma^\mu, \gamma_5\} = 0, \tag{165}$$

the invariance of the Lagragian under these two independent transformations is evident:

$$\psi \mapsto e^{-i\alpha}\psi \Rightarrow \mathcal{L} = i\overline{\psi}\partial\!\!\!/\psi \mapsto i\overline{\psi}e^{i\alpha}\partial\!\!\!/ e^{-i\alpha}\psi = i\overline{\psi}\partial\!\!\!/\psi = \mathcal{L} \tag{166}$$

$$\psi \mapsto e^{-i\beta\gamma_5}\psi \Rightarrow \mathcal{L} = i\overline{\psi}\partial\!\!\!/\psi \mapsto i\overline{\psi}e^{-i\beta\gamma_5}\gamma^\mu\partial_\mu e^{-i\beta\gamma_5}\psi = i\overline{\psi}\partial\!\!\!/\psi = \mathcal{L}. \tag{167}$$

Spinorial fields | Dirac equation

- As a consequence, there are two conserved currents,

$$j_V^\mu = \overline{\psi}\gamma^\mu\psi \quad (\text{vector current}), \quad j_A^\mu = \overline{\psi}\gamma^\mu\gamma_5\psi \quad (\text{axial current}). \tag{168}$$

- If $m \neq 0$ only the vector current is conserved. It is enough to use the Dirac equation to check it:

$$(\mathrm{i}\partial\!\!\!/ - m)\psi = 0 \Rightarrow \begin{array}{l} \mathrm{i}\gamma^\mu\partial_\mu\psi = m\psi \\ -\mathrm{i}\partial_\mu\overline{\psi}\gamma^\mu = m\overline{\psi}, \quad \text{because } \gamma^0\gamma^{\mu\dagger}\gamma^0 = \gamma^\mu \end{array} \tag{169}$$

$$\Rightarrow \begin{array}{l} \partial_\mu j_V^\mu = \partial_\mu(\overline{\psi}\gamma^\mu\psi) = \partial_\mu\overline{\psi}\gamma^\mu\psi + \overline{\psi}\gamma^\mu\partial_\mu\psi = \mathrm{i}m\overline{\psi}\psi - \mathrm{i}m\overline{\psi}\psi = 0 \\ \partial_\mu j_A^\mu = \partial_\mu(\overline{\psi}\gamma^\mu\gamma_5\psi) = \partial_\mu\overline{\psi}\gamma^\mu\gamma_5\psi + \overline{\psi}\gamma^\mu\gamma_5\partial_\mu\psi = \mathrm{i}m\overline{\psi}\gamma_5\psi + \mathrm{i}m\overline{\psi}\gamma_5\psi = 2\mathrm{i}m\overline{\psi}\gamma_5\psi. \end{array} \tag{170}$$

▷ This is because the mass term $m\overline{\psi}\psi$ is invariant only if $\theta_R = \theta_L$ because:

$$\overline{\psi}\psi = \psi_R^\dagger\psi_L + \psi_L^\dagger\psi_R. \tag{171}$$

Spinorial fields | Majorana mass

- A Majorana field is a self-conjugate Dirac field,

$$\psi_M = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}, \quad \psi_R = \zeta\mathrm{i}\sigma^2\psi_L^*, \quad |\zeta|^2 = 1. \tag{172}$$

▷ It is evident that ψ_M can be massive in spite of being made of a single Weyl spinor. It is enough to write

$$(\mathrm{i}\partial\!\!\!/ - m)\psi_M = 0 \Rightarrow \mathrm{i}\overline{\sigma}^\mu\partial_\mu\psi_L = m\psi_R = \mathrm{i}\zeta m\sigma^2\psi_L^* \tag{173}$$

leading to a massive Klein-Gordon equation for ψ_L,

$$(\Box + m^2)\psi_L = 0 \tag{174}$$

regardless of whether ψ_R is given by ψ_L or not.

Spinorial fields | Majorana mass

- We will not write the classical Lagrangian for ψ_M because its mass term would be proportional to

$$\overline{\psi}_M\psi_M = (\psi_L^\dagger, -\mathrm{i}\zeta^*\psi_L^\mathrm{T}\sigma^2)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\psi_L\\ \mathrm{i}\zeta\sigma^2\psi_L^*\end{pmatrix}$$
$$= \mathrm{i}\zeta\psi_L^\dagger\sigma^2\psi_L^* - \mathrm{i}\zeta^*\psi_L^\mathrm{T}\sigma^2\psi_L = -\mathrm{i}\zeta^*\psi_L^\mathrm{T}\sigma^2\psi_L + \text{h.c.} \qquad (175)$$

which is null unless the ψ_M components are taken as anticommuting quantities (Grassmann variables) because

$$\mathrm{i}\psi_L^\mathrm{T}\sigma^2\psi_L = \psi_L^1\psi_L^2 - \psi_L^2\psi_L^1. \qquad (176)$$

Spinorial fields | Majorana mass

- Even more interesting is that, although the Dirac Lagrangian is invariant under the group U(1) of global transformations

$$\psi_L \mapsto \mathrm{e}^{-\mathrm{i}\alpha}\psi_L, \quad \psi_R \mapsto \mathrm{e}^{-\mathrm{i}\alpha}\psi_R, \qquad (177)$$

this cannot be a symmetry of the Majorana field because its left- and right-handed components are conjugate according to (172).

- ▷ This means that a Majorana field cannot have U(1) charges, as the electric charge, the baryon number or the lepton number, for instance.
- ▷ Is the neutrino a Majorana fermion?

Electromagnetic field | Covariant form of Maxwell equations

- The electromagnetic field is described by the four-vector A^μ.

  Defining the tensor $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$,
  the electric field $\vec{E}$ and the magnetic field $\vec{B}$ are

$$E^i = -F^{0i} = -\partial_t A^i - \nabla^i A^0, \quad B^i = -\frac{1}{2}\epsilon^{ijk}F^{jk} = (\nabla \times \vec{A})^i, \tag{178}$$

  where $F^{jk} = -\epsilon^{jk\ell}B^\ell$, because $\epsilon^{ijk}\epsilon^{jk\ell} = \epsilon^{ijk}\epsilon^{\ell jk} = 2\delta^{i\ell}$.

  Namely:

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix} .$$

Electromagnetic field | Covariant form of Maxwell equations

- Applying the Euler-Lagrange equations to the Maxwell Lagrangian,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}(\vec{E}^2 - \vec{B}^2) \tag{179}$$

  that can be also written $\mathcal{L} = -\frac{1}{2}(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu)$, we obtain the equations of motion (Maxwell equations in vaccuum)

$$\boxed{\partial_\mu F^{\mu\nu} = 0} \quad \Leftrightarrow \quad \boxed{\nabla \cdot \vec{E} = 0, \quad \nabla \times \vec{B} = \partial_t \vec{E}} \tag{180}$$

  The remaining two Maxwell equations are obtained from the dual tensor
  $\widetilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$, whose four-divergence vanishes: $\partial_\mu \widetilde{F}^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma}\partial_\mu\partial_\rho A_\sigma = 0$,

$$\boxed{\partial_\mu \widetilde{F}^{\mu\nu} = 0} \quad \Leftrightarrow \quad \boxed{\nabla \cdot \vec{B} = 0, \quad \nabla \times \vec{E} = -\partial_t \vec{B}} \tag{181}$$

Electromagnetic field **Gauge invariance**

- The Maxwell Lagrangian is invariant under local transformations $\theta = \theta(x)$ of the form

$$A_\mu(x) \mapsto A_\mu(x) - \partial_\mu \theta(x) \quad \text{(U(1) gauge transformation)}. \tag{182}$$

The existence of this local symmetry implies that $A_\mu(x)$ provides a redundant description of the electromagnetic field, because we may use the freedom to choose the gauge to constrain $A_\mu(x)$.

▷ Usually one calls "symmetry" to a transformation that leaves invariant the Lagrangian (or rather the action). However, although a local symmetry implies a global symmetry, that has physical consequences as the charged conservation (Noether's theorem), the gauge invariance is not a symmetry of the physical system, because the physical states are not transformed.

▷ Remember that the gauge invariance is necessary in order to describe massless particles of spin 1 (two degrees of freedom) using vector fields of four components (two of them are spurious). The gauge symmetry is more appropriately a gauge freedom.

Electromagnetic field **Gauge invariance**

- We may take $A_0(x) = 0$ choosing

$$A_\mu(x) \mapsto A'_\mu(x) = A_\mu(x) - \partial_\mu \int^t \mathrm{d}t' \; A_0(t', \vec{x}), \tag{183}$$

and then $A'_0(x) = A_0(x) - A_0(x) = 0$.

- We can further do another transformation, not changing the component A_0,

$$A'_\mu(x) \mapsto A''_\mu(x) = A'_\mu(x) - \partial_\mu \theta(\vec{x}), \quad \theta(\vec{x}) \equiv -\int \frac{\mathrm{d}^3 y}{4\pi |\vec{x} - \vec{y}|} \frac{\partial A'^i(t, \vec{y})}{\partial y^i}. \tag{184}$$

Although it is not apparent, this θ does not depend on t, because

$$E^i = -F^{0i} = -\partial^0 A'^i + \cancel{\partial^i A'^0} = -\partial^0 A'^i \tag{185}$$

and as $\nabla \cdot \vec{E} = \partial_i E^i = 0$ in absence of sources we have that $\partial_0 \partial_i A'^i = 0$ and then $\partial_0 \theta = 0$. Therefore, also $A''_0(x) = 0$.

Electromagnetic field | Gauge invariance

▷ Let us see which consequences do the gauge transformations have:

$$\nabla^2\theta(\vec{x}) = -\int \mathrm{d}^3y\, \frac{\partial A'^i(t,\vec{y})}{\partial y^i}\nabla_x^2\left(\frac{1}{4\pi|\vec{x}-\vec{y}|}\right) = \frac{\partial A'^i(x)}{\partial x^i} = \nabla\cdot\vec{A}', \tag{186}$$

where we have used that

$$\nabla_x^2\left(\frac{1}{4\pi|\vec{x}-\vec{y}|}\right) = -\delta^3(\vec{x}-\vec{y}), \tag{187}$$

so

$$\partial^\mu A''_\mu(x) = \partial^\mu A'_\mu(x) - \partial^\mu\partial_\mu\theta(\vec{x}) \Rightarrow \nabla\cdot\vec{A}'' = \nabla\cdot\vec{A}' - \nabla^2\theta = 0. \tag{188}$$

Then we can also take $\nabla\cdot\vec{A}=0$. This choice,

$$A^0 = 0, \quad \nabla\cdot\vec{A} = 0 \tag{189}$$

that is only possible in absence of sources, is called radiation gauge.

Electromagnetic field | Gauge invariance

- Another choice, which is always possible, is

$$A_\mu \mapsto A'_\mu = A_\mu - \partial_\mu\theta, \quad \partial_\mu\partial^\mu\theta \equiv \partial_\mu A^\mu \tag{190}$$

in such a way that we can always take

$$\partial_\mu A^\mu = 0. \tag{191}$$

This is the so called Lorenz gauge.[a] Then

$$\partial_\mu F^{\mu\nu} = 0 \Rightarrow \partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = \Box A^\nu = 0. \tag{192}$$

We see that every component of A^μ satisfies a massless Klein-Gordon equation. The solutions are of the form (A^μ is a massless real field):

$$A_\mu(x) = \epsilon_\mu(k)\mathrm{e}^{-\mathrm{i}kx} + \epsilon^*_\mu(k)\mathrm{e}^{\mathrm{i}kx}, \quad k^2 = 0. \tag{193}$$

[a]Do not confuse L.V. Lorenz (Danish physicist and mathematician), who stated the Lorenz gauge, with H.A. Lorentz (Dutch physicist, Nobel prize in 1902), who proposed the Lorentz transformations. Also do not confuse with E.N. Lorenz (American mathematician and meteorologist), founder of chaos theory, who coined the term “butterfly effect” and discovered the Lorenz attractor.

▷ The condition (191) implies that the polarization vector $\epsilon^\mu(k)$ satisfies

$$k\epsilon = 0. \tag{194}$$

- In the radiation gauge, which is compatible with the Lorenz gauge, the field is transverse because its polarizations have $\epsilon^0 = 0$ and $\vec{k}\cdot\vec{\epsilon} = 0$.

- Clarification: Unlike the constraint $\nabla\cdot\vec{A} = 0$, that can only be imposed in absence of sources, the constraint $A^0 = 0$ can be always applied. However, usually one does not take this choice when there are sources.

▷ For example, consider an observer in front of a charge e at rest at a distance r. In this case one usually takes

$$A^\mu = (\phi, \vec{A}) = \left(\frac{e}{4\pi r}, 0\right), \tag{195}$$

leading to the electromagnetic field

$$\vec{E} = -\partial_t\vec{A} - \nabla\phi = \frac{e}{4\pi r^2}\hat{r}, \quad \vec{B} = \nabla\times\vec{A} = 0. \tag{196}$$

However, one could have chosen a gauge where

$$A'^\mu = (\phi', \vec{A}') = \left(0, -\frac{et}{4\pi r^2}\hat{r}\right), \tag{197}$$

leading to the same electromagnetic field,

$$\vec{E} = -\partial_t\vec{A}' - \nabla\phi' = \frac{e}{4\pi r^2}\hat{r}, \quad \vec{B} = \nabla\times\vec{A}' = 0. \tag{198}$$

Both choices are related by the gauge transformation

$$A'_\mu(x) = A_\mu(x) - \partial_\mu\theta(x), \quad \theta(x) = \frac{et}{4\pi r}. \tag{199}$$

Electromagnetic field | Gauge invariance

- Let us find now the energy-momentum tensor. Applying the Noether's theorem:

$$\theta^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\rho)}\partial^\nu A_\rho - g^{\mu\nu}\mathcal{L} = -F^{\mu\rho}\partial^\nu A_\rho + \frac{1}{4}g^{\mu\nu}F^2, \quad F^2 \equiv F_{\mu\nu}F^{\mu\nu}, \tag{200}$$

which is neither gauge invariant nor symmetric!

▷ We can symmetrize it adding $\partial_\rho(F^{\mu\rho}A^\nu)$, that satisfies $\partial_\mu\partial_\rho(F^{\mu\rho}A^\nu) = 0$ and turns it gauge invariant,

$$\begin{aligned} T^{\mu\nu} &= -F^{\mu\rho}\partial^\nu A_\rho + \frac{1}{4}g^{\mu\nu}F^2 + \partial_\rho(F^{\mu\rho}A^\nu) \\ &= F^{\mu\rho}F_\rho{}^\nu + \frac{1}{4}g^{\mu\nu}F^2\,, \quad \text{when } \partial_\rho F^{\mu\rho} = 0\,. \end{aligned} \tag{201}$$

▷ The conserved charges under spacetime translations are then

$$E = \int d^3x\; T^{00} = \frac{1}{2}\int d^3x\;(\vec{E}^2 + \vec{B}^2) \quad \text{(energy)} \tag{202}$$

$$\mathbb{P}^i = \int d^3x\; T^{0i} = \int d^3x\;(\vec{E}\times\vec{B})^i \quad \text{(Poynting vector).} \tag{203}$$

Electromagnetic field | Minimal coupling to matter

- In the presence of sources of the electromagnetic field (charges and currents) the Maxwell equations are

$$\boxed{\partial_\mu F^{\mu\nu} = j^\nu} \quad \Leftrightarrow \quad \boxed{\nabla\cdot\vec{E} = \rho, \quad \nabla\times\vec{B} = \partial_t\vec{E} + \vec{j}} \qquad j^\mu \equiv (\rho, \vec{j}) \tag{204}$$

$$\boxed{\partial_\mu \widetilde{F}^{\mu\nu} = 0} \quad \Leftrightarrow \quad \boxed{\nabla\cdot\vec{B} = 0, \quad \nabla\times\vec{E} = -\partial_t\vec{B}} \tag{205}$$

▷ Note that the last two equations are the same in absence of sources, because $\partial_\mu\widetilde{F}^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma}\partial_\mu\partial_\rho A_\sigma = 0$ in any case.
These equations come from minimizing the action

$$S = \int d^4x\,\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - j^\mu A_\mu\right) = \int d^4x\;\mathcal{L}(x)\,. \tag{206}$$

- This action is gauge invariant only if j^μ is a conserved current, $\partial_\mu j^\mu = 0$, because

$$j^\mu A_\mu \mapsto j^\mu A_\mu - j^\mu \partial_\mu \theta \tag{207}$$

and, since $\int d^4x \, \partial_\mu(\theta j^\mu) = 0 \Rightarrow \int d^4x \, j^\mu \partial_\mu \theta = -\int d^4x \, \theta \partial_\mu j^\mu = 0$, we have that

$$\int d^4x \, j^\mu A_\mu \mapsto \int d^4x \, j^\mu A_\mu \Leftrightarrow \partial_\mu j^\mu = 0. \tag{208}$$

▷ The gauge invariance is the guiding principle dictating the interactions.

- Let us see how the gauge principle works, applying it to the Dirac Lagrangian in the presence of the electromagnetic field.

▷ The *free* Dirac Lagrangian

$$\mathcal{L}_D = \overline{\psi}(i\partial\!\!\!/ - m)\psi \tag{209}$$

is *not invariant* under U(1) gauge transformations (local phase transformations),

$$\psi \mapsto e^{-iQ\theta(x)}\psi, \quad \overline{\psi} \mapsto \overline{\psi} e^{iQ\theta(x)}. \tag{210}$$

However, the Maxwell Lagrangian,

$$\mathcal{L}_A = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{211}$$

is invariant under gauge transformations

$$A_\mu \mapsto A_\mu + \frac{1}{e}\partial_\mu \theta(x). \tag{212}$$

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▷ We can get a total Lagrangian that is gauge invariant by replacing the usual derivative ∂_μ by the covariant derivative

$$D_\mu = \partial_\mu + \mathrm{i}eQA_\mu \tag{213}$$

because then

$$\begin{aligned} D_\mu\psi = (\partial_\mu + \mathrm{i}eQA_\mu)\psi \mapsto &(\partial_\mu + \mathrm{i}eQA_\mu + \mathrm{i}Q\partial_\mu\theta)\mathrm{e}^{-\mathrm{i}Q\theta}\psi \\ &= \mathrm{e}^{-\mathrm{i}Q\theta}(-\mathrm{i}Q\partial_\mu\theta + \partial_\mu + \mathrm{i}eQA_\mu + \mathrm{i}Q\partial_\mu\theta)\psi = \mathrm{e}^{-\mathrm{i}Q\theta}D_\mu\psi \end{aligned} \tag{214}$$

and the resulting Lagrangian,

$$\begin{aligned} \mathcal{L} &= \overline{\psi}(\mathrm{i}\not{D} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\ &= \overline{\psi}(\mathrm{i}\not{\partial} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - eQA_\mu\overline{\psi}\gamma^\mu\psi \end{aligned} \tag{215}$$

is gauge invariant.

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- In this way, we have introduced an interaction of the form $j^\mu A_\mu$ (minimal coupling) between the electromagnetic field and the fermionic current,

$$j^\mu = eQ\overline{\psi}\gamma^\mu\psi \tag{216}$$

allowing us to restore the local symmetry.

▷ Note that j^μ is a conserved current due the global invariance of $\mathcal{L}_D$ under U(1) transformations, leading to the conserved current:

$$Q = \int \mathrm{d}^3x\, j^0(x) = eQ\int \mathrm{d}^3x\, \overline{\psi}\gamma^0\psi = eQ\int \mathrm{d}^3x\, \psi^\dagger\psi \quad \text{(electric charge).} \tag{217}$$

▷ Other types of gauge invariant couplings are possible, but they involve field interaction terms of canonical dimension higher than four, that must be multiplied by coupling constants with dimension of mass raised to a negative power.
For instance, the magnetic dipole moment interaction: $\mathcal{L} = a\overline{\psi}\sigma^{\mu\nu}\psi F_{\mu\nu}$, $[a] = M^{-1}$.
Such couplings will naturally emerge when quantizing the theory, providing corrections to the minimal coupling to higher orders in perturbation theory.

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- In general, let $\{T^a\}$ be the generators of a gauge symmetry group, $\{W^a_\mu(x)\}$ the gauge bosons asociated to each generator and $\{\theta^a(x)\}$ the parameters of the transformation. It is easy to check that if the fields transform as

$$\text{(fundamental irrep)} \quad \Psi \mapsto U\Psi, \quad U = \exp\{-\mathrm{i}T^a\theta^a(x)\} \tag{218}$$

$$\text{(adjoint irrep)} \quad \widetilde{W}_\mu \mapsto U\widetilde{W}_\mu U^\dagger - \frac{\mathrm{i}}{g}(\partial_\mu U)U^\dagger, \quad \widetilde{W}_\mu \equiv T^a W^a_\mu, \tag{219}$$

(Ψ is a multiplet of fermionic fields) then introducing the covariant derivative

$$D_\mu = \partial_\mu - ig\widetilde{W}_\mu \tag{220}$$

one has

$$D_\mu\Psi \mapsto UD_\mu\Psi \tag{221}$$

and the resulting Lagrangian

$$\mathcal{L} = \overline{\Psi}(\mathrm{i}\not{D} - m)\Psi \tag{222}$$

remains invariant.

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▷ For a group of non Abelian symmetries, the invariant gauge field Lagrangian (211) must be extended to include, besides the kinetic terms, cubic and quartic self-interactions, fixed by the structure constants:

$$\mathcal{L}_{\mathrm{YM}} = -\frac{1}{2}\mathrm{tr}\left(\widetilde{W}_{\mu\nu}\widetilde{W}^{\mu\nu}\right) = -\frac{1}{4}W^a_{\mu\nu}W^{a,\mu\nu} \tag{223}$$

where $\mathcal{L}_{\mathrm{YM}} = \mathcal{L}_{\mathrm{kin}} + \mathcal{L}_{\mathrm{cubic}} + \mathcal{L}_{\mathrm{quartic}}$ is the Yang-Mills Lagrangian:

$$\mathcal{L}_{\mathrm{kin}} = -\frac{1}{4}(\partial_\mu W^a_\nu - \partial_\nu W^a_\mu)(\partial^\mu W^{a,\nu} - \partial^\nu W^{a,\mu}) \tag{224}$$

$$\mathcal{L}_{\mathrm{cubic}} = -\frac{1}{2}g f^{abc}\,(\partial_\mu W^a_\nu - \partial_\nu W^a_\mu)W^{b,\mu}W^{c,\nu} \tag{225}$$

$$\mathcal{L}_{\mathrm{quartic}} = -\frac{1}{4}g^2 f^{abe}f^{cde}\,W^a_\mu W^b_\nu W^{c,\mu}W^{d,\nu} \tag{226}$$

with

$$\widetilde{W}_{\mu\nu} \equiv \partial_\mu\widetilde{W}_\mu - \partial_\nu\widetilde{W}_\mu - \mathrm{i}g[\widetilde{W}_\mu, \widetilde{W}_\nu] \mapsto U\widetilde{W}_{\mu\nu}U^\dagger \tag{227}$$

$$\Rightarrow W^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g\,f^{abc}W^b_\mu W^c_\nu. \tag{228}$$

▷ In the particular case of the group U(1) of electromagnetism the only generator is a multiple of the identity:

$$T = Q \text{ (the charge of the field in units of the coupling } g = e\text{)}.$$

From now on we will call it Q_f, because it will be the electric charge (in units of e) of the fermion f annihilated by the quantum field ψ.

3. Quantization of free fields

Scalar fields | Fock space

- Remember that in order to quantize a classical system of coordinates q^i and momenta p^i in the Schrödinger picture we promote q^i and p^i to operators and impose the commutation rules (in natural units):

$$[q^i, p^j] = \mathrm{i}\delta_{ij}, \quad [q^i, q^j] = [p^i, p^j] = 0. \tag{1}$$

- In the Heisenberg picture, where operators depend on time,

$$q^j_H(t) = \mathrm{e}^{\mathrm{i}Ht} q^j \mathrm{e}^{-\mathrm{i}Ht}, \quad p^j_H(t) = \mathrm{e}^{\mathrm{i}Ht} p^j \mathrm{e}^{-\mathrm{i}Ht} \tag{2}$$

$$(\Rightarrow \quad \partial_t q^j_H = \mathrm{i}Hq^j_H - \mathrm{i}q^j_H H = -\mathrm{i}[q_H, H], \quad \text{if } \partial_t q^j = 0)$$

we impose the equal-time commutation rules,

$$[q^i_H(t), p^j_H(t)] = \mathrm{i}\delta_{ij}, \quad [q^i_H(t), q^j_H(t)] = [p^i_H(t), p^j_H(t)] = 0. \tag{3}$$

Scalar fields | Fock space

- In a field theory we have replaced $q^i_H(t)$ by $\phi(t,\vec{x})$ and $p^i_H(t)$ by $\Pi(t,\vec{x})$, so in order to quantize the fields we promote them to operators and impose[a]

$$\boxed{[\phi(t,\vec{x}),\Pi(t,\vec{y})] = \mathrm{i}\delta^3(\vec{x}-\vec{y}), \quad [\phi(t,\vec{x}),\phi(t,\vec{y})] = [\Pi(t,\vec{x}),\Pi(t,\vec{y})] = 0} \tag{4}$$

second quantization

▷ We will study first the case of the scalar field.

[a]This procedure is called canonical quantization. There is an alternative procedure, the Feynman path integral formalism, that is particularly useful to quantize gauge field theories.

Scalar fields | Fock space

- Consider a free real scalar field ,

$$\phi(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_{\vec{p}}}}(a_{\vec{p}}\mathrm{e}^{-\mathrm{i}px} + a^\dagger_{\vec{p}}\mathrm{e}^{\mathrm{i}px}), \quad p^0 = E_{\vec{p}} \equiv +\sqrt{m^2+\vec{p}^2}, \tag{5}$$

where now ϕ, $a_{\vec{p}}$ and $a^\dagger_{\vec{p}}$ are operators. Recalling

$$\Pi(t,\vec{y}) = \partial_t\phi(t,\vec{y}) = \int \frac{\mathrm{d}^3q}{(2\pi)^3}\left(-\mathrm{i}\sqrt{\frac{E_{\vec{q}}}{2}}\right)\left(a_{\vec{q}}\mathrm{e}^{-\mathrm{i}qy} - a^\dagger_{\vec{q}}\mathrm{e}^{\mathrm{i}qy}\right) \tag{6}$$

it is easy to check that (4) implies

$$\boxed{[a_{\vec{p}},a^\dagger_{\vec{q}}] = (2\pi)^3\delta^3(\vec{p}-\vec{q}), \quad [a_{\vec{p}},a_{\vec{q}}] = [a^\dagger_{\vec{p}},a^\dagger_{\vec{q}}] = 0} \tag{7}$$

▷ In fact,

$$[\phi(t,\vec{x}),\Pi(t,\vec{y})] = \mathrm{i}\int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_{\vec{p}}}}\int \frac{\mathrm{d}^3q}{(2\pi)^3}\sqrt{\frac{E_{\vec{q}}}{2}}(2\pi)^3\delta^3(\vec{p}-\vec{q})$$
$$\times\left(\mathrm{e}^{-\mathrm{i}(E_{\vec{p}}-E_{\vec{q}})t}\mathrm{e}^{\mathrm{i}(\vec{p}\cdot\vec{x}-\vec{q}\cdot\vec{y})}+\mathrm{e}^{\mathrm{i}(E_{\vec{p}}-E_{\vec{q}})t}\mathrm{e}^{-\mathrm{i}(\vec{p}\cdot\vec{x}-\vec{q}\cdot\vec{y})}\right)$$
$$=\frac{\mathrm{i}}{2}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\left(\mathrm{e}^{\mathrm{i}\vec{p}\cdot(\vec{x}-\vec{y})}+\mathrm{e}^{-\mathrm{i}\vec{p}\cdot(\vec{x}-\vec{y})}\right)=\mathrm{i}\delta^3(\vec{x}-\vec{y}) \tag{8}$$

where we have used

$$\delta^3(\vec{x})=\int\frac{\mathrm{d}^3p}{(2\pi)^3}\mathrm{e}^{\mathrm{i}\vec{p}\cdot\vec{x}},\quad \delta^3(-\vec{x})=\delta^3(\vec{x}). \tag{9}$$

▷ The commutation rules (7) remind us of the creation and annihilation operators of the energy modes $\hbar\omega$ of a harmonic oscillator with Hamiltonian

$$H=\frac{p^2}{2m}+\frac{1}{2}m\omega^2x^2, \tag{10}$$

whose solutions are found by introducing the operators
(we reinsert the $\hbar$ to refresh our memory):

$$x=\sqrt{\frac{\hbar}{2m\omega}}(a+a^\dagger),\quad p=-\mathrm{i}\sqrt{\frac{\hbar m\omega}{2}}(a-a^\dagger), \tag{11}$$

satisfying the commutation relations,

$$[x,p]=\mathrm{i}\hbar \Rightarrow [a,a^\dagger]=1. \tag{12}$$

From them we derive

$$H=\hbar\omega(a^\dagger a+\tfrac{1}{2}). \tag{13}$$

Defining the state of minimum energy (the vacuum) $|0\rangle$ as the one annihilated by the operator a and applying (12)

$$[H, a^\dagger] = \hbar\omega a^\dagger, \quad [H, a] = -\hbar\omega a, \tag{14}$$

we have that, normalizing $\langle 0|0\rangle = 1$,

$$a|0\rangle = 0 \Rightarrow a^\dagger a|n\rangle = n|n\rangle, \quad |n\rangle \equiv \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle \tag{15}$$

from where $a^\dagger a$ is the operator number of modes, $|0\rangle$ has energy $E_0 = \frac{1}{2}\hbar\omega$ (zero-point energy) and $|n\rangle$ has energy $E_n = \hbar\omega(n + \frac{1}{2})$.

The Hamiltonian eigenstates $\{|n\rangle\}$ form the system's Hilbert space, called the Fock space.

- Coming back to our field theory, we see that (7) are the commutation relations of an infinite set of harmonic oscillators, one per value of $\vec{p}$, except for a normalization factor, which is the (infinite) volume of the system, since

$$\lim_{\vec{p}\to\vec{q}}(2\pi)^3\delta^3(\vec{p}-\vec{q}) = \lim_{\vec{p}\to\vec{q}}\int d^3x\, e^{-i(\vec{p}-\vec{q})\cdot\vec{x}} = V(\to\infty). \tag{16}$$

▷ Then, we can construct the Fock space of states using the operators creation ($a^\dagger_{\vec{p}}$) and annihilation ($a_{\vec{p}}$) of modes of momentum $\vec{p}$, from (7) and $a_{\vec{p}}|0\rangle = 0$.
This way we obtain the multiparticle states:

$$|\vec{p}_1, \vec{p}_2, \ldots\rangle = \sqrt{2E_{\vec{p}_1}}\sqrt{2E_{\vec{p}_2}}\cdots a^\dagger_{\vec{p}_1} a^\dagger_{\vec{p}_2}\cdots|0\rangle\,. \tag{17}$$

Scalar fields | Fock space

▷ The state normalization has been conveniently chosen to be Lorentz invariant. In fact, taking to simplify the state of one particle of momentum $\vec{p}$,

$$|\vec{p}\rangle = \sqrt{2E_{\vec{p}}}\, a^\dagger_{\vec{p}} |0\rangle \Rightarrow \langle\vec{q}|\vec{p}\rangle = \sqrt{2E_{\vec{q}}}\sqrt{2E_{\vec{p}}}\, \langle 0|\, a_{\vec{q}} a^\dagger_{\vec{p}} |0\rangle = 2E_{\vec{p}}(2\pi)^3\delta^3(\vec{p}-\vec{q}) \tag{18}$$

we see that this scalar product is a invariant because performing e.g. a boost along the direction z,

$$E' = \gamma(E+\beta p_z), \quad p'_x = p_x, \quad p'_y = p_y, \quad p'_z = \gamma(\beta E + p_z) \tag{19}$$

we have

$$\delta^3(\vec{p}\,'-\vec{q}\,') = \frac{\delta^3(\vec{p}-\vec{q})}{\gamma\left(\beta\dfrac{\partial E}{\partial p_z}+1\right)} = \frac{E\delta^3(\vec{p}-\vec{q})}{\gamma(\beta p_z + E)} = \frac{E}{E'}\delta^3(\vec{p}-\vec{q})$$
$$\Rightarrow E_{\vec{p}\,'}\delta^3(\vec{p}\,'-\vec{q}\,') = E_{\vec{p}}\delta^3(\vec{p}-\vec{q})\ . \tag{20}$$

Scalar fields | Fock space

In the first step we have used

$$\delta(f(x)-f(x_0)) = \frac{\delta(x-x_0)}{\left|\dfrac{\mathrm{d}f}{\mathrm{d}x}(x=x_0)\right|}\ , \quad f(x) = p'_z(p_z) = \gamma(\beta E + p_z)\ , \tag{21}$$

and in the second,

$$\frac{\mathrm{d}E}{\mathrm{d}p_z} = \frac{p_z}{E}\ , \quad \text{because } E = \sqrt{m^2+\vec{p}^{\,2}}\ . \tag{22}$$

Scalar fields | Fock space

- Let us see now what is the energy of the multiparticle states. For that, we will express the Hamiltonian in terms of the creation and annihilation operators (at $t=0$ to simplify, because the Hamiltonian is a constant of motion anyway):

$$
\begin{aligned}
H &= \int \mathrm{d}^3x\, \mathcal{H}(x) = \int \mathrm{d}^3x\, \frac{1}{2}\left(\Pi^2 + (\nabla\phi)^2 + m^2\phi^2\right) \\
&= \frac{1}{2}\int \mathrm{d}^3x \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_{\vec p}}} \int \frac{\mathrm{d}^3q}{(2\pi)^3\sqrt{2E_{\vec q}}} \\
&\times \Big\{ -E_{\vec p}E_{\vec q}\left(a_{\vec p}a_{\vec q}\mathrm{e}^{\mathrm{i}(\vec p+\vec q)\cdot\vec x} + a^\dagger_{\vec p}a^\dagger_{\vec q}\mathrm{e}^{-\mathrm{i}(\vec p+\vec q)\cdot\vec x} - a_{\vec p}a^\dagger_{\vec q}\mathrm{e}^{\mathrm{i}(\vec p-\vec q)\cdot\vec x} - a^\dagger_{\vec p}a_{\vec q}\mathrm{e}^{-\mathrm{i}(\vec p-\vec q)\cdot\vec x}\right) \\
&\quad - \vec p\cdot\vec q\left(a_{\vec p}a_{\vec q}\mathrm{e}^{\mathrm{i}(\vec p+\vec q)\cdot\vec x} + a^\dagger_{\vec p}a^\dagger_{\vec q}\mathrm{e}^{-\mathrm{i}(\vec p+\vec q)\cdot\vec x} - a_{\vec p}a^\dagger_{\vec q}\mathrm{e}^{\mathrm{i}(\vec p-\vec q)\cdot\vec x} - a^\dagger_{\vec p}a_{\vec q}\mathrm{e}^{-\mathrm{i}(\vec p-\vec q)\cdot\vec x}\right) \\
&\quad + m^2\left(a_{\vec p}a_{\vec q}\mathrm{e}^{\mathrm{i}(\vec p+\vec q)\cdot\vec x} + a^\dagger_{\vec p}a^\dagger_{\vec q}\mathrm{e}^{-\mathrm{i}(\vec p+\vec q)\cdot\vec x} + a_{\vec p}a^\dagger_{\vec q}\mathrm{e}^{\mathrm{i}(\vec p-\vec q)\cdot\vec x} + a^\dagger_{\vec p}a_{\vec q}\mathrm{e}^{-\mathrm{i}(\vec p-\vec q)\cdot\vec x}\right)\Big\} \\
&= \frac{1}{2}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\, E_{\vec p}(a^\dagger_{\vec p}a_{\vec p} + a_{\vec p}a^\dagger_{\vec p}) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\, E_{\vec p}\left(a^\dagger_{\vec p}a_{\vec p} + \frac{1}{2}V\right). \qquad (23)
\end{aligned}
$$

Scalar fields | Fock space

▷ The second term is the sum of the zero-point energy of all oscillators,

$$
E_{\text{vac}} = V\frac{1}{2}\int \frac{\mathrm{d}^3p}{(2\pi)^3}E_{\vec p} \Rightarrow \rho_{\text{vac}} = \frac{E_{\text{vac}}}{V} = \frac{1}{2}\int \frac{\mathrm{d}^3p}{(2\pi)^3}E_{\vec p}. \qquad (24)
$$

It is not worrisome that the total energy of this infinite-size system is divergent, but, moreover, the vacuum energy density ρ_{vac} is infinite. This is also not a problem, because we are interested in energy differences,[a] so we can subtract the zero-point energy and declare that H is

$$
H \equiv \int \mathrm{d}^3x\ :\frac{1}{2}\left(\Pi^2 + (\nabla\phi)^2 + m^2\phi^2\right): = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\, E_{\vec p}a^\dagger_{\vec p}a_{\vec p} \qquad (25)
$$

where $:\mathcal{O}:$ is the normal ordering of $\mathcal{O}$, consisting in writing all creation operators to the right of the annihilation operators. Therefore,

$$
:a_{\vec p}a^\dagger_{\vec p}: \equiv\ a^\dagger_{\vec p}a_{\vec p}. \qquad (26)
$$

[a] This cannot be done if gravity is included, because then the vacuum energy is relevant. The zero-point energy is related to the cosmological constant. See discussion in Maggiore's book, p. 141.

▷ This way, the vacuum has zero energy and

$$H\,|\vec p_1\vec p_2\ldots\rangle = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\; E_{\vec p} a^\dagger_{\vec p} a_{\vec p} \sqrt{2E_{\vec p_1}}\sqrt{2E_{\vec p_2}}\cdots a^\dagger_{\vec p_1} a^\dagger_{\vec p_2}\cdots|0\rangle$$
$$= (E_{\vec p_1}+E_{\vec p_2}+\cdots)\,|\vec p_1\vec p_2\ldots\rangle\,, \tag{27}$$

where we have used $a_{\vec p} a^\dagger_{\vec p_i} = (2\pi)^3\delta^3(\vec p-\vec p_i) + a^\dagger_{\vec p_i} a_{\vec p}$ from (7) and $a_{\vec p}\,|0\rangle = 0$.

▷ As for the moment,

$$P^i = \int \mathrm{d}^3x\; :\partial^0\phi\partial^i\phi: = \int \mathrm{d}^3x \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_{\vec p}}} \int \frac{\mathrm{d}^3q}{(2\pi)^3\sqrt{2E_{\vec q}}}$$
$$\times :\Big\{ -E_{\vec p} q^i a_{\vec p} a_{\vec q} \mathrm{e}^{-\mathrm{i}(p+q)x} - E_{\vec p} q^i a^\dagger_{\vec p} a^\dagger_{\vec q} \mathrm{e}^{\mathrm{i}(p+q)x} + E_{\vec p} q^i a_{\vec p} a^\dagger_{\vec q} \mathrm{e}^{-\mathrm{i}(p-q)x} + E_{\vec p} q^i a^\dagger_{\vec p} a_{\vec q} \mathrm{e}^{\mathrm{i}(p-q)x}\Big\}$$
$$= \frac{1}{2}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\; p^i : (-a_{\vec p} a_{-\vec p} - a^\dagger_{\vec p} a^\dagger_{-\vec p} + a_{\vec p} a^\dagger_{\vec p} + a^\dagger_{\vec p} a_{\vec p}) := \int \frac{\mathrm{d}^3p}{(2\pi)^3}\; p^i a^\dagger_{\vec p} a_{\vec p}, \tag{28}$$

where the first two terms in the sum are null because they result from the integration of an odd function in a symmetric interval.

Therefore,

$$P^i\,|\vec p_1\vec p_2\ldots\rangle = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\; p^i a^\dagger_{\vec p} a_{\vec p} \sqrt{2E_{\vec p_1}}\sqrt{2E_{\vec p_2}}\cdots a^\dagger_{\vec p_1} a^\dagger_{\vec p_2}\cdots|0\rangle$$
$$= (p_1^i + p_2^i + \cdots)\,|\vec p_1\vec p_2\ldots\rangle\,. \tag{29}$$

Scalar fields | Fock space

- Note that the multiparticle states $|\vec{p}_1\vec{p}_2\ldots\rangle$ are symmetric under the exchange of any pair of particles, because the creation operators commute with one another.

▷ On the other hand, remember that from Noether's theorem scalar fields have zero spin, so quanta created and annihilated by a scalar field are spin-zero particles.

▷ As a consequence the spin-statistics connection stating that particles of integer spin (0, 1, 2, ...) are bosons, i.e. they obey the Bose-Einstein statistics, implies that their states are symmetric under particle exchange.

▷ We will see that imposing anticommutation rules for the quantization of spin $\frac{1}{2}$ fields, necessary to prevent that the Hamiltonian is unbounded from below, leads to multiparticle states that are antisymmetric under particle exchange, accordingly to fermions.

▷ Therefore, in quantum field theory the spin-statistics connection is not a postulate but a theorem.

Complex fields | Antiparticles

- If the scalar field is complex,

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_{\vec{p}}}} \left(a_{\vec{p}} e^{-ipx} + b^\dagger_{\vec{p}} e^{ipx} \right), \tag{30}$$

$$\phi^\dagger(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_{\vec{p}}}} \left(a^\dagger_{\vec{p}} e^{ipx} + b_{\vec{p}} e^{-ipx} \right). \tag{31}$$

Then,

$$\begin{array}{l} [\phi(t,\vec{x}),\Pi(t,\vec{y})] = i\delta^3(\vec{x}-\vec{y}) \\ [\phi(t,\vec{x}),\phi(t,\vec{y})] = [\Pi(t,\vec{x}),\Pi(t,\vec{y})] = 0 \end{array} \Rightarrow \begin{array}{l} [a_{\vec{p}},a^\dagger_{\vec{q}}] = [b_{\vec{p}},b^\dagger_{\vec{q}}] = (2\pi)^3\delta^3(\vec{p}-\vec{q}) \\ [a_{\vec{p}},a_{\vec{q}}] = [b_{\vec{p}},b_{\vec{q}}] = [a_{\vec{p}},b_{\vec{q}}] = [a_{\vec{p}},b^\dagger_{\vec{q}}] = 0. \end{array} \tag{32}$$

In analogy to the case of the real scalar field, we construct the Fock space from

$$a_{\vec{p}}\,|0\rangle = b_{\vec{p}}\,|0\rangle = 0, \tag{33}$$

applying $a^\dagger_{\vec{p}}$ and $b^\dagger_{\vec{p}}$ successively.

- It is easy to show that, taking the normal ordering,

$$H = \int \frac{d^3p}{(2\pi)^3} E_{\vec{p}} (a^\dagger_{\vec{p}} a_{\vec{p}} + b^\dagger_{\vec{p}} b_{\vec{p}}), \quad P^i = \int \frac{d^3p}{(2\pi)^3} p^i (a^\dagger_{\vec{p}} a_{\vec{p}} + b^\dagger_{\vec{p}} b_{\vec{p}}). \tag{34}$$

We see that the quanta of a complex scalar field are two species of equal mass created by $a^\dagger_{\vec{p}}$ and $b^\dagger_{\vec{p}}$, respectively.

- The conserved U(1) charge is

$$\begin{aligned}
Q = \mathrm{i} \int d^3x \; :\phi^\dagger \overleftrightarrow{\partial_0} \phi: &= \mathrm{i} \int d^3x \int \frac{d^3p}{(2\pi)^3 \sqrt{2E_{\vec{p}}}} \int \frac{d^3q}{(2\pi)^3 \sqrt{2E_{\vec{q}}}} \\
&\times : \Big\{ \left(a^\dagger_{\vec{p}} e^{\mathrm{i}px} + b_{\vec{p}} e^{-\mathrm{i}px} \right) \partial_0 \left(a_{\vec{q}} e^{-\mathrm{i}qx} + b^\dagger_{\vec{q}} e^{\mathrm{i}qx} \right) - \partial_0 \left(a^\dagger_{\vec{p}} e^{\mathrm{i}px} + b_{\vec{p}} e^{-\mathrm{i}px} \right) \left(a_{\vec{q}} e^{-\mathrm{i}qx} + b^\dagger_{\vec{q}} e^{\mathrm{i}qx} \right) \\
&= \int d^3x \int \frac{d^3p}{(2\pi)^3 \sqrt{2E_{\vec{p}}}} \int \frac{d^3q}{(2\pi)^3 \sqrt{2E_{\vec{q}}}} \\
&\times : \Big\{ \left(a^\dagger_{\vec{p}} e^{\mathrm{i}px} + b_{\vec{p}} e^{-\mathrm{i}px} \right) E_{\vec{q}} \left(a_{\vec{q}} e^{-\mathrm{i}qx} - b^\dagger_{\vec{q}} e^{\mathrm{i}qx} \right) + E_{\vec{p}} \left(a^\dagger_{\vec{p}} e^{\mathrm{i}px} - b_{\vec{p}} e^{-\mathrm{i}px} \right) \left(a_{\vec{q}} e^{-\mathrm{i}qx} + b^\dagger_{\vec{q}} e^{\mathrm{i}qx} \right) \\
&= \int d^3x \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} : \left(a^\dagger_{\vec{p}} a_{\vec{q}} e^{\mathrm{i}(q-p)x} - b_{\vec{p}} b^\dagger_{\vec{q}} e^{-\mathrm{i}(q-p)x} \right) : \\
&= \int \frac{d^3p}{(2\pi)^3} (a^\dagger_{\vec{p}} a_{\vec{p}} - b^\dagger_{\vec{p}} b_{\vec{p}}).
\end{aligned} \tag{35}$$

▷ Therefore, the state $a^\dagger_{\vec{p}} |0\rangle$ has charge $Q = +1$ and $b^\dagger_{\vec{p}} |0\rangle$ has charge $Q = -1$.

Scalar fields | Antiparticles

▷ We can now interpret the negative energy solutions of the Klein-Gordon equation:

- The coefficient of the positive energy solution of a complex ϕ becomes the annihilation operator of a particle (of a given charge) while the coefficient of the negative energy solution becomes the creation operator of its antiparticle (of opposite charge).
- For the field $\phi^\dagger$ it is the other way around, because the roles of particles and antiparticles are exchanged.
- If the field is real, $a_{\vec{p}} = b_{\vec{p}}$, then it creates and annihilates particles that coincide with their antiparticles.

Spin $\frac{1}{2}$ fields | Dirac field

- In order to quantize the Dirac field, which is a complex field,

$$\psi(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_{\vec{p}}}} \sum_{s=1,2} \left(a_{\vec{p},s} u^{(s)}(\vec{p}) \mathrm{e}^{-\mathrm{i}px} + b^\dagger_{\vec{p},s} v^{(s)}(\vec{p}) \mathrm{e}^{\mathrm{i}px} \right), \tag{36}$$

$$\psi^\dagger(x) = \int \frac{\mathrm{d}^3q}{(2\pi)^3\sqrt{2E_{\vec{q}}}} \sum_{r=1,2} \left(a^\dagger_{\vec{q},r} u^{(r)\dagger}(\vec{q}) \mathrm{e}^{\mathrm{i}qx} + b_{\vec{q},r} v^{(r)\dagger}(\vec{q}) \mathrm{e}^{-\mathrm{i}qx} \right), \tag{37}$$

we promote the coefficients $a_{\vec{p},s}$, $b_{\vec{p},s}$ and their complex conjugates to operators and their adjoints, as we did for the scalar field.

▷ Before imposing any (anti)commutation relation on them, let us see which is the Hamiltonian operator (taking again $t = 0$ to simplify, because we know it will be time independent):

$$
\begin{aligned}
\mathcal{H} &= \int \mathrm{d}^3x\, \theta^{00} = \int \mathrm{d}^3x\, \psi^\dagger \mathrm{i}\partial_0\psi = \int \mathrm{d}^3x \int \frac{\mathrm{d}^3q}{(2\pi)^3\sqrt{2E_{\vec q}}} \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_{\vec p}}} \\
&\times \sum_{r,s} \Big\{ a^\dagger_{\vec q,r} a_{\vec p,s} \mathrm{e}^{-\mathrm{i}(\vec q-\vec p)\cdot\vec x} u^{(r)\dagger}(\vec q) E_{\vec p} u^{(s)}(\vec p) - b_{\vec q,r} b^\dagger_{\vec p,s} \mathrm{e}^{\mathrm{i}(\vec q-\vec p)\cdot\vec x} v^{(r)\dagger}(\vec q) E_{\vec p} v^{(s)}(\vec p) \\
&\qquad - a^\dagger_{\vec q,r} b^\dagger_{\vec p,s} \mathrm{e}^{-\mathrm{i}(\vec q+\vec p)\cdot\vec x} u^{(r)\dagger}(\vec q) E_{\vec p} v^{(s)}(\vec p) + b_{\vec q,r} a_{\vec p,s} \mathrm{e}^{\mathrm{i}(\vec q+\vec p)\cdot\vec x} v^{(r)\dagger}(\vec q) E_{\vec p} u^{(s)}(\vec p) \Big\} \\
&= \frac{1}{2} \int \frac{\mathrm{d}^3p}{(2\pi)^3} \sum_{r,s} \Big\{ a^\dagger_{\vec p,r} a_{\vec p,s} u^{(r)\dagger}(\vec p) u^{(s)}(\vec p) - b_{\vec p,r} b^\dagger_{\vec p,s} v^{(r)\dagger}(\vec p) v^{(s)}(\vec p) \\
&\qquad\qquad - a^\dagger_{-\vec p,r} b^\dagger_{\vec p,s} u^{(r)\dagger}(-\vec p) v^{(s)}(\vec p) + b_{-\vec p,r} a_{\vec p,s} v^{(r)\dagger}(-\vec p) u^{(s)}(\vec p) \Big\} \\
&= \int \frac{\mathrm{d}^3p}{(2\pi)^3} E_{\vec p} \sum_s (a^\dagger_{\vec p,s} a_{\vec p,s} - b_{\vec p,s} b^\dagger_{\vec p,s}),
\end{aligned}
\tag{38}
$$

where we have used

$$
\begin{aligned}
u^{(r)\dagger}(\vec p) u^{(s)}(\vec p) = 2E_{\vec p}\delta_{rs}, \quad v^{(r)\dagger}(\vec p) v^{(s)}(\vec p) = 2E_{\vec p}\delta_{rs}, \\
u^{(r)\dagger}(-\vec p) v^{(s)}(\vec p) = v^{(r)\dagger}(-\vec p) u^{(s)}(\vec p) = 0.
\end{aligned}
\tag{39}
$$

▷ If now we imposed the same commutation rules as for the complex scalar field and applied the normal ordering (subtraction of the vacuum energy), we would get a Hamiltonian unbounded from below, because the states created by $b^\dagger_{\vec p}$, the antiparticles, for all momenta would contribute with an arbitrarily large negative energy. In order to produce a meaningful energy spectrum, we are forced to rather impose anticommutation rules:

$$
\{\psi(t,\vec x), \Pi_\psi(t,\vec y)\} = \mathrm{i}\delta^3(\vec x-\vec y)\,, \quad \{\psi(t,\vec x), \psi(t,\vec y)\} = \{\Pi_\psi(t,\vec x), \Pi_\psi(t,\vec y)\} = 0
$$

$$
\Rightarrow \begin{aligned}
&\{a_{\vec p,r}, a^\dagger_{\vec q,s}\} = \{b_{\vec p,r}, b^\dagger_{\vec q,s}\} = (2\pi)^3\delta^3(\vec p-\vec q)\delta_{rs} \\
&\{a_{\vec p,r}, a_{\vec q,s}\} = \{b_{\vec p,r}, b_{\vec q,s}\} = \{a_{\vec p,r}, b_{\vec q,s}\} = \{a_{\vec p,r}, b^\dagger_{\vec q,s}\} = 0.
\end{aligned}
\tag{40}
$$

Spin $\frac{1}{2}$ fields **Dirac field**

Therefore, we must consistently define the normal ordering for fermionic operators as

$$: a_{\vec{p},r} a^{\dagger}_{\vec{p},r} : \equiv -a^{\dagger}_{\vec{p},r} a_{\vec{p},r}, \quad : b_{\vec{p},r} b^{\dagger}_{\vec{p},r} : \equiv -b^{\dagger}_{\vec{p},r} b_{\vec{p},r}, \tag{41}$$

leading to the Hamiltonian

$$\mathcal{H} = \int \mathrm{d}^3x \ : \psi^{\dagger} \mathrm{i}\partial_0 \psi : = \int \frac{\mathrm{d}^3p}{(2\pi)^3} E_{\vec{p}} \sum_s (a^{\dagger}_{\vec{p},s} a_{\vec{p},s} + b^{\dagger}_{\vec{p},s} b_{\vec{p},s}). \tag{42}$$

Likewise, for the momentum operator one has

$$P^i = \int \mathrm{d}^3x \ : \theta^{0i} : = \int \mathrm{d}^3x \ : \psi^{\dagger} \mathrm{i}\partial^i \psi : = \int \frac{\mathrm{d}^3p}{(2\pi)^3} p^i \sum_s (a^{\dagger}_{\vec{p},s} a_{\vec{p},s} + b^{\dagger}_{\vec{p},s} b_{\vec{p},s}). \tag{43}$$

Spin $\frac{1}{2}$ fields **Dirac field**

- The angular momentum (conserved Noether charge associated to the invariance under rotations) has an orbital part (identical to that of the scalar field) and a spin part (in addition). Applying the general expressions for the Noether currents, one can show that the spin part in the chiral representation is

$$\vec{S} = \int \mathrm{d}^3x \ : \psi^{\dagger} \tfrac{1}{2} \vec{\Sigma} \psi : \quad \text{where } \Sigma^i = \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}. \tag{44}$$

Expressed in the Fock space, the spin about the z axis reads

$$\begin{aligned} S_z = \frac{1}{2} \int \mathrm{d}^3x \int \frac{\mathrm{d}^3p}{(2\pi)^3 \sqrt{2E_{\vec{p}}}} \int \frac{\mathrm{d}^3q}{(2\pi)^3 \sqrt{2E_{\vec{q}}}} \\ \times \sum_{r,s} : \Big\{ \mathrm{e}^{-\mathrm{i}(\vec{q}-\vec{p})\cdot\vec{x}} a^{\dagger}_{\vec{q},r} a_{\vec{p},s} u^{(r)\dagger}(\vec{q}) \Sigma^3 u^{(s)}(\vec{p}) + \mathrm{e}^{\mathrm{i}(\vec{q}-\vec{p})\cdot\vec{x}} b_{\vec{q},r} b^{\dagger}_{\vec{p},s} v^{(r)\dagger}(\vec{q}) \Sigma^3 v^{(s)}(\vec{p}) \\ + \mathrm{e}^{-\mathrm{i}(\vec{q}+\vec{p})\cdot\vec{x}} a^{\dagger}_{\vec{q},r} b^{\dagger}_{\vec{p},s} u^{(r)\dagger}(\vec{q}) \Sigma^3 v^{(s)}(\vec{p}) + \mathrm{e}^{\mathrm{i}(\vec{q}+\vec{p})\cdot\vec{x}} b_{\vec{q},r} a_{\vec{p},s} v^{(r)\dagger}(\vec{q}) \Sigma^3 u^{(s)}(\vec{p}) \Big\} : . \end{aligned} \tag{45}$$

▷ Then the spin J_z of a state created by $a^\dagger_{\vec{p},s}|0\rangle$ or $b^\dagger_{\vec{p},s}|0\rangle$ in its rest frame ($\vec{p}=0$) is obtained by applying S_z to those states. Remember:

$$u_L^{(s)}(0) = u_R^{(s)}(0) = \sqrt{m}\,\xi^{(s)}, \quad v_L^{(s)}(0) = -v_R^{(s)}(0) = \sqrt{m}\,\eta^{(s)} \tag{46}$$

$$\xi^{(1)} = \begin{pmatrix}1\\0\end{pmatrix}, \quad \xi^{(2)} = \begin{pmatrix}0\\1\end{pmatrix}, \quad \eta^{(1)} = \begin{pmatrix}0\\1\end{pmatrix}, \quad \eta^{(2)} = -\begin{pmatrix}1\\0\end{pmatrix}. \tag{47}$$

The last line in (45) vanishes applying (39) and we get

$$S_z\, a^\dagger_{0,1}|0\rangle = +\frac{1}{2}a^\dagger_{0,1}|0\rangle, \quad S_z\, a^\dagger_{0,2}|0\rangle = -\frac{1}{2}a^\dagger_{0,2}|0\rangle, \tag{48}$$

$$S_z\, b^\dagger_{0,1}|0\rangle = +\frac{1}{2}b^\dagger_{0,1}|0\rangle, \quad S_z\, b^\dagger_{0,2}|0\rangle = -\frac{1}{2}b^\dagger_{0,2}|0\rangle, \tag{49}$$

where we have used that $:b_{\vec{p},s}b^\dagger_{\vec{p},s}: = -b^\dagger_{\vec{p},s}b_{\vec{p},s}$. Note that, thanks to our convention of taking the antiparticle spinors $\eta^{(s)} = -i\sigma^2\xi^{(s)*}$ for a given s, which is an eigenvector of σ^3 with opposite eigenvalue to that of $\xi^{(s)}$, the states of particles and antiparticles with the same s are in the same spin state.

▷ As $[K_z, J_z] = 0$, the state resulting from performing a boost along the direction of axis z (the one used to define the spin) is still an eigenstate of spin. Remember that the projection of the spin along the direction of motion is called helicity. One can check using the explicit expressions of the spinors that

$$\hat{p}\cdot\vec{\Sigma}\,u^{(1)}(\vec{p}) = +u^{(1)}(\vec{p}), \qquad \hat{p}\cdot\vec{\Sigma}\,u^{(2)}(\vec{p}) = -u^{(2)}(\vec{p}) \tag{50}$$

$$\hat{p}\cdot\vec{\Sigma}\,v^{(1)}(\vec{p}) = -v^{(1)}(\vec{p}), \qquad \hat{p}\cdot\vec{\Sigma}\,v^{(2)}(\vec{p}) = +v^{(2)}(\vec{p}), \tag{51}$$

and hence the results of (48) y (49) can be extended to the helicity states:

$$\hat{p}\cdot\vec{S}\,a^\dagger_{\vec{p},1}|0\rangle = +\frac{1}{2}a^\dagger_{\vec{p},1}|0\rangle, \quad \hat{p}\cdot\vec{S}\,\,a^\dagger_{\vec{p},2}|0\rangle = -\frac{1}{2}a^\dagger_{\vec{p},2}|0\rangle \tag{52}$$

$$\hat{p}\cdot\vec{S}\,b^\dagger_{\vec{p},1}|0\rangle = +\frac{1}{2}b^\dagger_{\vec{p},1}|0\rangle, \quad \hat{p}\cdot\vec{S}\,\,b^\dagger_{\vec{p},2}|0\rangle = -\frac{1}{2}b^\dagger_{\vec{p},2}|0\rangle, \tag{53}$$

namely, states of particle and antiparticle with the same s have the same helicity. Remember that the helicities are Lorentz invariant only for massless states (and then they are called chiralities).

- As for the U(1) charge:

$$Q = \int \mathrm{d}^3x \ :\psi^\dagger\psi: = \int \frac{\mathrm{d}^3p}{(2\pi)^3} \sum_s (a^\dagger_{\vec{p},s} a_{\vec{p},s} - b^\dagger_{\vec{p},s} b_{\vec{p},s}). \tag{54}$$

Therefore, the quantum field ψ annihilates particles and creates antiparticles of equal mass, spin $\frac{1}{2}$ and opposite charge.

- Let us see now what is the meaning of the spin eigenstate labels $s = 1, 2$.
  We associate s with the spin of the fermion about a given direction. Consider an arbitrary direction $\hat{n}(\theta, \varphi)$. Then the spin eigenstates in that direction are

$$\xi^{(s)} = (\xi(\uparrow), \xi(\downarrow))$$

$$\xi(\uparrow) \equiv D^{\frac{1}{2}}(\theta,\varphi)\xi^{(1)} = \begin{pmatrix} \cos\frac{\theta}{2} \\ \mathrm{e}^{\mathrm{i}\varphi}\sin\frac{\theta}{2} \end{pmatrix}, \qquad \text{since } (\hat{n}\cdot\vec{\sigma})\xi(\uparrow) = +\xi(\uparrow), \tag{55}$$

$$\xi(\downarrow) \equiv D^{\frac{1}{2}}(\theta,\varphi)\xi^{(2)} = \begin{pmatrix} -\mathrm{e}^{-\mathrm{i}\varphi}\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix}, \qquad \text{since } (\hat{n}\cdot\vec{\sigma})\xi(\downarrow) = -\xi(\downarrow). \tag{56}$$

(In particular, $\xi^{(s)} = (\xi^{(1)}, \xi^{(2)})$ are the spin eigenstates in the z axis.)

▷ Now, the state with opposite eigenvalue to that of any ξ is $\eta = -\mathrm{i}\sigma^2\xi^*$, because if $(\hat{n}\cdot\vec{\sigma})\xi = \xi$ then

$$(\hat{n}\cdot\vec{\sigma})\eta = (\hat{n}\cdot\vec{\sigma})(-\mathrm{i}\sigma^2\xi^*) = \mathrm{i}\sigma^2\hat{n}\cdot\vec{\sigma}^*\xi^* = -(-\mathrm{i}\sigma^2\xi^*) = -\eta, \tag{57}$$

where we have used that $\vec{\sigma}\sigma^2 = -\sigma^2\vec{\sigma}^*$. Therefore, we can also denote

$$\xi^{(-s)} \equiv \eta^{(s)} = -\mathrm{i}\sigma^2\xi^{(s)*} = (\xi(\downarrow), -\xi(\uparrow)) \tag{58}$$

to remind us that these are eigenstates with opposite eigenvalue to the one given.

– Note, by the way, that a double inversion of the spin of ξ takes it to

$$-\mathrm{i}\sigma^2\eta^* = -\mathrm{i}\sigma^2(-\mathrm{i}\sigma^2\xi^*)^* = \sigma^2\sigma^{2*}\xi = -\xi \tag{59}$$

which does not coincide with ξ, reflecting that a rotation of 2π does not take a spin $\frac{1}{2}$ system back to its original state (a rotation of 4π is needed).

▷ To summarize, take the spinors introduced in the previous chapter:

$$u^{(s)}(\vec{p}) = \begin{pmatrix} \sqrt{p\sigma}\ \xi^{(s)} \\ \sqrt{p\bar{\sigma}}\ \xi^{(s)} \end{pmatrix}, \quad v^{(s)}(\vec{p}) = \begin{pmatrix} \sqrt{p\sigma}\ \xi^{(-s)} \\ -\sqrt{p\bar{\sigma}}\ \xi^{(-s)} \end{pmatrix}. \tag{60}$$

We have seen that, given the field $\psi(x)$ of (36), the operator $a_{\vec{p},s}$ annihilates particles whose spinor $u^{(s)}(\vec{p})$ contains the $\xi^{(s)}$ and the operator $b^\dagger_{\vec{p},s}$ creates antiparticles whose spinor $v^{(s)}(\vec{p})$ contains the $\xi^{(-s)}$.

This will simplify things. For instance, we will see that the charge conjugate of a field $\psi(x)$ exchanges particles for antiparticles preserving the same state s, namely, in the same spin state or the same helicity.

Spin $\frac{1}{2}$ fields | Weyl fields (massless)

▷ And if we take the ultrarelativistic limit ($E \gg m$) to

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_{\vec{p}}}} \sum_{s=1,2} \left(a_{\vec{p},s}u^{(s)}(\vec{p})e^{-ipx} + b^\dagger_{\vec{p},s}v^{(s)}(\vec{p})e^{ipx}\right)$$

remember that

$$u^{(1)} = \begin{pmatrix} 0 \\ u_R \end{pmatrix}, \quad u^{(2)} = \begin{pmatrix} u_L \\ 0 \end{pmatrix}, \quad v^{(1)} = \begin{pmatrix} v_L \\ 0 \end{pmatrix}, \quad v^{(2)} = \begin{pmatrix} 0 \\ v_R \end{pmatrix}. \tag{61}$$

Therefore:

- The field ψ_L:
  - annihilates particles of helicity $h = -\frac{1}{2}$ (spinor $u^{(2)}$)
  - and creates antiparticles of helicity $h = +\frac{1}{2}$ (spinor $v^{(1)}$).
- The field ψ_R:
  - annihilates particles of helicity $h = +\frac{1}{2}$ (spinor $u^{(1)}$)
  - and creates antiparticles of helicity $h = -\frac{1}{2}$ (spinor $v^{(2)}$).

Spin $\frac{1}{2}$ fields | Charge conjugation

- The charge conjugate of the classical Dirac field (in the chiral representation) is

$$\psi^c(x) = \begin{pmatrix} -i\sigma^2\psi_R^*(x) \\ i\sigma^2\psi_L^*(x) \end{pmatrix} = -i\gamma^2\psi^*(x) \quad (= -i\gamma^2\gamma^0\overline{\psi}^T(x)). \tag{62}$$

▷ Let us see the result of the charge conjugation operation on one-particle states. For that we need to introduce a unitary operator C, with $C^2 = 1$, transforming the creation operators as follows,

$$Ca_{\vec{p},s}C = \eta_C b_{\vec{p},s}\ , \quad Cb_{\vec{p},s}C = \eta_C a_{\vec{p},s}\ , \tag{63}$$

where $\eta_C^2 = 1$, so that $\eta_C = \pm 1$. Then, from (60) we have

$$\begin{aligned}[v^{(s)}(\vec{p})]^* &= \begin{pmatrix} \sqrt{p\sigma}(-\mathrm{i}\sigma^2\xi^{(s)*}) \\ -\sqrt{p\bar{\sigma}}(-\mathrm{i}\sigma^2\xi^{(s)*}) \end{pmatrix}^* = \begin{pmatrix} -\mathrm{i}\sigma^2\sqrt{p\bar{\sigma}^*}\ \xi^{(s)*} \\ \mathrm{i}\sigma^2\sqrt{p\sigma^*}\ \xi^{(s)*} \end{pmatrix}^* \\ &= \begin{pmatrix} 0 & -\mathrm{i}\sigma^2 \\ \mathrm{i}\sigma^2 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{p\sigma}\ \xi^{(s)} \\ \sqrt{p\bar{\sigma}}\ \xi^{(s)} \end{pmatrix} = -\mathrm{i}\gamma^2 u^{(s)}(\vec{p}),\end{aligned} \tag{64}$$

(first we have used $\sigma^2\vec{\sigma}\sigma^2 = -\vec{\sigma}^*$ with $(\sigma^2)^2 = 1$ and then $\sigma^{2*} = -\sigma^2$) from where

$$u^{(s)}(\vec{p}) = -\mathrm{i}\gamma^2[v^{(s)}(\vec{p})]^*, \quad v^{(s)}(\vec{p}) = -\mathrm{i}\gamma^2[u^{(s)}(\vec{p})]^* \tag{65}$$

(because $(\gamma^2)^2 = -1$.)

▷ This way the operator version of (62) is satisfied:

$$\begin{aligned}C\psi(x)C &= \eta_C \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_{\vec{p}}}} \sum_s \left(b_{\vec{p},s}u^{(s)}(\vec{p})\mathrm{e}^{-\mathrm{i}px} + a^\dagger_{\vec{p},s}v^{(s)}(\vec{p})\mathrm{e}^{\mathrm{i}px}\right) \\ &= -\mathrm{i}\eta_C\gamma^2 \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_{\vec{p}}}} \sum_s \left(b_{\vec{p},s}[v^{(s)}(\vec{p})]^*\mathrm{e}^{-\mathrm{i}px} + a^\dagger_{\vec{p},s}[u^{(s)}(\vec{p})]^*\mathrm{e}^{\mathrm{i}px}\right) \\ &= -\mathrm{i}\eta_C\gamma^2\psi^*(x).\end{aligned} \tag{66}$$

Therefore, the charge conjugation exchanges particles for antiparticles preserving the spin state, i.e. keeping the helicities.

Spin $\frac{1}{2}$ fields | Charge conjugation

▷ For Majorana fields, that at classical level satisfy

$$\psi_M^c(x) = \zeta^* \psi_M(x) \tag{67}$$

one has that[a]

$$\begin{aligned} &C\psi_M(x)C = \psi_M(x) \\ \Rightarrow &\eta_C \int \frac{\mathrm{d}^3 p}{(2\pi)^3 \sqrt{2E_{\vec{p}}}} \sum_s \left(b_{\vec{p},s} u^{(s)}(\vec{p}) \mathrm{e}^{-\mathrm{i}px} + a^\dagger_{\vec{p},s} v^{(s)}(\vec{p}) \mathrm{e}^{\mathrm{i}px} \right) \\ &= \int \frac{\mathrm{d}^3 p}{(2\pi)^3 \sqrt{2E_{\vec{p}}}} \sum_s \left(a_{\vec{p},s} u^{(s)}(\vec{p}) \mathrm{e}^{-\mathrm{i}px} + b^\dagger_{\vec{p},s} v^{(s)}(\vec{p}) \mathrm{e}^{\mathrm{i}px} \right) \\ \Rightarrow &a_{\vec{p},s} = \eta_C \, b_{\vec{p},s} \, , \end{aligned} \tag{68}$$

namely, particle and antiparticle coincide, because

$$C a^\dagger_{\vec{p},s} |0\rangle = C a^\dagger_{\vec{p},s} C C |0\rangle = \eta_C b^\dagger_{\vec{p},s} |0\rangle = a^\dagger_{\vec{p},s} |0\rangle \, , \tag{69}$$

where we have taken $C|0\rangle \equiv |0\rangle$.

[a] The complex phase ζ^* to the right of the equality (67) is incorporated in the definition (63) introduced to the right of equality (68), so we can say that $\eta_C = \zeta$ and hences it must be real, because $\eta_C = \pm 1$.

Complex scalar fields | Charge conjugation

- For a complex scalar field, ignore spinors and spinorial indices:

$$C\phi(x)C = \eta_C \phi^*(x) \Rightarrow C a^\dagger_{\vec{p}} C = \eta_C b^\dagger_{\vec{p}} \, , \quad C b^\dagger_{\vec{p}} C = \eta_C a^\dagger_{\vec{p}} . \tag{70}$$

▷ The Majorana field is the analogous to the real scalar field.

Spin $\frac{1}{2}$ fields | Parity

- The parity transformed of the classical Dirac field (in the chiral representation) is

$$\psi(x) = \begin{pmatrix} \psi_L(x) \\ \psi_R(x) \end{pmatrix} \mapsto \begin{pmatrix} \psi_R(\tilde{x}) \\ \psi_L(\tilde{x}) \end{pmatrix} = \gamma^0 \begin{pmatrix} \psi_L(\tilde{x}) \\ \psi_R(\tilde{x}) \end{pmatrix} = \gamma^0 \psi(\tilde{x}), \quad \text{where } \tilde{x} = (t, -\vec{x}). \tag{71}$$

▷ We need that on one-particle states the parity acts as follows:

$$P a_{\vec{p},s} P = \eta_a a_{-\vec{p},s} \,, \quad P b_{\vec{p},s} P = \eta_b b_{-\vec{p},s} \,, \tag{72}$$

where P is a unitary operator, with $P^2 = 1$, and η_a, η_b being phases that we will call intrinsic parities of particles and antiparticles, respectively.

We will assume $P\,|0\rangle = |0\rangle$.

Because observables depend on an even number of fermionic oparators, from the constraint $P^2 = 1$ we may take $\eta_a^2, \eta_b^2 = \pm 1$
(the minus sign with be needed for the Majorana fields).

Spin $\frac{1}{2}$ fields | Parity

- Then,

$$\begin{aligned} P\psi(t,\vec{x})P &= \int \frac{d^3p}{(2\pi)^3\sqrt{2E_{\vec{p}}}} \sum_s \left(\eta_a a_{-\vec{p},s} u^{(s)}(\vec{p}) e^{-ipx} + \eta_b^* b^\dagger_{-\vec{p},s} v^{(s)}(\vec{p}) e^{ipx} \right) \\ &= \int \frac{d^3p}{(2\pi)^3\sqrt{2E_{\vec{p}}}} \sum_s \left(\eta_a a_{\vec{p},s} u^{(s)}(-\vec{p}) e^{-ip\tilde{x}} + \eta_b^* b^\dagger_{\vec{p},s} v^{(s)}(-\vec{p}) e^{ip\tilde{x}} \right) \\ &= \gamma^0 \int \frac{d^3p}{(2\pi)^3\sqrt{2E_{\vec{p}}}} \sum_s \left(\eta_a a_{\vec{p},s} u^{(s)}(\vec{p}) e^{-ip\tilde{x}} - \eta_b^* b^\dagger_{\vec{p},s} v^{(s)}(\vec{p}) e^{ip\tilde{x}} \right) \\ &= \eta_a \gamma^0 \psi(t,-\vec{x}), \quad \text{if } \eta_a = -\eta_b^*, \end{aligned} \tag{73}$$

where we have first changed $\vec{p}$ for $-\vec{p}$, which implies replacing px by $p\tilde{x}$ and

$$u^{(s)}(-\vec{p}) = \gamma^0 u^{(s)}(\vec{p}), \quad v^{(s)}(-\vec{p}) = -\gamma^0 v^{(s)}(\vec{p}). \tag{74}$$

If it is a Majorana field, then $a = b$ and the constraint $\eta_a = -\eta_a^*$ forces $\eta_a = \pm i$ ($\eta_a^2 = -1$), as we had anticipated. Otherwise the intrinsic parity of a fermion $\eta_a = \pm 1$ is opposite to that of its antifermion, $\eta_b = -\eta_a$.

Spin $\frac{1}{2}$ fields | Parity

- The value of η_a (o η_b) is irrelevant for any observable involving only fermions (or antifermions), but the sign difference has consequences if *both* fermions and antifermions are present.

  (See for instance the positronium system, a bound state of electron and positron.)

Scalar fields | Parity

- For a scalar field, ignoring spinors and spinorial indices, it is straightfoward that

$$P\phi(t,\vec{x})P = \eta_a\phi(t,-\vec{x}), \quad \text{si } \eta_a = \eta_b. \tag{75}$$

  Namely, the intrinsic parities of a spin-zero particle and and its antiparticle are the same.

Spin $\frac{1}{2}$ fields | Time inversion

- We need that the time inversion T changes

$$t \mapsto -t, \quad \vec{p} \mapsto -\vec{p}, \quad \vec{J} \mapsto -\vec{J}. \tag{76}$$

Note first that if H is invariant under time inversion then T must be an antiunitary operator because $T\mathrm{e}^{-\mathrm{i}Ht} = \mathrm{e}^{\mathrm{i}Ht}T$.

Then for any couple of states,

$$\langle Ta|Tb\rangle = \langle a|b\rangle^* = \langle b|a\rangle\,, \quad T(z\,|a\rangle) = z^* T\,|a\rangle\,. \tag{77}$$

The action of T is defined by

$$Ta_{\vec{p},s}T = a_{-\vec{p},-s}\,, \quad Tb_{\vec{p},s}T = b_{-\vec{p},-s}\,, \tag{78}$$

where

$$a_{-\vec{p},-s} \equiv (a_{-\vec{p},2}, -a_{-\vec{p},1})\,, \quad b_{-\vec{p},-s} \equiv (b_{-\vec{p},2}, -b_{-\vec{p},1}). \tag{79}$$

Spin $\frac{1}{2}$ fields | Time inversion

▷ Therefore,

$$\begin{aligned}
T\psi(t,\vec{x})T &= \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_{\vec{p}}}} \sum_s T\left(a_{\vec{p},s}u^{(s)}(\vec{p})\mathrm{e}^{-\mathrm{i}px} + b^\dagger_{\vec{p},s}v^{(s)}(\vec{p})\mathrm{e}^{\mathrm{i}px}\right)T \\
&= \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_{\vec{p}}}} \sum_s \left(a_{-\vec{p},-s}[u^{(s)}(\vec{p})]^*\mathrm{e}^{\mathrm{i}px} + b^\dagger_{-\vec{p},-s}[v^{(s)}(\vec{p})]^*\mathrm{e}^{-\mathrm{i}px}\right) \\
&= \gamma^1\gamma^3 \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_{\vec{p}}}} \sum_s \left(a_{-\vec{p},-s}u^{(-s)}(-\vec{p})\mathrm{e}^{\mathrm{i}px} + b^\dagger_{-\vec{p},-s}v^{(-s)}(-\vec{p})\mathrm{e}^{-\mathrm{i}px}\right) \\
&= \gamma^1\gamma^3 \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_{\vec{p}}}} \sum_s \left(a_{\vec{p},s}u^{(s)}(\vec{p})\mathrm{e}^{-\mathrm{i}px'} + b^\dagger_{\vec{p},s}v^{(s)}(\vec{p})\mathrm{e}^{\mathrm{i}px'}\right) \\
&= \gamma^1\gamma^3\,\psi(-t,\vec{x})\,,
\end{aligned} \tag{80}$$

Spin $\frac{1}{2}$ fields | Time inversion

where first we have used

$$u^{(-s)}(-\vec{p}) = \begin{pmatrix} \sqrt{p\sigma}(-i\sigma^2\xi^{(s)*}) \\ \sqrt{p\bar{\sigma}}(-i\sigma^2\xi^{(s)*}) \end{pmatrix} = \begin{pmatrix} -i\sigma^2\sqrt{p\sigma^*}\;\xi^{(s)*} \\ -i\sigma^2\sqrt{p\bar{\sigma}^*}\;\xi^{(s)*} \end{pmatrix}$$

$$= -i\begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix} [u^{(s)}(\vec{p})]^* = -\gamma^1\gamma^3[u^{(s)}(\vec{p})]^*$$

$$\Rightarrow [u^{(s)}(\vec{p})]^* = \gamma^1\gamma^3 u^{(-s)}(-\vec{p}), \tag{81}$$

$$[v^{(s)}(\vec{p})]^* = \gamma^1\gamma^3 v^{(-s)}(-\vec{p}), \tag{82}$$

and then we have changed s for $-s$ and $\vec{p}$ for $-\vec{p}$ that amounts to the substitution of px by $-px'$ with $x' = (-t, \vec{x})$.

Scalar fields | Time inversion

- For a scalar field, it is easy to show that

$$T\phi(t,\vec{x})T = \phi(-t,\vec{x}). \tag{83}$$

C, P, T of fermion bilinears

- From the transformation properties of $\psi(x)$ we can get those of the fermion bilinears:

	C	P	T	CPT
$S(x) = \overline{\psi}(x)\psi(x)$	$S(x)$	$S(\tilde{x})$	$S(-\tilde{x})$	$S(-x)$
$P(x) = \overline{\psi}(x)\gamma_5\psi(x)$	$P(x)$	$-P(\tilde{x})$	$P(-\tilde{x})$	$-P(-x)$
$V^\mu(x) = \overline{\psi}(x)\gamma^\mu\psi(x)$	$-V^\mu(x)$	$V_\mu(\tilde{x})$	$V_\mu(-\tilde{x})$	$-V^\mu(-x)$
$A^\mu(x) = \overline{\psi}(x)\gamma^\mu\gamma_5\psi(x)$	$A^\mu(x)$	$-A_\mu(\tilde{x})$	$A_\mu(-\tilde{x})$	$-A^\mu(-x)$
$T^{\mu\nu}(x) = \overline{\psi}(x)\sigma^{\mu\nu}\psi(x)$	$-T^{\mu\nu}(x)$	$T_{\mu\nu}(\tilde{x})$	$-T_{\mu\nu}(-\tilde{x})$	$T^{\mu\nu}(-x)$

with $\tilde{x} = (t, -\vec{x})$.

- Remember that in the radiation gauge,

$$A^0(x) = 0, \quad \nabla \cdot \vec{A} = 0, \tag{84}$$

the three non zero components of $A^\mu(x)$ satisfy a massless Klein-Gordon equation,

$$\Box A^i = 0. \tag{85}$$

The solutions are of the form

$$\vec{A}(x) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3 \sqrt{2\omega_{\vec{k}}}} \sum_{\lambda=1,2} \left(\vec{\epsilon}(\vec{k},\lambda) a_{\vec{k},\lambda} \mathrm{e}^{-\mathrm{i}kx} + \vec{\epsilon}^{\,*}(\vec{k},\lambda) a^*_{\vec{k},\lambda} \mathrm{e}^{\mathrm{i}kx} \right) \tag{86}$$

with

$$\begin{aligned} k^\mu = (\omega_{\vec{k}}, \vec{k}), \quad \omega_{\vec{k}} = |\vec{k}| \quad &\Leftarrow \quad k^2 = 0, \\ \vec{k} \cdot \vec{\epsilon}(\vec{k},\lambda) = 0 \quad &\Leftarrow \quad \nabla \cdot \vec{A} = 0 \end{aligned} \tag{87}$$

and $\vec{\epsilon}(\vec{k},1)$, $\vec{\epsilon}(\vec{k},2)$ two polarization vectors orthogonal to each other.

▷ Let us find the conjugate momenta,

$$\Pi^0(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0 A_0)} = 0, \quad \text{since } \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{88}$$

$$\Pi^i(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0 A_i)} = -F^{0i}(x) = -\partial^0 A^i(x) = E^i(x) \quad \text{(electric field).} \tag{89}$$

We see that $A^0(x) = 0$ (in this gauge) and $\Pi^0(x) = 0$ (in general), so they are not dynamical variables.

▷ In order to quantize the electromagnetic field we promote, as so far, $\vec{A}(x)$ to an operator imposing

$$[a_{\vec{k},\lambda}, a^\dagger_{\vec{q},\lambda'}] = (2\pi)^3 \delta^3(\vec{k} - \vec{q}) \delta_{\lambda\lambda'} , \quad [a_{\vec{k},\lambda}, a_{\vec{q},\lambda'}] = [a^\dagger_{\vec{k},\lambda}, a^\dagger_{\vec{q},\lambda'}] = 0, \tag{90}$$

where $a^\dagger_{\vec{k},\lambda}$ ($a_{\vec{k},\lambda}$) are operators on the Fock space creating (annihilating) photons.

▷ Note that previous commutation relations imply

$$
\begin{aligned}
[A^i(t,\vec{x}),E^j(t,\vec{y})] = -\mathrm{i}\int \frac{\mathrm{d}^3k}{(2\pi)^3\sqrt{2\omega_{\vec{k}}}}\int \frac{\mathrm{d}^3q}{(2\pi)^3\sqrt{2\omega_{\vec{q}}}}\omega_{\vec{q}}(2\pi)^3\delta^3(\vec{k}-\vec{q})\delta_{\lambda\lambda'} \\
\times\Big\{\mathrm{e}^{-\mathrm{i}(\omega_{\vec{k}}-\omega_{\vec{q}})t}\mathrm{e}^{\mathrm{i}(\vec{k}\cdot\vec{x}-\vec{q}\cdot\vec{y})}\sum_{\lambda,\lambda'}\epsilon^i(\vec{k},\lambda)\epsilon^{j*}(\vec{q},\lambda') \\
+\mathrm{e}^{\mathrm{i}(\omega_{\vec{k}}-\omega_{\vec{q}})t}\mathrm{e}^{-\mathrm{i}(\vec{k}\cdot\vec{x}-\vec{q}\cdot\vec{y})}\sum_{\lambda,\lambda'}\epsilon^{i*}(\vec{k},\lambda)\epsilon^{j}(\vec{q},\lambda')\Big\} \\
= -\frac{\mathrm{i}}{2}\int\frac{\mathrm{d}^3k}{(2\pi)^3}\Big\{\mathrm{e}^{\mathrm{i}\vec{k}\cdot(\vec{x}-\vec{y})}\sum_{\lambda}\epsilon^i(\vec{k},\lambda)\epsilon^{j*}(\vec{k},\lambda) \\
+\mathrm{e}^{-\mathrm{i}\vec{k}\cdot(\vec{x}-\vec{y})}\sum_{\lambda}\epsilon^{i*}(\vec{k},\lambda)\epsilon^{j}(\vec{k},\lambda)\Big\} \\
= -\mathrm{i}\int\frac{\mathrm{d}^3k}{(2\pi)^3}\mathrm{e}^{\mathrm{i}\vec{k}\cdot(\vec{x}-\vec{y})}\frac{1}{2}\sum_{\lambda}\Big\{\epsilon^i(\vec{k},\lambda)\epsilon^{j*}(\vec{k},\lambda)+\epsilon^{i*}(-\vec{k},\lambda)\epsilon^{j}(-\vec{k},\lambda)\Big\} \\
= -\mathrm{i}\int\frac{\mathrm{d}^3k}{(2\pi)^3}\mathrm{e}^{\mathrm{i}\vec{k}\cdot(\vec{x}-\vec{y})}\left(\delta^{ij}-\frac{k^ik^j}{\vec{k}^2}\right) \equiv \mathrm{i}g^{ij}\delta^3_{\mathrm{tr}}(\vec{x}-\vec{y}).
\end{aligned} \tag{91}
$$

▷ In the last step we have used that the term in braces must be covariant under rotations and hence it is a combination of the rank-two tensors under rotations that can be written using δ^{ij} and k^i, that is,

$$
A\delta^{ij}+B\frac{k^ik^j}{\vec{k}^2} = \frac{1}{2}\sum_{\lambda}\Big\{\epsilon^i(\vec{k},\lambda)\epsilon^{j*}(\vec{k},\lambda)+\epsilon^{i*}(-\vec{k},\lambda)\epsilon^{j}(-\vec{k},\lambda)\Big\}\,. \tag{92}
$$

Multiplying by k^i, and from (87), we have that

$$
Ak^j + Bk^j = 0 \Rightarrow A = -B\,, \tag{93}
$$

and taking, for example, $\vec{k}=(0,0,\omega_{\vec{k}})$, $\vec{\epsilon}(\vec{k},1)=(1,0,0)$ and $\vec{\epsilon}(\vec{k},2)=(0,1,0)$ it is enough to check the term $i=j=1$ to find

$$
A = \frac{1}{2}(1+1) = 1\,. \tag{94}
$$

▷ If not for the term $k^i k^j/\vec{k}^2$ in (91) we would have got

$$-\mathrm{i}\delta^{ij}\int\frac{\mathrm{d}^3k}{(2\pi)^3}\mathrm{e}^{\mathrm{i}\vec{k}\cdot(\vec{x}-\vec{y})} = -\mathrm{i}\delta^{ij}\delta^3(\vec{x}-\vec{y}) = \mathrm{i}g^{ij}\delta^3(\vec{x}-\vec{y})\ . \tag{95}$$

This term is responsible for the transversality condition of the electromagnetic field to be satisfied in the radiation gauge ($\vec{k}\cdot\vec{\epsilon}=0$), coming from $\nabla\cdot\vec{A}=0$ and also $\nabla\cdot\vec{E}=0$. Then,

$$[\nabla\cdot\vec{A}(t,\vec{x}),E^j(t,\vec{y})] = \nabla_x[\vec{A}(t,\vec{x}),E^j(t,\vec{y})] = -\mathrm{i}\int\frac{\mathrm{d}^3k}{(2\pi)^3}\mathrm{e}^{\mathrm{i}\vec{k}\cdot(\vec{x}-\vec{y})}(k^j-k^j) = 0\ , \tag{96}$$

$$[A^i(t,\vec{x}),\nabla\cdot\vec{E}(t,\vec{y})] = \nabla_y[A^i(t,\vec{x}),\vec{E}(t,\vec{y})] = -\mathrm{i}\int\frac{\mathrm{d}^3k}{(2\pi)^3}\mathrm{e}^{\mathrm{i}\vec{k}\cdot(\vec{x}-\vec{y})}(k^i-k^i) = 0\ . \tag{97}$$

That is why we have introduced in (91) the transverse delta $\delta^3_{\mathrm{tr}}(\vec{x}-\vec{y})$.

- We are ready to construct the Fock space by acting with $a^\dagger_{\vec{k},\lambda}$ on the vacuum defined by $a_{\vec{k},\lambda}|0\rangle=0$. Applying the normal ordering to the classical expression, for the reasons we already know, we obtain

$$H = \frac{1}{2}\int\mathrm{d}^3x\ :\vec{E}^2+\vec{B}^2: = \int\frac{\mathrm{d}^3k}{(2\pi)^3}\sum_{\lambda=1,2}\omega_{\vec{k}}\ a^\dagger_{\vec{k},\lambda}a_{\vec{k},\lambda}\ ,$$
$$\vec{\mathbb{P}} = \int\mathrm{d}^3x\ :\vec{E}\times\vec{B}: = \int\frac{\mathrm{d}^3k}{(2\pi)^3}\sum_{\lambda=1,2}\vec{k}\ a^\dagger_{\vec{k},\lambda}a_{\vec{k},\lambda}\ . \tag{98}$$

▷ Therefore $\sqrt{2\omega_{\vec{k}}}\,a^\dagger_{\vec{k},\lambda}|0\rangle$ is the state of a massless particle with energy $\omega_{\vec{k}}$ and momentum $\vec{k}$. It has two possible polarizations $\lambda=1,2$ that we will analyze next.

- Applying Noether's theorem one can find that the quantity conserved under rotations is (show it!)

$$M^{ij} = \int d^3x\ \partial_0 A^k (x^i \partial^j - x^j \partial^i) A_k + \int d^3x\ (A^i \partial_0 A^j - A^j \partial_0 A^i). \tag{99}$$

The first term is the orbital angular momentum and the second is the spin part. Let us focus on the spin:

$$\begin{aligned} S^{ij} &= \int d^3x\ :A^i \partial_0 A^j - A^j \partial_0 A^i: \\ &= i \int \frac{d^3q}{(2\pi)^3} \sum_{\lambda',\lambda''} \Big(\epsilon^i(\vec{q},\lambda'')\epsilon^{j*}(\vec{q},\lambda') - \epsilon^{i*}(\vec{q},\lambda')\epsilon^j(\vec{q},\lambda'') \Big)\, a^\dagger_{\vec{q},\lambda'} a_{\vec{q},\lambda''}. \end{aligned} \tag{100}$$

Then, using (90),

$$a_{\vec{q},\lambda''} a^\dagger_{\vec{k},\lambda} |0\rangle = [a_{\vec{q},\lambda''}, a^\dagger_{\vec{k},\lambda}] |0\rangle = (2\pi)^3 \delta^3(\vec{q}-\vec{k}) \delta_{\lambda,\lambda''} |0\rangle \tag{101}$$

we get

$$S^{ij} a^\dagger_{\vec{k},\lambda} |0\rangle = i \sum_{\lambda'} \Big(\epsilon^i(\vec{k},\lambda)\epsilon^{j*}(\vec{k},\lambda') - \epsilon^{i*}(\vec{k},\lambda')\epsilon^j(\vec{k},\lambda) \Big)\, a^\dagger_{\vec{k},\lambda'} |0\rangle\,. \tag{102}$$

▷ Take now $\vec{k} = (0,0,\omega_{\vec{k}})$ and find the spin in the direction of the z axis, $S^3 = S^{12}$. Let us choose the basis of linearly polarized states $\vec{\epsilon}(\vec{k},1) = (1,0,0)$ and $\vec{\epsilon}(\vec{k},2) = (0,1,0)$, that is, $\epsilon^i(\vec{k},\lambda) = \delta^i_\lambda$. Then,

$$S^3 a^\dagger_{\vec{k},\lambda} |0\rangle = i \sum_{\lambda'} (\delta^1_\lambda \delta^2_{\lambda'} - \delta^1_{\lambda'} \delta^2_\lambda) a^\dagger_{\vec{k},\lambda'} |0\rangle \Rightarrow \begin{array}{l} S^3 a^\dagger_{\vec{k},1} |0\rangle = +i a^\dagger_{\vec{k},2} |0\rangle \\ S^3 a^\dagger_{\vec{k},2} |0\rangle = -i a^\dagger_{\vec{k},1} |0\rangle \end{array} \tag{103}$$

We see that the linear polarizations are not helicity eigenstates. However, circular polarizations are helicity eigenstates:

$$\vec{\epsilon}(\vec{k},\pm) = \frac{1}{\sqrt{2}} (\vec{\epsilon}(\vec{k},1) \pm i \vec{\epsilon}(\vec{k},2)) \tag{104}$$

because

$$S^3 a^\dagger_{\vec{k},+} |0\rangle = +a^\dagger_{\vec{k},+} |0\rangle\,, \quad a^\dagger_{\vec{k},+} = \frac{1}{\sqrt{2}} (a^\dagger_{\vec{k},1} + i a^\dagger_{\vec{k},2})\,, \tag{105}$$

$$S^3 a^\dagger_{\vec{k},-} |0\rangle = -a^\dagger_{\vec{k},-} |0\rangle\,, \quad a^\dagger_{\vec{k},-} = \frac{1}{\sqrt{2}} (a^\dagger_{\vec{k},1} - i a^\dagger_{\vec{k},2})\,. \tag{106}$$

▷ Then, the states $\sqrt{2\omega_{\vec{k}}}\, a^\dagger_{\vec{k},\pm}\,|0\rangle$ describe massless particles of spin 1 and helicity ± 1.

- A final comment is in order: although the Lorentz covariance is broken by the choice of frame in this gauge, one can check that the Poincaré generators written in terms of creation and annihilation operators satisfy the Lie algebra, as they have to.

- We would like to be able to impose a covariant quantization,

$$[A^\mu(t,\vec{x}),\Pi^\nu(t,\vec{y})] = \mathrm{i}g^{\mu\nu}\delta^3(\vec{x}-\vec{y})\ , \quad [A^\mu(t,\vec{x}),A^\nu(t,\vec{y})] = 0 \tag{107}$$

However this is not possible because, as we have seen in (88) and (89), $\Pi^0(x) = 0$. Instead, if the Lagrangian was

$$\mathcal{L}' = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(\partial_\mu A^\mu)^2\ , \tag{108}$$

which is not Maxwell Lagrangian, we would have

$$\Pi^\mu(x) = \frac{\partial\mathcal{L}'}{\partial(\partial_0 A_\mu)} \Rightarrow \begin{array}{l} \Pi^0(x) = -\partial_\mu A^\mu(x) \\ \Pi^i(x) = -F^{0i} = E^i(x) \quad \text{(as before)} \end{array} \tag{109}$$

▷ Rewriting

$$\mathcal{L}' = -\frac{1}{2}(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu) - \frac{1}{2} g^{\mu\nu} \partial_\mu A_\nu \partial_\alpha A^\alpha \tag{110}$$

the Euler-Lagrange equations are

$$\begin{aligned} \frac{\partial \mathcal{L}'}{\partial A_\nu} - \partial_\mu \frac{\partial \mathcal{L}'}{\partial(\partial_\mu A_\nu)} &= 0 \Rightarrow \partial_\mu \left(F^{\mu\nu} + g^{\mu\nu} \partial_\alpha A^\alpha \right) = 0 \\ &\Rightarrow \Box A^\nu - \partial^\nu \partial_\mu A^\mu + \partial^\nu \partial_\mu A^\mu = 0 \\ &\Rightarrow \Box A^\nu = 0 \,, \end{aligned} \tag{111}$$

so A^μ is massless, where we have used

$$\begin{aligned} \partial_\mu F^{\mu\nu} &= \partial_\mu (\partial^\mu A^\nu - \partial^\nu A^\mu) = \Box A^\nu - \partial^\nu \partial_\mu A^\mu \\ \partial_\mu (g^{\mu\nu} \partial_\alpha A^\alpha) &= \partial^\nu \partial_\mu A^\mu \,, \end{aligned} \tag{112}$$

whose solutions are

$$A^\mu(x) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3 \sqrt{2\omega_{\vec{k}}}} \sum_{\lambda=0}^{3} \left(\epsilon^\mu(\vec{k},\lambda) a_{\vec{k},\lambda} \mathrm{e}^{-ikx} + \epsilon^{\mu *}(\vec{k},\lambda) a^\dagger_{\vec{k},\lambda} \mathrm{e}^{ikx} \right) \,. \tag{113}$$

▷ Because this time we have not imposed $\epsilon^0 = 0$ or $k_\mu \epsilon^\mu = 0$, the field A^μ has four degrees of freedom, labeled by $\lambda = 0, 1, 2, 3$.
Obviously the Lagrangian $\mathcal{L}'$ is not gauge invariant.
In particular, it we take $k^\mu = (k, 0, 0, k)$ then $\epsilon^\mu(\vec{k},\lambda) = \delta^\mu_\lambda$:
$\epsilon^\mu(\vec{k},0) = (1,0,0,0)$,
$\epsilon^\mu(\vec{k},1) = (0,1,0,0)$,
$\epsilon^\mu(\vec{k},2) = (0,0,1,0)$,
$\epsilon^\mu(\vec{k},3) = (0,0,0,1)$.
Only $\epsilon^\mu(\vec{k},1)$ and $\epsilon^\mu(\vec{k},2)$ satisfy $k_\mu \epsilon^\mu = 0$.

▷ It is easy to check that the commutation rules (107) imply

$$[a_{\vec{k},\lambda}, a^\dagger_{\vec{q},\lambda'}] = \zeta_\lambda \delta_{\lambda\lambda'} (2\pi)^3 \delta^3(\vec{k} - \vec{q}) \,, \quad [a_{\vec{k},\lambda}, a_{\vec{q},\lambda'}] = [a^\dagger_{\vec{k},\lambda}, a^\dagger_{\vec{q},\lambda'}] = 0 \,, \tag{114}$$

where

$$\zeta_0 = -1 \,, \quad \zeta_1 = \zeta_2 = \zeta_3 = 1 \,. \tag{115}$$

Electromagnetic field | Covariant quantization

- The one-particle states,

$$\left|\vec{k},\lambda\right\rangle = \sqrt{2\omega_{\vec{k}}}\, a^\dagger_{\vec{k},\lambda} \left|0\right\rangle \tag{116}$$

have a negative norm for $\lambda = 0$, because

$$\left\langle\vec{q},\lambda\middle|\vec{k},\lambda\right\rangle = 2\omega_{\vec{k}} \left\langle 0\right| a_{\vec{q},\lambda} a^\dagger_{\vec{k},\lambda} \left|0\right\rangle = 2\omega_{\vec{k}} \left\langle 0\right| [a_{\vec{q},\lambda}, a^\dagger_{\vec{k},\lambda}] \left|0\right\rangle = \zeta_\lambda 2\omega_{\vec{k}} (2\pi)^3 \delta^3(\vec{k}-\vec{q})\ . \tag{117}$$

This is not acceptable, because the norms are interpreted as probabilities.
In any case, $\mathcal{L}'$ is not the Lagrangian of electromagnetism and, if it were, the states $\left|\vec{k},0\right\rangle$ and $\left|\vec{k},3\right\rangle$ are not physical.

Electromagnetic field | Covariant quantization

- We can think of recovering the electromagnetism by requiring that on the physical states,

$$\left\langle \text{phys}'\right| \partial_\mu A^\mu \left|\text{phys}\right\rangle = 0\ . \tag{118}$$

Then, instead of taking $\partial_\mu A^\mu = 0$ at the Lagrangian level, we will suppose that the Lagrangian is $\mathcal{L}'$ but we impose the constraint above on the physical states.
This is known as the Gupta-Bleuler quantization.

Let us see that in fact this is sufficient to eliminate from the Fock space all unphysical states.

▷ For that purpose, let us denote

$$\partial_\mu A^\mu = (\partial_\mu A^\mu)^+ + (\partial_\mu A^\mu)^- \tag{119}$$

where we have separated the states of positive and negative energy,

$$\begin{aligned}(\partial_\mu A^\mu)^+ &= -\mathrm{i}\int \frac{\mathrm{d}^3k}{(2\pi)^3\sqrt{2\omega_{\vec{k}}}} \sum_{\lambda=0}^{3} k_\mu \epsilon^\mu(\vec{k},\lambda) a_{\vec{k},\lambda} \mathrm{e}^{-\mathrm{i}kx} \\ (\partial_\mu A^\mu)^- &= \mathrm{i}\int \frac{\mathrm{d}^3k}{(2\pi)^3\sqrt{2\omega_{\vec{k}}}} \sum_{\lambda=0}^{3} k_\mu \epsilon^{\mu*}(\vec{k},\lambda) a^\dagger_{\vec{k},\lambda} \mathrm{e}^{\mathrm{i}kx} \,.\end{aligned} \tag{120}$$

As $(\partial_\mu A^\mu)^- = [(\partial_\mu A^\mu)^+]^\dagger$, the constraint (118) is satisfied as long as

$$(\partial_\mu A^\mu)^+ |\mathrm{phys}\rangle = 0 \,. \tag{121}$$

Moreover, as $(\partial_\mu A^\mu)^+$ is a linear operator, if $|\mathrm{phys}_1\rangle$ and $|\mathrm{phys}_2\rangle$ are physical states then an arbitrary combination $\alpha\,|\mathrm{phys}_1\rangle + \beta\,|\mathrm{phys}_2\rangle$ is also physical.

▷ Then, given a physical one-particle state

$$|\psi\rangle = \sum_\lambda c_\lambda a^\dagger_{\vec{k},\lambda} |0\rangle \tag{122}$$

the constraint (121) implies

$$\begin{aligned} 0 = (\partial_\mu A^\mu)^+ |\psi\rangle &= -\mathrm{i}\int \frac{\mathrm{d}^3q}{(2\pi)^3\sqrt{2\omega_{\vec{q}}}} \sum_{\lambda,\lambda'} c_\lambda q_\mu \epsilon^\mu(\vec{q},\lambda') a_{\vec{q},\lambda'} a^\dagger_{\vec{k},\lambda} |0\rangle \\ \Rightarrow -\frac{\mathrm{i}}{\sqrt{2\omega_{\vec{k}}}} \sum_\lambda \zeta_\lambda c_\lambda k_\mu \epsilon^\mu(\vec{k},\lambda) |0\rangle = 0 &\Rightarrow \mathrm{i}\sqrt{\frac{\omega_{\vec{k}}}{2}}(c_0+c_3)|0\rangle = 0 \Rightarrow c_0 + c_3 = 0 \end{aligned}$$

if $k^\mu = (\omega_{\vec{k}}, 0, 0, \omega_{\vec{k}})$ and $\epsilon^\mu(k,\lambda) = \delta^\mu_\lambda$. (123)

As a consequence, a physical state is:

- An arbitrary combination $|\psi_T\rangle$ of tranverse states created by $a^\dagger_{\vec{k},1}$ and $a^\dagger_{\vec{k},2}$, as expected.
- And it is also physical a combination of the form ($c_0 + c_3 = 0$):

$$|\phi\rangle = (a^\dagger_{\vec{k},0} - a^\dagger_{\vec{k},3})\,|0\rangle \;. \tag{124}$$

▷ Therefore, the subspace of physical one-particle states of momentum $\vec{k}$ is of the form

$$|\psi\rangle = |\psi_T\rangle + c\,|\phi\rangle \;, \quad |\psi_T\rangle = \sum_{\lambda=1,2} c_\lambda a^\dagger_{\vec{k},\lambda}\,|0\rangle \;, \tag{125}$$

where we will see that, first

$$\langle\phi|\phi\rangle = 0\,, \quad \langle\psi_T|\phi\rangle = 0 \Rightarrow \langle\psi|\psi\rangle = \langle\psi_T|\psi_T\rangle \tag{126}$$

and, second, $|\psi\rangle$ and $|\psi_T\rangle$ have the same energy, momentum, angular momentum, etc. Thus, we can introduce an equivalence relation

$$|\psi\rangle \sim |\psi_T\rangle \quad \text{if} \quad |\psi\rangle = |\psi_T\rangle + c\,|\phi\rangle \tag{127}$$

and choose any $|\psi\rangle$ in the class of $|\psi_T\rangle$, transverse or not, because this choice has no physical consequences.

▷ Let us prove the first statement:

$$\begin{aligned}\langle\phi|\phi\rangle &= \langle 0|\,(a_{\vec{k},0} - a_{\vec{k},3})(a^\dagger_{\vec{k},0} - a^\dagger_{\vec{k},3})\,|0\rangle = \langle 0|\,(a_{\vec{k},0}a^\dagger_{\vec{k},0} + a_{\vec{k},3}a^\dagger_{\vec{k},3})\,|0\rangle \\ &= \langle 0|\,([a_{\vec{k},0}, a^\dagger_{\vec{k},0}] + [a_{\vec{k},3}, a^\dagger_{\vec{k},3}])\,|0\rangle = 0 \end{aligned} \tag{128}$$

$$\langle\psi_T|\phi\rangle = \langle 0|\,(c_1^* a_{\vec{k},1} + c_2^* a_{\vec{k},2})(a^\dagger_{\vec{k},0} - a^\dagger_{\vec{k},3})\,|0\rangle = 0\,. \tag{129}$$

Electromagnetic field | Covariant quantization

▷ And now let us prove the second: the energy and momentum are given by

$$H = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\omega_{\vec{k}}\left(-a^\dagger_{\vec{k},0}a_{\vec{k},0} + \sum_{\lambda=1,2,3} a^\dagger_{\vec{k},\lambda}a_{\vec{k},\lambda}\right) \tag{130}$$

$$\vec{\mathbb{P}} = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\vec{k}\left(-a^\dagger_{\vec{k},0}a_{\vec{k},0} + \sum_{\lambda=1,2,3} a^\dagger_{\vec{k},\lambda}a_{\vec{k},\lambda}\right) . \tag{131}$$

If we calculate the matrix elements of these operators, that always contain the combination $(-a^\dagger_{\vec{k},0}a_{\vec{k},0} + a^\dagger_{\vec{k},3}a_{\vec{k},3})$ between two physical states, we must take into account that on a physical state $|\psi\rangle = |\psi_T\rangle + c\,|\phi\rangle$,

$$(a_{\vec{k},0} - a_{\vec{k},3})\,|\psi\rangle = c(a_{\vec{k},0} - a_{\vec{k},3})\,|\phi\rangle = c(a_{\vec{k},0} - a_{\vec{k},3})(a^\dagger_{\vec{k},0} - a^\dagger_{\vec{k},3})\,|0\rangle = 0 \tag{132}$$

and, then

$$\begin{aligned}\langle \mathrm{phys}'|\,(-a^\dagger_{\vec{k},0}a_{\vec{k},0} + a^\dagger_{\vec{k},3}a_{\vec{k},3})\,|\mathrm{phys}\rangle &= \langle \mathrm{phys}'|\,(-a^\dagger_{\vec{k},0}a_{\vec{k},0} + a^\dagger_{\vec{k},0}(a_{\vec{k},0} - a_{\vec{k},3}) + a^\dagger_{\vec{k},3}a_{\vec{k},3})\,|\mathrm{phys}\rangle \\ = \langle \mathrm{phys}'|\,(-a^\dagger_{\vec{k},0}a_{\vec{k},3} + a^\dagger_{\vec{k},3}a_{\vec{k},3})\,|\mathrm{phys}\rangle &= -\,\langle \mathrm{phys}'|\,(a^\dagger_{\vec{k},0} - a^\dagger_{\vec{k},3})a_{\vec{k},3})\,|\mathrm{phys}\rangle = 0\,, \end{aligned} \tag{133}$$

i.e. to energy and momentum contribute only the transverse oscillators.

Electromagnetic field | C, P, T

- Finally, let us find the transformation properties of $A^\mu(x)$ under C, P and T.

▷ As $C\overline{\psi}\gamma^\mu\psi C = -\overline{\psi}\gamma^\mu\psi$, for C to be a symmetry of the QED Lagrangian we need that $\mathcal{L}_{\mathrm{QED}} \supset q\overline{\psi}\gamma^\mu\psi A_\mu$ remains invariant, namely

$$CA^\mu(x)C = -A^\mu(x) \equiv \eta_C A^\mu(x) \Leftrightarrow Ca_{\vec{k},\pm}C = \eta_C a_{\vec{k},\pm}\,, \tag{134}$$

from where the charge conjugation of the photon is $\eta_C = -1$.

▷ Regarding P, as $\vec{A}(x)$ is a vector we have (consistently with P conserved in QED)

$$PA^\mu(t,x)P = A_\mu(t,-\vec{x}) \Leftrightarrow Pa_{\vec{k},\pm}P = \eta_P a_{-\vec{k},\pm}\,, \tag{135}$$

from where the intrinsic parity of the photon is $\eta_P = -1$, as it corresponds to a state with $J = 1$, consistent with a parity $(-1)^L = -1$ for a system of orbital angular momentum $L = 1$, whose wave function is given by $Y_L^M(\theta,\phi)$.

▷ And regarding T (conserved in QED),

$$TA^\mu(t,\vec{x})T = A_\mu(-t,\vec{x}) \Leftrightarrow Ta_{\vec{k},\pm}T = a_{-\vec{k},\mp}\,, \tag{136}$$

the same as for the vector $T[\overline{\psi}(t,\vec{x})\gamma^\mu\psi(t,\vec{x})]T = \overline{\psi}(-t,\vec{x})\gamma_\mu\psi(-t,\vec{x})$.

- This completes the transformation properties under C, P and T of scalar (70, 75, 83), spinorial[a] (66, 73, 80) and vector[b] fields (134, 135, 136), which are the building blocks used to construct Lagrangians describing elementary particle physics. The interactions involve Lorentz invariant products of these fields and their derivatives.

▷ We know that weak interactions violate C, P, CP and T, but strong and electromagnetic interactions conserve these three discrete symmetries.

▷ The CPT theorem states that any *local* quantum field theory (*Hermitian* and *Lorentz invariant* Lagrangian) is invariant under the combined action of CPT,

$$CPT\ \mathcal{L}(x)\ CPT = \mathcal{L}(-x)\ . \tag{137}$$

This can be checked on any Hermitian scalar combination of femion bilinears, scalar and vector fields and their derivatives.

[a] In particular, it is useful to derive the C, P and T properties of the fermion bilinears from these properties of spinorial fields. Do it!

[b] For a complex vector field the charge conjugation is not (134) but $CA^\mu(x)C = -A^{\mu *}(x)$.

4. Field interactions and Feynman diagrams

The S matrix

- We have quantized free fields.
  Now we will assume interactions:

$$H = H_0 + H_{\text{int}} \,, \quad H_{\text{int}} = \int \mathrm{d}^3x \; \mathcal{H}_{\text{int}}(x) = -\int \mathrm{d}^3x \; \mathcal{L}_{\text{int}}(x)$$

(if $\mathcal{L}_{\text{int}}$ does not contain field derivatives)

  For example: in QED, $\mathcal{L}_{\text{int}} = e\overline{\psi}\gamma^\mu\psi A_\mu$ and in the $\lambda\phi^4$ theory, $\mathcal{L}_{\text{int}} = -\frac{\lambda}{4!}\phi^4$

▷ We will always assume small coupling constant ⇒ perturbation

(the relevant parameter in QED: $\alpha = e^2/(4\pi) \approx 1/137 \ll 1$)

- Our aim: find transition probabilities in a scattering process

The S matrix

- In the Schrödinger picture states depend on time:

  $|a(t)\rangle$ is the time evolution until a time t of an initial state $|a\rangle \equiv |a(t_i)\rangle$
  labeled by compatible observables with eigenvalues a (e.g. $\vec{p}_i$ and $\vec{s}_i$)
  $|b\rangle \equiv |b(t_f)\rangle$ in the final state after the scattering

▷ The probability amplitude that $|a\rangle$ evolves to $|b\rangle$ is then

$$\langle b|a(t_f)\rangle = \langle b|\, \mathrm{e}^{-\mathrm{i}H(t_f-t_i)}\, |a\rangle$$

We call S matrix to the evolution operator $\mathrm{e}^{-\mathrm{i}H(t_f-t_i)}$ in the limit $(t_f - t_i) \to \infty$, where H is the Hamiltonian of the field theory.

The scattering amplitude is given by

$$\langle b|\, S\, |a\rangle = \lim_{(t_f-t_i)\to\infty} \langle b|\, \mathrm{e}^{-\mathrm{i}H(t_f-t_i)}\, |a\rangle$$

The S matrix

▷ Note that if $\langle a|a\rangle = 1$ and $|n\rangle$ is a complete state basis, $\sum_n |n\rangle\langle n| = 1$, we have

$$1 = \sum_n |\, \langle n|\, S\, |a\rangle\, |^2 = \sum_n \langle a|\, S^\dagger\, |n\rangle\langle n|\, S\, |a\rangle = \langle a|\, S^\dagger S\, |a\rangle$$

meaning that $SS^\dagger = 1$, so S is unitary. Therefore, the unitarity of S expresses the probability conservation. It is convenient to write

$$\left.\begin{array}{l} S \equiv 1 + \mathrm{i}T \\ SS^\dagger = 1 \end{array}\right. \Rightarrow -\mathrm{i}(T - T^\dagger) = TT^\dagger$$

Then, defining $T_{ba} = \langle b|\, T\, |a\rangle$ we have

$$-\mathrm{i}(T_{ba} - T^*_{ab}) = \sum_n T_{bn} T^*_{an} \Rightarrow 2\,\mathrm{Im} T_{aa} = \sum_n |T_{an}|^2$$

that leads to the optical theorem

The S matrix

- In the Heisenberg picture the operators, not the states, depend on time. This is more appropriate for the QFT, where the fields are time-dependent operators $\phi(t,\vec{x})$, $\psi(t,\vec{x})$, $A_\mu(t,\vec{x})$ ...

▷ The states $|a\rangle \equiv |a(t_i)\rangle$ and $|b\rangle \equiv |b(t_f)\rangle$ are in the Heisenberg picture $|a\rangle_H = \mathrm{e}^{\mathrm{i}Ht}\,|a(t)\rangle$ and $|b\rangle_H = \mathrm{e}^{\mathrm{i}Ht}\,|b(t)\rangle$, independent of time

▷ Then, defining the Heisenberg picture states $|a;t_i\rangle = \mathrm{e}^{\mathrm{i}Ht_i}\,|a\rangle$ and $|b;t_f\rangle = \mathrm{e}^{\mathrm{i}Ht_f}\,|b\rangle$ the S matrix reads

$$\langle b|\,S\,|a\rangle = \lim_{(t_f-t_i)\to\infty} \langle b|\,\mathrm{e}^{-\mathrm{i}H(t_f-t_i)}\,|a\rangle = \lim_{(t_f-t_i)\to\infty} \langle b;t_f|a;t_i\rangle$$

The LSZ reduction formula

- Next we will see the S matrix between initial and final states of the same species labeled by their momenta (assume for simplicity that they are spinless),

$$\langle \vec{p}_1\vec{p}_2\cdots\vec{p}_n|\,S\,\left|\vec{k}_1\vec{k}_2\cdots\vec{k}_m\right\rangle = \left\langle \vec{p}_1\vec{p}_2\cdots\vec{p}_n;t_f\middle|\vec{k}_1\vec{k}_2\cdots\vec{k}_m;t_i\right\rangle$$

where it is understood that $t_i\to-\infty$ and $t_f\to+\infty$, **can be expressed as a function of the vacuum expectation values of time ordered products of fields** (that we will soon define).

▷ To that end, let us first note that for a free real scalar field,

$$\phi_{\text{free}}(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_{\vec{p}}}}(a_{\vec{p}}\mathrm{e}^{-\mathrm{i}px} + a^\dagger_{\vec{p}}\mathrm{e}^{\mathrm{i}px})$$

then

$$\sqrt{2E_{\vec{k}}}\,a_{\vec{k}} = \mathrm{i}\int \mathrm{d}^3x\;\mathrm{e}^{\mathrm{i}kx}\,\overleftrightarrow{\partial_0}\,\phi_{\text{free}}(x)\;,\quad \sqrt{2E_{\vec{k}}}\,a^\dagger_{\vec{k}} = -\mathrm{i}\int \mathrm{d}^3x\;\mathrm{e}^{-\mathrm{i}kx}\,\overleftrightarrow{\partial_0}\,\phi_{\text{free}}(x)\;. \quad (1)$$

The LSZ reduction formula

▷ In fact,

$$
\begin{aligned}
\mathrm{i}\int \mathrm{d}^3x\ \mathrm{e}^{\mathrm{i}kx}\ \overleftrightarrow{\partial_0}\ \phi_{\text{free}}(x) &= \mathrm{i}\int \mathrm{d}^3x \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_{\vec{p}}}} \left(a_{\vec{p}}\mathrm{e}^{\mathrm{i}kx}\ \overleftrightarrow{\partial_0}\ \mathrm{e}^{-\mathrm{i}px} + a^\dagger_{\vec{p}}\mathrm{e}^{\mathrm{i}kx}\ \overleftrightarrow{\partial_0}\ \mathrm{e}^{\mathrm{i}px} \right) \\
&= \mathrm{i}\int \mathrm{d}^3x \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_{\vec{p}}}} \left(-\mathrm{i}a_{\vec{p}}(E_{\vec{p}}+E_{\vec{k}})\mathrm{e}^{\mathrm{i}(k-p)x} + \mathrm{i}a^\dagger_{\vec{p}}(E_{\vec{p}}-E_{\vec{k}})\mathrm{e}^{\mathrm{i}(k+p)x} \right) \\
&= \sqrt{2E_{\vec{k}}}\ a_{\vec{k}}\ .
\end{aligned}
$$

The LSZ reduction formula

- One expects that

$$
\phi(x) \xrightarrow[t\to-\infty]{} Z^{1/2}\phi_{\text{in}}(x)\ , \quad \phi(x) \xrightarrow[t\to+\infty]{} Z^{1/2}\phi_{\text{out}}(x)\ , \tag{2}
$$

where $\phi_{\text{in}}(x)$ and $\phi_{\text{out}}(x)$ are free fields (before and after the interaction, respectively) and Z is a constant factor called wave function renormalization (whose meaning will be understood later). Then, using (1),

$$
\sqrt{2E_{\vec{k}}}\ a^{\dagger(\text{in})}_{\vec{k}} = -\mathrm{i}Z^{-1/2} \lim_{t\to-\infty} \int \mathrm{d}^3x\ \mathrm{e}^{-\mathrm{i}kx}\ \overleftrightarrow{\partial_0}\ \phi(x)\ , \tag{3}
$$

$$
\sqrt{2E_{\vec{k}}}\ a^{\dagger(\text{out})}_{\vec{k}} = -\mathrm{i}Z^{-1/2} \lim_{t\to+\infty} \int \mathrm{d}^3x\ \mathrm{e}^{-\mathrm{i}kx}\ \overleftrightarrow{\partial_0}\ \phi(x)\ . \tag{4}
$$

Therefore

$$
\begin{aligned}
\left\langle \vec{p}_1\vec{p}_2\cdots\vec{p}_n; t_f \middle| \vec{k}_1\vec{k}_2\cdots\vec{k}_m; t_i \right\rangle &= \sqrt{2E_{\vec{k}_1}}\ \langle \vec{p}_1\vec{p}_2\cdots\vec{p}_n; t_f |\ a^{\dagger(\text{in})}_{\vec{k}_1} \left| \vec{k}_2\cdots\vec{k}_m; t_i \right\rangle \\
&= -\mathrm{i}Z^{-1/2} \lim_{t\to-\infty} \int \mathrm{d}^3x\ \mathrm{e}^{-\mathrm{i}k_1x} \langle \vec{p}_1\vec{p}_2\cdots\vec{p}_n; t_f |\ \overleftrightarrow{\partial_0}\ \phi(x) \left| \vec{k}_2\cdots\vec{k}_m; t_i \right\rangle\ .
\end{aligned} \tag{5}
$$

The LSZ reduction formula

▷ It is convenient to write previous expression in a covariant fashion noting that

$$\sqrt{2E_{\vec{k}_1}}\,\langle \vec{p}_1\vec{p}_2\cdots\vec{p}_n;t_f|\,a^{\dagger(\text{in})}_{\vec{k}_1}\,\left|\vec{k}_2\cdots\vec{k}_m;t_i\right\rangle$$
$$=\sqrt{2E_{\vec{k}_1}}\,\langle \vec{p}_1\vec{p}_2\cdots\vec{p}_n;t_f|\left(a^{\dagger(\text{in})}_{\vec{k}_1}-a^{\dagger(\text{out})}_{\vec{k}_1}\right)\left|\vec{k}_2\cdots\vec{k}_m;t_i\right\rangle$$

because $a^{\dagger(\text{out})}_{\vec{k}_1}$ acts on $\langle \vec{p}_1\vec{p}_2\cdots\vec{p}_n;t_f|$ annihilating a particle in the final state of momentum $\vec{k}_1$ and, given that in the scattering process there are no spectator particles (no $\vec{k}_i$ coincides with a $\vec{p}_j$), this operation vanishes.

This is because we are actually computing the part iT of the S matrix.

The LSZ reduction formula

▷ And, on the other hand, from (3) and (4),

$$\begin{aligned}
\sqrt{2E_{\vec{k}}}\left(a^{\dagger(\text{in})}_{\vec{k}}-a^{\dagger(\text{out})}_{\vec{k}}\right) &= \mathrm{i}Z^{-1/2}\int \mathrm{d}^4x\,\partial_0\left(\mathrm{e}^{-ikx}\overleftrightarrow{\partial_0}\,\phi\right)\\
&= \mathrm{i}Z^{-1/2}\int \mathrm{d}^4x\,\partial_0\left(\mathrm{e}^{-ikx}\partial_0\phi-\phi\partial_0\mathrm{e}^{-ikx}\right)\\
&= \mathrm{i}Z^{-1/2}\int \mathrm{d}^4x\,\left[\mathrm{e}^{-ikx}\partial_0^2\phi+\cancel{(\partial_0\mathrm{e}^{-ikx})\partial_0\phi}-\cancel{\partial_0\phi\partial_0\mathrm{e}^{-ikx}}-\phi\partial_0^2\mathrm{e}^{-ikx}\right]\\
&= \mathrm{i}Z^{-1/2}\int \mathrm{d}^4x\,\left[\mathrm{e}^{-ikx}\partial_0^2\phi-\phi(\nabla^2-m^2)\mathrm{e}^{-ikx}\right]\\
&= \mathrm{i}Z^{-1/2}\int \mathrm{d}^4x\,\mathrm{e}^{-ikx}\left(\partial_0^2\phi-\nabla^2\phi+m^2\phi\right)\\
&= \mathrm{i}Z^{-1/2}\int \mathrm{d}^4x\,\mathrm{e}^{-ikx}(\Box+m^2)\phi(x)\,, \qquad (6)
\end{aligned}$$

where in the first equality we used

$$\left(\lim_{t\to-\infty} - \lim_{t\to+\infty}\right)\int \mathrm{d}^3x\ f(t,\vec{x}) = -\int_{-\infty}^{\infty}\mathrm{d}t\ \frac{\partial}{\partial t}\int \mathrm{d}^3x\ f(t,\vec{x}) = -\int \mathrm{d}^4x\ \partial_t f(t,\vec{x})\ , \tag{7}$$

with $f(t,\vec{x}) = -\mathrm{i}Z^{-1/2}\mathrm{e}^{-\mathrm{i}kx}\overset{\leftrightarrow}{\partial_0}\phi$,

in the next-to-last one we have replaced

$$\phi\partial_0^2\mathrm{e}^{-\mathrm{i}kx} = \phi(\nabla^2 - m^2)\mathrm{e}^{-\mathrm{i}kx}\ ,$$

because $k^2 = m^2$, and in the last one we have used

$$\int \mathrm{d}^3x\ \nabla(\mathrm{e}^{-\mathrm{i}kx}\nabla\phi) = 0 \Rightarrow \int \mathrm{d}^3x\ (\nabla\mathrm{e}^{-\mathrm{i}kx})\nabla\phi = -\int \mathrm{d}^3x\ \mathrm{e}^{-\mathrm{i}kx}\nabla^2\phi$$

from where

$$\begin{aligned}
0 = \int \mathrm{d}^3x\ \nabla^2(\mathrm{e}^{-\mathrm{i}kx}\phi) &= \int \mathrm{d}^3x\ \nabla\left[(\nabla\mathrm{e}^{-\mathrm{i}kx})\phi + \mathrm{e}^{-\mathrm{i}kx}\nabla\phi\right] \\
&= \int \mathrm{d}^3x\ \left[(\nabla^2\mathrm{e}^{-\mathrm{i}kx})\phi + 2(\nabla\mathrm{e}^{-\mathrm{i}kx})\nabla\phi + \mathrm{e}^{-\mathrm{i}kx}\nabla^2\phi\right] \\
&= \int \mathrm{d}^3x\ \left[(\nabla^2\mathrm{e}^{-\mathrm{i}kx})\phi - \mathrm{e}^{-\mathrm{i}kx}\nabla^2\phi\right] \\
&\Rightarrow \int \mathrm{d}^3x\ \phi\nabla^2\mathrm{e}^{-\mathrm{i}kx} = \int \mathrm{d}^3x\ \mathrm{e}^{-\mathrm{i}kx}\nabla^2\phi\ .
\end{aligned}$$

The LSZ reduction formula

▷ Then, we can in fact write (5) in a covariant fashion,

$$\left\langle \vec{p}_1\vec{p}_2\cdots\vec{p}_n;t_f \middle| \vec{k}_1\vec{k}_2\cdots\vec{k}_m;t_i \right\rangle$$
$$= \mathrm{i}Z^{-1/2}\int \mathrm{d}^4x\ \mathrm{e}^{-\mathrm{i}k_1x}(\Box+m^2)\,\langle \vec{p}_1\vec{p}_2\cdots\vec{p}_n;t_f|\,\phi(x)\left|\vec{k}_2\cdots\vec{k}_m;t_i\right\rangle\ .$$

We will next proceed by iterating previous procedure until all initial and final state particles are eliminated, leaving only a product of fields acting on vacuum.

The LSZ reduction formula

▷ For that purpose, we write now

$$\langle \vec{p}_1\vec{p}_2\cdots\vec{p}_n;t_f|\,\phi(x)\left|\vec{k}_2\cdots\vec{k}_n;t_i\right\rangle = \sqrt{2E_{\vec{p}_1}}\,\langle \vec{p}_2\cdots\vec{p}_n;t_f|\,a^{(\mathrm{out})}_{\vec{p}_1}\phi(x)\left|\vec{k}_2\cdots\vec{k}_m;t_i\right\rangle$$
$$= \sqrt{2E_{\vec{p}_1}}\,\langle \vec{p}_2\cdots\vec{p}_n;t_f|\,T\{(a^{(\mathrm{out})}_{\vec{p}_1}-a^{(\mathrm{in})}_{\vec{p}_1})\phi(x)\}\left|\vec{k}_2\cdots\vec{k}_m;t_i\right\rangle \qquad (8)$$

where we have used that $a^{(\mathrm{in})}_{\vec{p}_1}\left|\vec{k}_2\cdots\vec{k}_m;t_i\right\rangle = 0$ and we were forced to introduce the time ordered product,

$$T\{\phi(y)\phi(x)\} = \begin{cases} \phi(y)\phi(x)\ , & y^0 > x^0 \\ \phi(x)\phi(y)\ , & y^0 < x^0 \end{cases}$$

which implies

$$T\{a^{(\mathrm{in})}_{\vec{p}}\phi(x)\} = \phi(x)a^{(\mathrm{in})}_{\vec{p}}\ ,\quad T\{a^{(\mathrm{out})}_{\vec{p}}\phi(x)\} = a^{(\mathrm{out})}_{\vec{p}}\phi(x)\ .$$

The LSZ reduction formula

▷ From (6) we have

$$\sqrt{2E_{\vec{p}}}(a^{(\text{out})}_{\vec{p}} - a^{(\text{in})}_{\vec{p}}) = \mathrm{i}Z^{-1/2}\int \mathrm{d}^4y\ \mathrm{e}^{\mathrm{i}py}(\Box_y + m^2)\phi(y)$$

and substituting this in (8) one gets

$$\begin{aligned}&\left\langle \vec{p}_1\vec{p}_2\cdots\vec{p}_n;t_f\right|\phi(x)\left|\vec{k}_2\cdots\vec{k}_m;t_i\right\rangle\\ &\qquad= \mathrm{i}Z^{-1/2}\int \mathrm{d}^4y\ \mathrm{e}^{\mathrm{i}p_1y}(\Box_y + m^2)\left\langle \vec{p}_2\cdots\vec{p}_n;t_f\right| T\{\phi(y)\phi(x)\}\left|\vec{k}_2\cdots\vec{k}_m;t_i\right\rangle\ .\end{aligned}$$

Then, we already see that

$$\begin{aligned}&\left\langle \vec{p}_1\vec{p}_2\cdots\vec{p}_n;t_f\middle|\vec{k}_1\vec{k}_2\cdots\vec{k}_m;t_i\right\rangle\\ &= \left(\mathrm{i}Z^{-1/2}\right)^{m+n}\int\left(\prod_{i=1}^m \mathrm{d}^4x_i\ \mathrm{e}^{-\mathrm{i}k_ix_i}\right)\left(\prod_{j=1}^n \mathrm{d}^4y_j\ \mathrm{e}^{\mathrm{i}p_jy_j}\right)\\ &\quad\times(\Box_{x_1}+m^2)\cdots(\Box_{y_n}+m^2)\left\langle 0\right| T\{\phi(x_1)\cdots\phi(x_m)\phi(y_1)\cdots\phi(y_n)\}\left|0\right\rangle\ .\end{aligned}$$

The LSZ reduction formula

▷ If now we define the N-point Green's function,

$$G(x_1,\dots,x_N) = \left\langle 0\right| T\{\phi(x_1)\cdots\phi(x_N)\}\left|0\right\rangle\ ,$$

and we write it in terms of its Fourier transform $\widetilde{G}$,

$$G(x_1,\dots,x_N) = \int\left(\prod_{i=1}^N \frac{\mathrm{d}^4\tilde{q}_i}{(2\pi)^4}\mathrm{e}^{-\mathrm{i}\tilde{q}_ix_i}\right)\widetilde{G}(\tilde{q}_1,\dots,\tilde{q}_N)\ ,$$

we see that (substituting $\Box\mathrm{e}^{\pm\mathrm{i}qx} = -q^2\mathrm{e}^{\pm\mathrm{i}qx}$)

$$\begin{aligned}&\left\langle \vec{p}_1\vec{p}_2\cdots\vec{p}_n;t_f\middle|\vec{k}_1\vec{k}_2\cdots\vec{k}_m;t_i\right\rangle\\ &= \left(-\mathrm{i}Z^{-1/2}\right)^{m+n}\int\left(\prod_{i=1}^m \mathrm{d}^4x_i\frac{\mathrm{d}^4\tilde{k}_i}{(2\pi)^4}\mathrm{e}^{-\mathrm{i}(\tilde{k}_i+k_i)x_i}(\tilde{k}_i^2-m^2)\right)\\ &\quad\times\int\left(\prod_{j=1}^n \mathrm{d}^4y_j\frac{\mathrm{d}^4\tilde{p}_j}{(2\pi)^4}\mathrm{e}^{-\mathrm{i}(\tilde{p}_j-p_j)y_j}(\tilde{p}_j^2-m^2)\right)\widetilde{G}(\tilde{k}_1,\dots,\tilde{k}_m,\tilde{p}_1,\dots,\tilde{p}_n)\\ &= \left(-\mathrm{i}Z^{-1/2}\right)^{m+n}\left(\prod_{i=1}^m(k_i^2-m^2)\right)\left(\prod_{j=1}^n(p_j^2-m^2)\right)\widetilde{G}(-k_1,\dots,-k_m,p_1,\dots,p_n)\end{aligned}$$

The LSZ reduction formula

and solving for $\widetilde{G}(-k_1,\ldots,-k_m,p_1,\ldots,p_n)$,

$$\boxed{\begin{aligned}&\left(\prod_{i=1}^{m}\frac{\mathrm{i}\sqrt{Z}}{k_i^2-m^2}\right)\left(\prod_{j=1}^{n}\frac{\mathrm{i}\sqrt{Z}}{p_j^2-m^2}\right)\langle\vec{p}_1\vec{p}_2\cdots\vec{p}_n|\,\mathrm{i}T\left|\vec{k}_1\vec{k}_2\cdots\vec{k}_m\right\rangle\\&=\int\left(\prod_{i=1}^{m}\mathrm{d}^4x_i\;\mathrm{e}^{-\mathrm{i}k_ix_i}\right)\int\left(\prod_{j=1}^{n}\mathrm{d}^4y_j\;\mathrm{e}^{+\mathrm{i}p_jy_j}\right)\langle 0|\,T\{\phi(x_1)\cdots\phi(x_m)\phi(y_1)\cdots\phi(y_n)\}\,|0\rangle\end{aligned}} \tag{9}$$

This is the LSZ reduction formula (Lehmann-Symanzik-Zimmermann). Remember that for a physical particle the relation $p^2-m^2=0$ is fulfilled (it is said to be on-shell or to be on its mass shell). Then, the right-hand side of the LSZ formula will have poles when the incoming or the outgoing particles are on-shell, that (as expected and we will see) will cancel the poles of the prefactor of the S matrix element on the left-hand side, so the S matrix has a finite value.

Perturbation theory

- The fields ϕ of the LSZ formula are solutions of $H=H_0+H_{\text{int}}$ and hence they are not combinations of plane waves, whose coefficients were interpreted as operators creation and annihilation of particles at the quantum level.
- However, we can define the field in the interaction picture,

$$\phi_I(t,\vec{x})\equiv\mathrm{e}^{\mathrm{i}H_0(t-t_0)}\phi(t_0,\vec{x})\mathrm{e}^{-\mathrm{i}H_0(t-t_0)}\;, \tag{10}$$

  which is a field coinciding with the field $\phi(t,\vec{x})$ of the Heisenberg picture only at a reference time $t=t_0$, that is by definition a free field,

$$\phi_I(t,\vec{x})=\int\frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_{\vec{p}}}}(a_{\vec{p}}\mathrm{e}^{-\mathrm{i}px}+a_{\vec{p}}^\dagger\mathrm{e}^{\mathrm{i}px})\;,$$

  whose time evolution is then determined by the free Hamiltonian H_0.

Perturbation theory

▷ Remember that a field in the Heisenberg picture evolves with time as

$$\phi(t,\vec{x}) = \mathrm{e}^{\mathrm{i}H(t-t_0)}\phi(t_0,\vec{x})\mathrm{e}^{-\mathrm{i}H(t-t_0)} \ .$$

So, from (10)

$$\phi(t_0,\vec{x}) = \mathrm{e}^{-\mathrm{i}H_0(t-t_0)}\phi_I(t,\vec{x})\mathrm{e}^{\mathrm{i}H_0(t-t_0)}$$

we see that $\phi(x)$ and $\phi_I(x)$ are related by

$$\phi(t,\vec{x}) = \mathrm{e}^{\mathrm{i}H(t-t_0)}\mathrm{e}^{-\mathrm{i}H_0(t-t_0)}\phi_I(t,\vec{x})\mathrm{e}^{\mathrm{i}H_0(t-t_0)}\mathrm{e}^{-\mathrm{i}H(t-t_0)} = U^\dagger(t,t_0)\phi_I(t,\vec{x})U(t,t_0) \ ,$$

$$U(t,t_0) \equiv \mathrm{e}^{\mathrm{i}H_0(t-t_0)}\mathrm{e}^{-\mathrm{i}H(t-t_0)} \ .$$

Perturbation theory

▷ We will write now ϕ perturbatively as a function of ϕ_I. For that, note that

$$\begin{aligned} \mathrm{i}\frac{\partial}{\partial t}U(t,t_0) &= \mathrm{e}^{\mathrm{i}H_0(t-t_0)}(H-H_0)\mathrm{e}^{-\mathrm{i}H(t-t_0)} \\ &= \mathrm{e}^{\mathrm{i}H_0(t-t_0)}H_{\text{int}}\,\mathrm{e}^{-\mathrm{i}H_0(t-t_0)}\mathrm{e}^{\mathrm{i}H_0(t-t_0)}\mathrm{e}^{-\mathrm{i}H(t-t_0)} \\ &= H_I(t)U(t,t_0) \end{aligned} \tag{11}$$

where we have introduced the Hamiltonian in the interaction picture[a]

$$H_I(t) \equiv \mathrm{e}^{\mathrm{i}H_0(t-t_0)}H_{\text{int}}\,\mathrm{e}^{-\mathrm{i}H_0(t-t_0)} \ .$$

[a] Note that in general $[H_0,H]=[H_0,H_{\text{int}}]\neq 0$.

Perturbation theory

▷ The solution of the differential equation (11) with the boundary condition $U(t,t) = 1$ is (check it by replacing the solution in the equation):

$$\begin{aligned} U(t,t_0) &= 1 + (-\mathrm{i}) \int_{t_0}^{t} \mathrm{d}t_1\ H_I(t_1) + (-\mathrm{i})^2 \int_{t_0}^{t} \mathrm{d}t_1 \int_{t_0}^{t_1} \mathrm{d}t_2\ H_I(t_1) H_I(t_2) \\ &\quad + (-\mathrm{i})^3 \int_{t_0}^{t} \mathrm{d}t_1 \int_{t_0}^{t_1} \mathrm{d}t_2 \int_{t_0}^{t_2} \mathrm{d}t_3\ H_I(t_1) H_I(t_2) H_I(t_3) + \dots \\ &= 1 + (-\mathrm{i}) \int_{t_0}^{t} \mathrm{d}t_1\ H_I(t_1) + (-\mathrm{i})^2 \frac{1}{2} \int_{t_0}^{t} \mathrm{d}t_1 \int_{t_0}^{t} \mathrm{d}t_2\ T\{H_I(t_1) H_I(t_2)\} \\ &\quad + (-\mathrm{i})^3 \frac{1}{3!} \int_{t_0}^{t} \mathrm{d}t_1 \int_{t_0}^{t} \mathrm{d}t_2 \int_{t_0}^{t} \mathrm{d}t_3\ T\{H_I(t_1) H_I(t_2) H_I(t_3)\} + \dots \\ &= T\left\{ \exp\left[-\mathrm{i} \int_{t_0}^{t} \mathrm{d}t'\ H_I(t') \right] \right\} \end{aligned}$$

(Dyson series)

Perturbation theory

▷ Another way of writing U which allows us to derive useful properties is

$$U(t,t') = \mathrm{e}^{\mathrm{i}H_0(t-t_0)} \mathrm{e}^{-\mathrm{i}H(t-t')} \mathrm{e}^{-\mathrm{i}H_0(t'-t_0)}$$

that indeed satisfies $U(t,t) = 1$ and (11) because

$$\begin{aligned} \mathrm{i}\frac{\partial}{\partial t} U(t,t') &= \mathrm{e}^{\mathrm{i}H_0(t-t_0)} (H - H_0) \mathrm{e}^{-\mathrm{i}H(t-t')} \mathrm{e}^{-\mathrm{i}H_0(t'-t_0)} \\ &= \mathrm{e}^{\mathrm{i}H_0(t-t_0)} H_{\text{int}}\ \mathrm{e}^{-\mathrm{i}H_0(t-t_0)} \mathrm{e}^{\mathrm{i}H_0(t-t_0)} \mathrm{e}^{-\mathrm{i}H(t-t')} \mathrm{e}^{-\mathrm{i}H_0(t'-t_0)} \\ &= H_I(t) U(t,t') . \end{aligned}$$

▷ From this one obtains easily that U is unitary and

$$\begin{aligned} U(t_1,t_2) U(t_2,t_3) &= \mathrm{e}^{\mathrm{i}H_0(t_1-t_0)} \mathrm{e}^{-\mathrm{i}H(t_1-t_2)} \mathrm{e}^{-\mathrm{i}H_0(t_2-t_0)} \mathrm{e}^{\mathrm{i}H_0(t_2-t_0)} \mathrm{e}^{-\mathrm{i}H(t_2-t_3)} \mathrm{e}^{-\mathrm{i}H_0(t_3-t_0)} \\ &= U(t_1,t_3) \\ \Rightarrow & U(t_1,t_3) U^\dagger(t_2,t_3) = U(t_1,t_2) \\ \Rightarrow & U^\dagger(t_2,t_1) = U(t_1,t_2) . \end{aligned}$$

Perturbation theory

▷ Let us see how to calculate $\langle 0|\,\phi(x_1)\cdots\phi(x_n)\,|0\rangle$, where we take the x_i already time ordered ($t_1 \geq t_2 \geq \ldots \geq t_n$),

$$
\begin{aligned}
&\langle 0|\,\phi(x_1)\cdots\phi(x_n)\,|0\rangle \\
&= \langle 0|\,U^\dagger(t_1,t_0)\phi_I(x_1)U(t_1,t_0)U^\dagger(t_2,t_0)\phi_I(x_2)U(t_2,t_0)\cdots U^\dagger(t_n,t_0)\phi_I(x_n)U(t_n,t_0)\,|0\rangle \\
&= \langle 0|\,U^\dagger(t_1,t_0)\phi_I(x_1)U(t_1,t_2)\phi_I(x_2)U(t_2,t_3)\cdots U(t_{n-1},t_n)\phi_I(x_n)U(t_n,t_0)\,|0\rangle \\
&= \langle 0|\,U^\dagger(t,t_0)U(t,t_1)\phi_I(x_1)U(t_1,t_2)\cdots U(t_{n-1},t_n)\phi_I(x_n)U(t_n,-t)U(-t,t_0)\,|0\rangle \\
&= \langle 0|\,U^\dagger(t,t_0)\;T\{\phi_I(x_1)\cdots\phi_I(x_n)U(t,t_1)U(t_1,t_2)\cdots U(t_n,-t)\}\;U(-t,t_0)\,|0\rangle \\
&= \langle 0|\,U^\dagger(t,t_0)\;T\left\{\phi_I(x_1)\cdots\phi_I(x_n)\exp\left[-\mathrm{i}\int_{-t}^{t}\mathrm{d}t'\;H_I(t')\right]\right\}\;U(-t,t_0)\,|0\rangle
\end{aligned}
$$

and where we have introduced $t \geq t_1 \geq t_2 \geq \ldots \geq t_n \geq -t$ and substituted

$$
U^\dagger(t_1,t_0) = U^\dagger(t,t_0)U(t,t_1)\;, \quad U(t_n,t_0) = U(t_n,-t)U(-t,t_0)
$$

$$
U(t,t_1)U(t_1,t_2)\cdots U(t_n,-t) = U(t,-t) = T\left\{\exp\left[-\mathrm{i}\int_{-t}^{t}\mathrm{d}t'\;H_I(t')\right]\right\}\;.
$$

Perturbation theory

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▷ Taking now $t_0 = -t$ with $t \to \infty$ and substituting the adjoint of

$$
U(\infty,-\infty)\,|0\rangle = \mathrm{e}^{\mathrm{i}\alpha}\,|0\rangle\;, \quad \mathrm{e}^{\mathrm{i}\alpha} = \langle 0|\,T\left\{\exp\left[-\mathrm{i}\int_{-\infty}^{\infty}\mathrm{d}t'\;H_I(t')\right]\right\}|0\rangle
$$

we have finally that

$$
\boxed{\langle 0|\,T\{\phi(x_1)\cdots\phi(x_n)\}\,|0\rangle = \frac{\langle 0|\,T\left\{\phi_I(x_1)\cdots\phi_I(x_n)\exp\left[-\mathrm{i}\int\mathrm{d}^4x\;\mathcal{H}_I(x)\right]\right\}|0\rangle}{\langle 0|\,T\left\{\exp\left[-\mathrm{i}\int\mathrm{d}^4x\;\mathcal{H}_I(x)\right]\right\}|0\rangle}} \tag{12}
$$

▷ Series expanding the exponentials in this expression and using the Wick theorem, to be introduced next, we will be able to calculate order by order in **perturbation theory** the scattering amplitude from the LSZ formula (9) resorting on Feynman diagrams, also to be seen below.

Perturbation theory

▷ It is worth noting that the functional dependence of $\mathcal{H}_I$ on ϕ_I is the same as that of $\mathcal{H}_{\text{int}}$ on ϕ. For example,

$$
\begin{aligned}
\mathcal{H}_{\text{int}} &= \frac{\lambda}{4!}\phi^4 \\
\mathcal{H}_I &= \mathrm{e}^{\mathrm{i}H_0(t-t_0)}\frac{\lambda}{4!}\phi^4\mathrm{e}^{-\mathrm{i}H_0(t-t_0)} \\
&= \frac{\lambda}{4!}\left(\mathrm{e}^{\mathrm{i}H_0(t-t_0)}\phi\mathrm{e}^{-\mathrm{i}H_0(t-t_0)}\right)\left(\mathrm{e}^{\mathrm{i}H_0(t-t_0)}\phi\mathrm{e}^{-\mathrm{i}H_0(t-t_0)}\right) \\
&\quad\times\left(\mathrm{e}^{\mathrm{i}H_0(t-t_0)}\phi\mathrm{e}^{-\mathrm{i}H_0(t-t_0)}\right)\left(\mathrm{e}^{\mathrm{i}H_0(t-t_0)}\phi\mathrm{e}^{-\mathrm{i}H_0(t-t_0)}\right) = \frac{\lambda}{4!}\phi_I^4 \ .
\end{aligned}
$$

Feynman propagator

- Let us find the Feynman propagator, defined by

$$\langle 0|\, T\{\phi_I(x)\phi_I(y)\}\, |0\rangle \ .$$

From now on the index I will be omitted and we will always refer to fields in the interaction picture, that can be decomposed as $\phi(x) = \phi^+(x) + \phi^-(x)$ with

$$\phi^+(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_{\vec{p}}}}\, a_{\vec{p}}\mathrm{e}^{-\mathrm{i}px} \ , \quad \phi^-(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_{\vec{p}}}}\, a^\dagger_{\vec{p}}\mathrm{e}^{\mathrm{i}px} \ .$$

Feynman propagator

▷ Remember that $\phi^+ |0\rangle = 0$ and $\langle 0| \phi^- = 0$. Then,[a]

$$\text{if } x^0 - y^0 > 0 :$$
$$\begin{aligned} T\{\phi(x)\phi(y)\} &= \phi(x)\phi(y) \\ &= \phi^+(x)\phi^+(y) + \phi^+(x)\phi^-(y) + \phi^-(x)\phi^+(y) + \phi^-(x)\phi^-(y) \\ &= :\phi(x)\phi(y): + [\phi^+(x), \phi^-(y)] , \end{aligned}$$

where we have substituted

$$\begin{aligned} \phi^+(x)\phi^-(y) &= \phi^-(y)\phi^+(x) + [\phi^+(x), \phi^-(y)] \\ &=: \phi^+(x)\phi^-(y) : +[\phi^+(x), \phi^-(y)] \end{aligned} \tag{13}$$

[a] If $x^0 = y^0$ the fields are already time ordered, so then

$$T\{\phi(x)\phi(y)\} = \phi(x)\phi(y) =: \phi(x)\phi(y) : +[\phi^+(x), \phi^-(y)]$$

because in this case $[\phi^+(x), \phi^-(y)] = [\phi^+(y), \phi^-(x)]$, as can be explicitly checked in (17, 18).

Feynman propagator

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▷ Likewise,

$$\text{if } x^0 - y^0 < 0 :$$
$$\begin{aligned} T\{\phi(x)\phi(y)\} &= \phi(y)\phi(x) \\ &= \phi^+(y)\phi^+(x) + \phi^+(y)\phi^-(x) + \phi^-(y)\phi^+(x) + \phi^-(y)\phi^-(x) \\ &= :\phi(x)\phi(y): + [\phi^+(y), \phi^-(x)] , \end{aligned}$$

because $:\phi(x)\phi(y): = :\phi(y)\phi(x):$.

$$\text{Then,} \qquad T\{\phi(x)\phi(y)\} = :\phi(x)\phi(y): + D_F(x-y)$$

where

$$\boxed{D_F(x-y) = \theta(x^0 - y^0)\Delta(x-y) + \theta(y^0 - x^0)\Delta(y-x)} \tag{14}$$

and since $[a_{\vec{p}}, a^\dagger_{\vec{q}}] = (2\pi)^3\delta^3(\vec{p} - \vec{q})$,

$$\Delta(x-y) = [\phi^+(x), \phi^-(y)] = \int \frac{d^3p}{(2\pi)^3 2E_{\vec{p}}}\, e^{-ip(x-y)} .$$

Feynman propagator

▷ So the Feynman propagator is

$$\langle 0|\, T\{\phi(x)\phi(y)\}\, |0\rangle = \langle 0|\, (:\phi(x)\phi(y): + D_F(x-y))\, |0\rangle = D_F(x-y)$$

Let us see that we can write (Feynman prescription)

$$\boxed{D_F(x-y) = \int \frac{\mathrm{d}^4p}{(2\pi)^4} \frac{\mathrm{i}}{p^2 - m^2 + \mathrm{i}\varepsilon}\, \mathrm{e}^{-\mathrm{i}p(x-y)}\,, \quad \text{with } \varepsilon \to 0^+\,.} \tag{15}$$

▷ In fact,

$$\int \frac{\mathrm{d}^4p}{(2\pi)^4} \frac{\mathrm{i}}{p^2 - m^2 + \mathrm{i}\varepsilon}\, \mathrm{e}^{-\mathrm{i}p(x-y)} = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\, \mathrm{e}^{\mathrm{i}\vec{p}\cdot(\vec{x}-\vec{y})} \int_{-\infty}^{\infty} \frac{\mathrm{d}p^0}{2\pi} \frac{\mathrm{i}\mathrm{e}^{-\mathrm{i}p^0(x^0-y^0)}}{(p^0)^2 - E_{\vec{p}}^2 + \mathrm{i}\varepsilon} \tag{16}$$

where we have written $p^2 - m^2 = (p^0)^2 - \vec{p}^2 - m^2 = (p^0)^2 - E_{\vec{p}}^2$
remembering that $E_{\vec{p}} \equiv +\sqrt{m^2 + \vec{p}^2}$.

Feynman propagator

▷ On the other hand, note that

$$\Delta(x-y) = \int \frac{\mathrm{d}^3p}{(2\pi)^3 2E_{\vec{p}}}\, \mathrm{e}^{-\mathrm{i}p(x-y)} = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\, \mathrm{e}^{\mathrm{i}\vec{p}\cdot(\vec{x}-\vec{y})}\, \frac{\mathrm{e}^{-\mathrm{i}E_{\vec{p}}(x^0-y^0)}}{2E_{\vec{p}}} \tag{17}$$

$$\Delta(y-x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3 2E_{\vec{p}}}\, \mathrm{e}^{+\mathrm{i}p(x-y)} = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\, \mathrm{e}^{\mathrm{i}\vec{p}\cdot(\vec{x}-\vec{y})}\, \frac{\mathrm{e}^{+\mathrm{i}E_{\vec{p}}(x^0-y^0)}}{2E_{\vec{p}}} \tag{18}$$

(in the second line we have changed $\vec{p}$ by $-\vec{p}$).

▷ So it is enough to show that

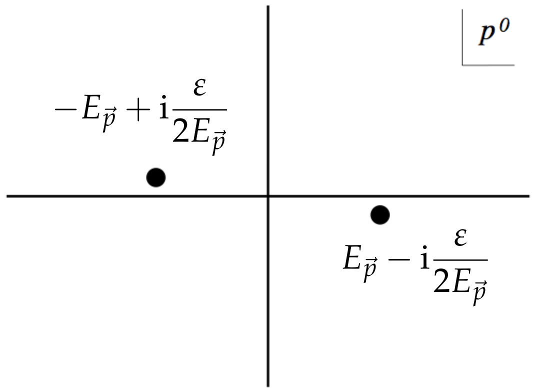
$$\int_{-\infty}^{\infty} \frac{\mathrm{d}p^0}{2\pi} \frac{\mathrm{i}\mathrm{e}^{-\mathrm{i}p^0(x^0-y^0)}}{(p^0)^2 - E_{\vec{p}}^2 + \mathrm{i}\varepsilon} = \theta(x^0-y^0)\, \frac{\mathrm{e}^{-\mathrm{i}E_{\vec{p}}(x^0-y^0)}}{2E_{\vec{p}}} + \theta(y^0-x^0)\, \frac{\mathrm{e}^{+\mathrm{i}E_{\vec{p}}(x^0-y^0)}}{2E_{\vec{p}}} \tag{19}$$

where note that, when $\varepsilon \to 0$,

$$(p^0)^2 - E_{\vec{p}}^2 + \mathrm{i}\varepsilon = \left[p^0 + \left(E_{\vec{p}} - \mathrm{i}\frac{\varepsilon}{2E_{\vec{p}}}\right)\right]\left[p^0 - \left(E_{\vec{p}} - \mathrm{i}\frac{\varepsilon}{2E_{\vec{p}}}\right)\right] \tag{20}$$

Feynman propagator

▷ To calculate the previous integral with respect to p^0 one has to choose the appropriate contour in the complex p^0 plane. The factor $\mathrm{i}\varepsilon$ slightly moves the poles away from the real axis. The pole $p^0 = E_{\vec{p}}$ is moved downward, $p^0 = E_{\vec{p}} - \mathrm{i}\varepsilon/(2E_{\vec{p}})$ and the pole $p^0 = -E_{\vec{p}}$ is shifted upward, $p^0 = -E_{\vec{p}} + \mathrm{i}\varepsilon/(2E_{\vec{p}})$.



Feynman propagator

▷ Then if $x^0 - y^0 > 0$ it is convenient to close the contour in the lower half plane, around the pole $p^0 = E_{\vec{p}} - \mathrm{i}0^+$ clockwise so that

$$\oint f(z)\,\mathrm{d}z = -2\pi\mathrm{i}\,\mathrm{Res}(f, z = z_0) \qquad \text{if } (x^0 - y^0) > 0$$

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}p^0}{2\pi} \frac{\mathrm{i}e^{-\mathrm{i}p^0(x^0-y^0)}}{(p^0)^2 - E_{\vec{p}}^2 + \mathrm{i}\varepsilon} = -2\pi\mathrm{i} \lim_{p^0 \to E_{\vec{p}}} (p^0 - E_{\vec{p}}) \frac{\mathrm{i}e^{-\mathrm{i}p^0(x^0-y^0)}}{(2\pi)(p^0 + E_{\vec{p}})(p^0 - E_{\vec{p}})} = \frac{e^{-\mathrm{i}E_{\vec{p}}(x^0-y^0)}}{2E_{\vec{p}}} .$$

And if $x^0 - y^0 < 0$ it is convenient to close the contour in the upper half plane, around the pole $p^0 = -E_{\vec{p}} + \mathrm{i}0^+$ counterclockwise so that

$$\oint f(z)\,\mathrm{d}z = 2\pi\mathrm{i}\,\mathrm{Res}(f, z = z_0) \qquad \text{if } (x^0 - y^0) < 0$$

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}p^0}{2\pi} \frac{\mathrm{i}e^{-\mathrm{i}p^0(x^0-y^0)}}{(p^0)^2 - E_{\vec{p}}^2 + \mathrm{i}\varepsilon} = 2\pi\mathrm{i} \lim_{p^0 \to -E_{\vec{p}}} (p^0 + E_{\vec{p}}) \frac{\mathrm{i}e^{-\mathrm{i}p^0(x^0-y^0)}}{(2\pi)(p^0 + E_{\vec{p}})(p^0 - E_{\vec{p}})} = \frac{e^{+\mathrm{i}E_{\vec{p}}(x^0-y^0)}}{2E_{\vec{p}}} .$$

Feynman propagator

▷ Then,

$$D_F(x-y) = \int \frac{\mathrm{d}^4p}{(2\pi)^4} \frac{\mathrm{i}}{p^2-m^2+\mathrm{i}\varepsilon}\, \mathrm{e}^{-\mathrm{i}p(x-y)}$$
$$= \int \frac{\mathrm{d}^3p}{(2\pi)^3 2E_{\vec{p}}}\, \mathrm{e}^{\mathrm{i}\vec{p}\cdot(\vec{x}-\vec{y})} \left[\theta(x^0-y^0)\mathrm{e}^{-\mathrm{i}E_{\vec{p}}(x^0-y^0)} + \theta(y^0-x^0)\mathrm{e}^{\mathrm{i}E_{\vec{p}}(x^0-y^0)}\right]$$
$$= \theta(x^0-y^0)\Delta(x-y) + \theta(y^0-x^0)\Delta(y-x)\,,$$

as we wanted to show.[a]

▷ The expression (15) is very useful because from it one can directly read the Feynman propagator in momentum space, $\widetilde{D}_F(p)$,

$$D_F(x-y) \equiv \int \frac{\mathrm{d}^4p}{(2\pi)^4}\, \mathrm{e}^{-\mathrm{i}p(x-y)} \widetilde{D}_F(p) \Rightarrow \widetilde{D}_F(p) = \frac{\mathrm{i}}{p^2-m^2+\mathrm{i}\varepsilon}\,.$$

[a]We can now unterstand why we had introduced a factor of 2 in the covariant normalization of states.

Feynman propagator

- Also note that $D_F(x-y)$ is a Green's function of the Klein-Gordon operator $(\Box_x + m^2)$ because

$$(\Box_x + m^2) D_F(x-y) = \int \frac{\mathrm{d}^4p}{(2\pi)^4} \frac{\mathrm{i}}{p^2-m^2+\mathrm{i}\varepsilon}(-p^2+m^2)\, \mathrm{e}^{-\mathrm{i}p(x-y)} = -\mathrm{i}\delta^4(x-y)$$

(regardless of the prescription adopted to circumvent the poles) which justifies that we have called $\langle 0|\, T\{\phi(x_1)\cdots\phi(x_N)\}\, |0\rangle$ the N-point Green's function.

▷ Further note that the Feynman propagator is not the only Green's function of the Klein-Gordon operator, because others are obtained by changing the prescription.

Feynman propagator | Causality

- The Feynman propagator $D_F(x-y)$ provides the probability amplitude that a particle created in y freely propagates to x where it is annihilated, if $x^0 - y^0 > 0$, or propagates from y to x, if $x^0 - y^0 < 0$. In fact, if $x^0 - y^0 > 0$ then

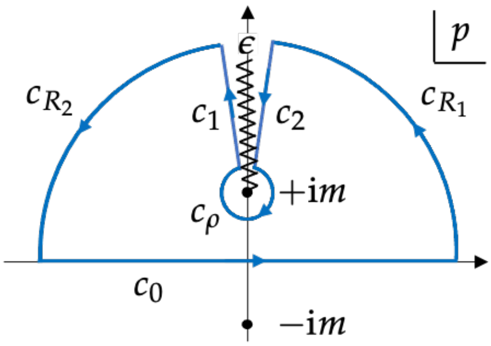
$$\langle 0|\, \phi(x)\phi(y)\, |0\rangle = D_F(x-y) = \Delta(x-y) = [\phi^+(x), \phi^-(y)] = \langle 0|\, \phi^+(x)\phi^-(y)\, |0\rangle \ .$$

▷ Let us see that, apparently, a problem arises: the probability of propagation of a particle from y to x with $(x-y)^2 < 0$ (spacelike interval), namely, beyond its light cone is not zero but falls exponentially for large distances.

▷ In fact, one can choose in this case a reference frame in which $(x-y) = (0, \vec{r})$ and then (here $p \equiv |\vec{p}|$ and $r \equiv |\vec{r}|$):

$$\begin{aligned}\Delta(x-y) &= \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{\mathrm{e}^{\mathrm{i}\vec{p}\cdot\vec{r}}}{2E_{\vec{p}}} = \frac{1}{(2\pi)^3} \int_0^{2\pi} \mathrm{d}\varphi \int_{-1}^{1} \mathrm{d}\cos\theta \int_0^\infty \mathrm{d}p\, p^2 \frac{\mathrm{e}^{\mathrm{i}pr\cos\theta}}{2\sqrt{p^2+m^2}} \\ &= \frac{1}{(2\pi)^2} \int_0^\infty \mathrm{d}p\, \frac{p^2}{2\sqrt{p^2+m^2}} \frac{\mathrm{e}^{\mathrm{i}pr} - \mathrm{e}^{-\mathrm{i}pr}}{\mathrm{i}pr} = -\frac{\mathrm{i}}{(2\pi)^2 2r} \int_{-\infty}^{\infty} \mathrm{d}p\, \frac{p\mathrm{e}^{\mathrm{i}pr}}{\sqrt{p^2+m^2}} \ .\end{aligned}$$

Feynman propagator | Causality



- We can calculate this integral in the complex p plane following the path in the figure (the integrand has branch cuts starting at the poles $p = \pm im$).
  Applying Cauchy's residue theorem:

$$0 = \left(\int_{c_0} + \int_{c_{R_1}} + \int_{c_{R_2}} + \int_{c_\rho} + \int_{c_1} + \int_{c_2} \right) \mathrm{d}p\, f(p).$$

In the limit $\varepsilon \to 0$, $\rho \to 0$, $R \to \infty$ the integrals over c_ρ, c_{R_1} and c_{R_2} vanish.[a]

[a] The integral over c_ρ vanishes $\rho \to 0$ because $\lim\limits_{p \to im} (p - im) f(p) = 0$.

▷ Then,

$$\Delta(x-y) = \lim_{\substack{\varepsilon,\rho\to 0\\ R\to\infty}} \int_{C_0} \mathrm{d}p\, f(p) = -\lim_{\substack{\varepsilon,\rho\to 0\\ R\to\infty}} \left(\int_{C_1} + \int_{C_2}\right) \mathrm{d}p\, f(p)$$
$$= \frac{\mathrm{i}}{(2\pi)^2 2r}\left(\int_{\mathrm{i}m}^{\mathrm{i}\infty} \mathrm{d}p\, p\mathrm{e}^{\mathrm{i}pr}(p^2+m^2)^{-1/2} + \int_{\mathrm{i}\infty}^{\mathrm{i}m} \mathrm{d}p\, p\mathrm{e}^{\mathrm{i}pr}(p^2+m^2)^{-1/2}\mathrm{e}^{-\frac{1}{2}2\pi\mathrm{i}}\right)$$
$$= \frac{\mathrm{i}}{(2\pi)^2 2r} 2\int_{\mathrm{i}m}^{\mathrm{i}\infty} \mathrm{d}p\, \frac{p\mathrm{e}^{\mathrm{i}pr}}{\sqrt{p^2+m^2}},$$

where the factor $\mathrm{e}^{-\frac{1}{2}2\pi\mathrm{i}} = -1$ takes into account that the second integration is done to the other side of the branch cut,[a] and we have exchanged the integration limits flipping the global sign. It is then useful to do the change of variable $p = \mathrm{i}\rho$,

$$\Delta(x-y) = \frac{\mathrm{i}}{(2\pi)^2 2r} 2\mathrm{i}^2 \int_m^\infty \mathrm{d}\rho\, \frac{\rho\mathrm{e}^{-\rho r}}{\mathrm{i}\sqrt{\rho^2-m^2}} = -\frac{1}{4\pi^2 r}\int_m^\infty \mathrm{d}\rho\, \frac{\rho\mathrm{e}^{-\rho r}}{\sqrt{\rho^2-m^2}}\,.$$

[a] Remember that $\log z = \log|z| + \mathrm{i}\arg(z)$ has a discontinuity of $2\pi i$ when crossing the branch cut and one can write $(p^2+m^2)^{-1/2} = \exp\{-\frac{1}{2}\log(p^2+m^2)\}$.

▷ And finally changing $\rho = mt$,

$$\Delta(x-y) = -\frac{m}{4\pi^2 r}\int_1^\infty \mathrm{d}t\, \frac{t\mathrm{e}^{-mrt}}{\sqrt{t^2-1}} = -\frac{m}{4\pi^2 r}K_1(mr) \xrightarrow[mr\gg 1]{} -\frac{m}{4\pi^2 r}\sqrt{\frac{\pi}{2mr}}\mathrm{e}^{-mr}\,, \tag{21}$$

where we have used the limit of the modified Bessel function K_1.

- This result seems to indicate that causality is violated. However this is not the case. In quantum mechanics what matters is whether two observables measured at points x and y commute when there is a spacelike interval between them, i.e. when $(x-y)^2 < 0$. It they commute then they are not correlated and one is not affected by the other.

Feynman propagator | Causality

- In practice, the causality principle is preserved whenever the commutator of two fields at two points separated by a spacelike interval vanishes.

  Let us see that in this case the commutator is zero, as wanted.

$$
\begin{aligned}
[\phi(x),\phi(y)] &= \int \frac{d^3p}{(2\pi)^3\sqrt{2E_{\vec p}}} \int \frac{d^3q}{(2\pi)^3\sqrt{2E_{\vec q}}} \left[\left(a_{\vec p}e^{-ipx}+a^\dagger_{\vec p}e^{ipx}\right),\left(a_{\vec q}e^{-iqy}+a^\dagger_{\vec q}e^{iqy}\right)\right] \\
&= \int \frac{d^3p}{(2\pi)^3\sqrt{2E_{\vec p}}} \int \frac{d^3q}{(2\pi)^3\sqrt{2E_{\vec q}}} \left\{e^{-i(px-qy)}[a_{\vec p},a^\dagger_{\vec q}] + e^{i(px-qy)}[a^\dagger_{\vec p},a_{\vec q}]\right\} \\
&= \int \frac{d^3p}{(2\pi)^3} \left\{e^{-ip(x-y)} - e^{ip(x-y)}\right\} = \Delta(x-y) - \Delta(y-x)\,, \qquad (22)
\end{aligned}
$$

  where we have used $[a_{\vec p},a^\dagger_{\vec q}] = (2\pi)^3\delta^3(\vec p-\vec q)$.

Feynman propagator | Causality

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▷ Now, if $(x-y)^2<0$ we can choose a reference frame in which $(x-y)=(0,\vec r)$ and then $(y-x)=(0,-\vec r)$, and recalling that for points separated by a spacelike interval $\Delta(x-y)$ depends only on the modulus of $\vec r$ (21) we have that $\Delta(x-y)=\Delta(y-x)$ and

$$[\phi(x),\phi(y)] = 0\,, \quad \text{if } (x-y)^2<0\,,$$

as we wanted to show.[a]

[a]If $(x-y)^2>0$ (timelike interval) we can choose a frame in which $(x-y)=(t,0)$ and then

$$\Delta(x-y) = \int \frac{d^3p}{(2\pi)^3}\frac{e^{-iE_{\vec p}t}}{2E_{\vec p}} = \frac{4\pi}{(2\pi)^3}\int_0^\infty dp\, p^2 \frac{e^{-i\sqrt{p^2+m^2}\,t}}{2\sqrt{p^2+m^2}} = \frac{1}{4\pi^2}\int_m^\infty dE\sqrt{E^2-m^2}e^{-iEt} \sim e^{-imt}\ (t\to\infty)$$

so $\Delta(x-y)-\Delta(y-x)\neq 0$ in this case.

- At this point several important comments are in order:
  1. For a complex scalar field the propagator is defined as

$$\begin{aligned} D_F(x-y) &= \langle 0| T\{\phi(x)\phi^\dagger(y)\} |0\rangle \\ &= \theta(x^0-y^0)\langle 0|\phi(x)\phi^\dagger(y)|0\rangle + \theta(y^0-x^0)\langle 0|\phi^\dagger(y)\phi(x)|0\rangle \end{aligned} \tag{23}$$

  expressing the probability amplitude that a particle created at y freely propagates to x where it is annihilated, if $x^0 - y^0 > 0$, or else the probability amplitude that an antiparticle created in x freely propagates to y where it is annihilated, if $x^0 - y^0 < 0$.
  Remember that for a real field particle and antiparticle coincide.

Feynman propagator **Causality**

- At this point several important comments are in order:
  2. To better understand the meaning of the two contributions to the Feynman propagator (14) canceling in (22) when $(x-y)^2 < 0$, let us find the complex scalar field commutator,

$$\begin{aligned} [\phi(x),\phi^\dagger(y)] &= \int \frac{d^3p}{(2\pi)^3\sqrt{2E_{\vec p}}} \int \frac{d^3q}{(2\pi)^3\sqrt{2E_{\vec q}}} \left[\left(a_{\vec p}e^{-ipx} + b^\dagger_{\vec p}e^{ipx}\right), \left(b_{\vec q}e^{-iqy} + a^\dagger_{\vec q}e^{iqy}\right)\right] \\ &= \int \frac{d^3p}{(2\pi)^3\sqrt{2E_{\vec p}}} \int \frac{d^3q}{(2\pi)^3\sqrt{2E_{\vec q}}} \left\{e^{-i(px-qy)}[a_{\vec p}, a^\dagger_{\vec q}] + e^{i(px-qy)}[b^\dagger_{\vec p}, b_{\vec q}]\right\} \\ &= \langle 0|\phi(x)\phi^\dagger(y)|0\rangle - \langle 0|\phi^\dagger(y)\phi(x)|0\rangle = \Delta(x-y) - \Delta(y-x) \end{aligned} \tag{24}$$

  where we see that $\Delta(x-y)$ is the probability amplitude that a particle created in y propagates to x while $\Delta(y-x)$ is the probability amplitude that an antiparticle created in x propagates to y.

Feynman propagator | Causality

- At this point several important comments are in order:
  - ▷ So if antiparticles did not exist the causality principle would be violated! as both contributions are needed for the cancellation, that occurs because the commutators (22) (or (24 for complex fields) yield identical results beyond the light cone preventing correlations between observations causally disconnected.

3. Finally, note that above it was key that scalar fields satisfy commutation rather than anticommutation relations, or otherwise the causality principle would not be held. And it can be shown that fermionic fields must anticommute for the same reason. This brings to light the tight link between the spin-statistics theorem and causality at the quantum level.

Wick theorem

- We have seen that the time ordered product of two fields in the interaction picture is $T\{\phi(x_1)\phi(x_2)\} = :\phi(x_1)\phi(x_2): + D_F(x_1 - x_2)$. Let us find now the time ordered product of n fields $\phi_i \equiv \phi(x_i)$.

- The Wick theorem, to be proven next, states that

$$T\{\phi_1 \cdots \phi_n\} = :\phi_1 \cdots \phi_n: + \left(\begin{array}{l} \text{every combination of normal ordering} \\ \text{and contractions of two fields} \end{array} \right) \tag{25}$$

where contractions of two fields $\phi(x_i)$ and $\phi(x_j)$ mean

$$\overbrace{\phi(x_i)\phi(x_j)} = D_F(x_i - x_j) \ , \quad \text{or in short} \quad \overbrace{\phi_i\phi_j} = D_{ij} \ ,$$

Wick theorem

and every combination of normal ordering and contractions of two fields means, for example,

$$
\begin{aligned}
T\{\phi_1\phi_2\phi_3\phi_4\} = {} & :\phi_1\phi_2\phi_3\phi_4: + :\Big(\wick{\c1\phi_1\c1\phi_2\phi_3\phi_4} + \wick{\c1\phi_1\phi_2\c1\phi_3\phi_4} + \wick{\c1\phi_1\phi_2\phi_3\c1\phi_4} + \wick{\phi_1\c1\phi_2\c1\phi_3\phi_4} \\
& + \wick{\phi_1\c1\phi_2\phi_3\c1\phi_4} + \wick{\phi_1\phi_2\c1\phi_3\c1\phi_4} + \wick{\c1\phi_1\c1\phi_2\c2\phi_3\c2\phi_4} + \wick{\c1\phi_1\c2\phi_2\c1\phi_3\c2\phi_4} + \wick{\c1\phi_1\c2\phi_2\c2\phi_3\c1\phi_4}\Big):,
\end{aligned}
$$

where

$$
:\wick{\c1\phi_1\phi_2\c1\phi_3\phi_4}: = \wick{\c1\phi_1\c1\phi_3} :\phi_2\phi_4: = D_{13} :\phi_2\phi_4:, \qquad :\wick{\c1\phi_1\c1\phi_2\c2\phi_3\c2\phi_4}: = D_{12}D_{34}, \qquad \text{etc.}
$$

Then, the vacuum expectation value of the time ordered product of fields receives contributions only from terms where all fields are contracted, for example,

$$
\begin{aligned}
\langle 0|\, T\{\phi_1\phi_2\phi_3\phi_4\}\, |0\rangle &= \wick{\c1\phi_1\c1\phi_2\c2\phi_3\c2\phi_4} + \wick{\c1\phi_1\c2\phi_2\c1\phi_3\c2\phi_4} + \wick{\c1\phi_1\c2\phi_2\c2\phi_3\c1\phi_4} \\
&= D_{12}D_{34} + D_{13}D_{24} + D_{14}D_{23}
\end{aligned}
$$

and it vanishes for an odd number of fields.

Wick theorem

▷ To prove the Wick theorem one proceeds by induction. We already know that for $n = 2$ it is fulfilled. Assume now that it is true for $n-1$ fields.
Then, starting with fields already time ordered ($x_1^0 \geq ... \geq x_n^0$),

$$
\begin{aligned}
T\{\phi_1\phi_2\cdots\phi_n\} &= \phi_1\phi_2\cdots\phi_n = \phi_1 T\{\phi_2\cdots\phi_n\} \\
&= (\phi_1^+ + \phi_1^-) : \left\{ \phi_2\cdots\phi_n + \begin{pmatrix} \text{every contraction of two} \\ \text{fields not involving } \phi_1 \end{pmatrix} \right\} : . \qquad (26)
\end{aligned}
$$

On the other hand,

$$
\phi_1^- :\{\phi_2\cdots\phi_n\}: = :\{\phi_1^-\phi_2\cdots\phi_n\}:
$$

because ϕ_1^- indroduces $a^\dagger$ on the left, already normal ordered, and

$$
\begin{aligned}
\phi_1^+ :\{\phi_2\cdots\phi_n\}: &= :\{\phi_2\cdots\phi_n\}: \phi_1^+ + [\phi_1^+, :\{\phi_2\cdots\phi_n\}:] \\
&= :\{\phi_1^+\phi_2\cdots\phi_n\}: + :\left(\wick{\c1\phi_1\c1\phi_2}\phi_3\cdots + \ldots + \begin{pmatrix} \text{simple contractions} \\ \text{involving } \phi_1 \end{pmatrix}\right):
\end{aligned}
$$

Wick theorem

▷ Let us check the latter statement with an example:

$$
\begin{aligned}
[\phi_1^+,\ :\phi_2\phi_3:\ ] &= [\phi_1^+, \phi_2^+\phi_3^+ + \phi_2^-\phi_3^- + \phi_2^-\phi_3^+ + \phi_3^-\phi_2^+] \\
&= \phi_2^+\cancel{[\phi_1^+,\phi_3^+]}^{=0} + \cancel{[\phi_1^+,\phi_2^+]}^{=0}\phi_3^+ + \phi_2^-[\phi_1^+,\phi_3^-] + [\phi_1^+,\phi_2^-]\phi_3^- + \phi_2^-\cancel{[\phi_1^+,\phi_3^+]}^{=0} \\
&\quad + [\phi_1^+,\phi_2^-]\phi_3^+ + \phi_3^-\cancel{[\phi_1^+,\phi_2^+]}^{=0} + [\phi_1^+,\phi_3^-]\phi_2^+ \\
&= \phi_2^-\overbracket{\phi_1\phi_3} + \overbracket{\phi_1\phi_3}\phi_2^+ + \overbracket{\phi_1\phi_2}\phi_3^- + \overbracket{\phi_1\phi_2}\phi_3^+ =: (\overbracket{\phi_1\phi_2}\phi_3 + \overbracket{\phi_1\phi_2\phi_3})\ :
\end{aligned}
$$

where we have used $[A, BC] = B[A, C] + [A, B]C$ and $\overbracket{\phi_i\phi_j} = [\phi_i^+, \phi_j^-]$, as $x_i^0 \geq x_j^0$. Then,

$$
(\phi_1^+ + \phi_1^-) : \{\phi_2 \cdots \phi_n\} : =\ : \phi_1\phi_2\cdots\phi_n : +\ : \begin{pmatrix} \text{simple contractions} \\ \text{involving } \phi_1 \end{pmatrix} : \ . \tag{27}
$$

Repeating the procedure for $(\phi_1^+ + \phi_1^-)\!: \begin{pmatrix} \text{every contraction of two} \\ \text{fields not involving } \phi_1 \end{pmatrix}\!:$

we will find the terms with the remaining contractions (double, triple and so on).

Feynman diagrams

- The LSZ reduction formula allows us to write the S matrix in terms of vacuum expectation values of time ordered field products in the interaction picture,

$$
\langle 0|\, T\left\{\phi(x_1)\phi(x_2)\cdots\phi(x_n)\exp\left[-\mathrm{i}\int \mathrm{d}^4x\ \mathcal{H}_I(x)\right]\right\}|0\rangle\ ,
$$

that can be calculated order by order in perturbation theory (PT), by series expanding the exponential.

Feynman diagrams

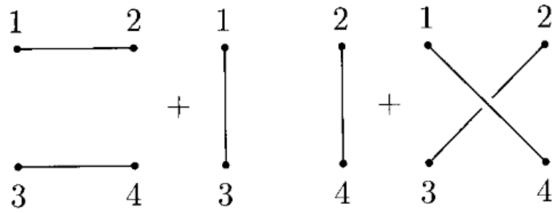
- At zeroth order (no interactions) we just need $\langle 0| T\{\phi(x_1)\cdots\phi(x_n)\} |0\rangle$, that applying Wick theorem, involves products of propagators of particles between different spacetime points $x_i \neq x_j$.

  This provides a physical view with a simple graphical representation:

$$\langle 0| T\{\phi_1\phi_2\} |0\rangle = \phi_1\phi_2 = D_{12}$$

$=$ 

$$\langle 0| T\{\phi_1\phi_2\phi_3\phi_4\} |0\rangle = \phi_1\phi_2\phi_3\phi_4 + \phi_1\phi_2\phi_3\phi_4 + \phi_1\phi_2\phi_3\phi_4 = D_{12}D_{34} + D_{13}D_{24} + D_{14}D_{23}$$

$=$ 

and so on. These are the Feynamn diagrams in position space.

Feynman diagrams

- From the first order in PT we find local interactions involving products of fields at the same spacetime point x, also with a simple graphical representation in terms of Feynam diagrams.

  The perturbative calculation is very complex but it can be organized with the help of Feynman rules. Let us illustrate the procedure with an example.

Feynman diagrams | $2 \to 2$ | ($\lambda\phi^4$)

▷ Consider the $2 \to 2$ scattering
(two particles in the initial state and two in the final state)
in the self-interacting scalar field theory $\lambda\phi^4$.
The LSZ formula written in terms of fields in the interaction picture is:

$$\prod_{i=1}^{2} \frac{\mathrm{i}\sqrt{Z}}{p_i^2 - m^2} \prod_{j=1}^{2} \frac{\mathrm{i}\sqrt{Z}}{k_j^2 - m^2} \langle \vec{p}_1 \vec{p}_2 | \, \mathrm{i}T \left| \vec{k}_1 \vec{k}_2 \right\rangle$$

$$= \int \prod_{i=1}^{4} \mathrm{d}^4 x_i \, \mathrm{e}^{\mathrm{i}(p_1 x_1 + p_2 x_2 - k_1 x_3 - k_2 x_4)} \frac{\langle 0 | \, T \left\{ \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \exp\left[-\mathrm{i}\frac{\lambda}{4!} \int \mathrm{d}^4 x \, \phi^4(x) \right] \right\} |0\rangle}{\langle 0 | \, T \left\{ \exp\left[-\mathrm{i}\frac{\lambda}{4!} \int \mathrm{d}^4 x \, \phi^4(x) \right] \right\} |0\rangle} \tag{28}$$

Feynman diagrams | $2 \to 2$ | ($\lambda\phi^4$)

– Zeroth order. In absence of interactions the denominator is 1. The numerator is

$$\begin{aligned}
N_0 &= \int \prod_{i=1}^{4} \mathrm{d}^4 x_i \, \mathrm{e}^{\mathrm{i}(p_1 x_1 + p_2 x_2 - k_1 x_3 - k_2 x_4)} \langle 0 | \, T\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\} \, |0\rangle \\
&= \int \prod_{i=1}^{4} \mathrm{d}^4 x_i \, \mathrm{e}^{\mathrm{i}(p_1 x_1 + p_2 x_2 - k_1 x_3 - k_2 x_4)} (D_{12}D_{34} + D_{13}D_{24} + D_{14}D_{23}) \\
&= \int \mathrm{d}^4 x \, \mathrm{d}^4 X \mathrm{d}^4 y \, \mathrm{d}^4 Y \, \mathrm{e}^{\mathrm{i}(p_1+p_2)X + \mathrm{i}(p_1-p_2)\frac{x}{2} - \mathrm{i}(k_1+k_2)Y - \mathrm{i}(k_1-k_2)\frac{y}{2}} \, D_F(x) D_F(y) \\
&\quad + \int \mathrm{d}^4 x \, \mathrm{d}^4 X \, \mathrm{d}^4 y \, \mathrm{d}^4 Y \, \mathrm{e}^{\mathrm{i}(p_1-k_1)X + \mathrm{i}(p_1+k_1)\frac{x}{2} + \mathrm{i}(p_2-k_2)Y + \mathrm{i}(p_2+k_2)\frac{y}{2}} \, D_F(x) D_F(y) \\
&\quad + \int \mathrm{d}^4 x \, \mathrm{d}^4 X \mathrm{d}^4 y \, \mathrm{d}^4 Y \, \mathrm{e}^{\mathrm{i}(p_1-k_2)X + \mathrm{i}(p_1+k_2)\frac{x}{2} - \mathrm{i}(k_1-p_2)Y - \mathrm{i}(k_1+p_2)\frac{y}{2}} \, D_F(x) D_F(y) \\
&= (2\pi)^4 \delta^4(p_1+p_2)(2\pi)^4 \delta^4(k_1+k_2) \frac{\mathrm{i}}{p_1^2 - m^2 + \mathrm{i}\varepsilon} \frac{\mathrm{i}}{k_1^2 - m^2 + \mathrm{i}\varepsilon} \\
&\quad + (2\pi)^4 \delta^4(p_1-k_1)(2\pi)^4 \delta^4(p_2-k_2) \frac{\mathrm{i}}{p_1^2 - m^2 + \mathrm{i}\varepsilon} \frac{\mathrm{i}}{p_2^2 - m^2 + \mathrm{i}\varepsilon} \\
&\quad + (2\pi)^4 \delta^4(p_1-k_2)(2\pi)^4 \delta^4(p_2-k_1) \frac{\mathrm{i}}{p_1^2 - m^2 + \mathrm{i}\varepsilon} \frac{\mathrm{i}}{k_1^2 - m^2 + \mathrm{i}\varepsilon}
\end{aligned} \tag{29}$$

Feynman diagrams $2 \to 2$ $(\lambda\phi^4)$

where in the first term of the third equality we have changed:

$$
\begin{aligned}
&x = x_1 - x_2\,, \quad y = x_3 - x_4 && x_1 = X + \frac{x}{2}\,, \quad x_3 = Y + \frac{y}{2} \\
& && \Rightarrow \\
&X = \frac{x_1 + x_2}{2}\,, \quad Y = \frac{x_3 + x_4}{2} && x_2 = X - \frac{x}{2}\,, \quad x_4 = Y - \frac{y}{2}
\end{aligned}
$$

$$
\mathrm{d}x_1\,\mathrm{d}x_2 = \begin{vmatrix} \dfrac{\partial x_1}{\partial x} & \dfrac{\partial x_1}{\partial X} \\ \dfrac{\partial x_2}{\partial x} & \dfrac{\partial x_2}{\partial X} \end{vmatrix} \mathrm{d}x\,\mathrm{d}X = \begin{vmatrix} \frac{1}{2} & 1 \\ -\frac{1}{2} & 1 \end{vmatrix} \mathrm{d}x\,\mathrm{d}X = \mathrm{d}x\,\mathrm{d}X\,, \quad \text{etc.}\,,
$$

the second term is analogous to the first if $x_2 \leftrightarrow x_3$ implying $p_2 \leftrightarrow -k_1$; and the third term is analogous to the first if $x_2 \leftrightarrow x_4$ implying $p_2 \leftrightarrow -k_2$.

In (29) there are terms with just two poles, not enough to cancel the four poles in the left-hand side of (28), so

$$
\langle \vec{p}_1 \vec{p}_2 |\, \mathrm{i}T \left| \vec{k}_1 \vec{k}_2 \right\rangle = 0 \quad \text{at zeroth order.}
$$

This result (vanishing amplitude) is general for disconnected diagrams (those with at least one external point no connected to the others).

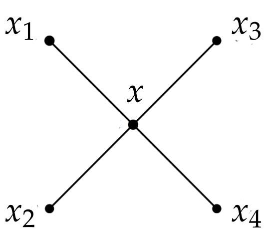
Feynman diagrams $2 \to 2$ $(\lambda\phi^4)$

– First order. Expanding the exponential of the numerator at $\mathcal{O}(\lambda)$ we obtain products of fields evaluated at the same spacetime point that, applying Wick theorem, leads to an interaction vertex.

The only way to get connected diagrams is by contracting each $\phi(x_i)$ with $\phi(x)$:

$$
\langle 0 |\, T\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\phi^4(x)\}\, |0\rangle_c = 4!\ :\phi_1\phi_2\phi_3\phi_4\phi_x\phi_x\phi_x\phi_x:
$$

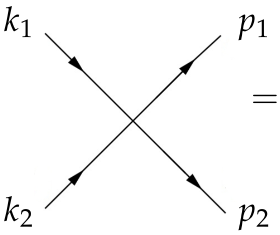
$=$



There are 4! possible combinations of such contractions, all identical: $\phi(x_1)$ with one of the 4 $\phi(x)$, $\phi(x_2)$ with one of the 3 $\phi(x)$ remaining, $\phi(x_3)$ with one of the 2 $\phi(x)$ remaining and $\phi(x_4)$ with the $\phi(x)$ remaining. The 4! cancels the 4! introduced in the definition of the coupling constant (hence its convenience).

Feynman diagrams $2 \to 2$ $(\lambda\phi^4)$

Thus, at first order, the only relevant contribution to the numerator of the $2 \to 2$ amplitude is given by the following Feynman diagram in momentum space:



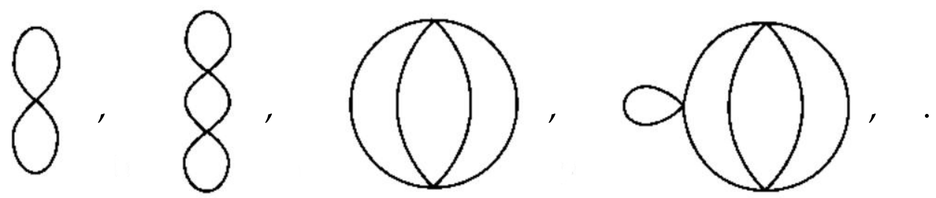
$$
\begin{aligned}
&= \int \prod_{i=1}^{4} \mathrm{d}^4 x_i \; \mathrm{e}^{\mathrm{i}(p_1 x_1 + p_2 x_2 - k_1 x_3 - k_2 x_4)} \\
&\quad \times \left(-\frac{\mathrm{i}\lambda}{4!}\right) 4! \int \mathrm{d}^4 x \; D_F(x_1 - x) D_F(x_2 - x) D_F(x_3 - x) D_F(x_4 - x) \\
&= -\mathrm{i}\lambda \int \prod_{i=1}^{4} \mathrm{d}^4 y_i \; \mathrm{d}^4 x \; \mathrm{e}^{\mathrm{i}(p_1 + p_2 - k_1 - k_2)x} \mathrm{e}^{\mathrm{i}(p_1 y_1 + p_2 y_2 - k_1 y_3 - k_2 y_4)} D_F(y_1) D_F(y_2) D_F(y_3) D_F(y_4) \\
&= -\mathrm{i}\lambda (2\pi)^4 \delta^4(p_1 + p_2 - k_1 - k_2) \widetilde{D}_F(p_1) \widetilde{D}_F(p_2) \widetilde{D}_F(k_1) \widetilde{D}_F(k_2) \qquad (30)
\end{aligned}
$$

where the change of variables $y_i = x_i - x$ has been performed.

Feynman diagrams $2 \to 2$ $(\lambda\phi^4)$

– Let us now fin the denominator $\langle 0 | T \left\{ \exp\left[-\mathrm{i}\frac{\lambda}{4!} \int \mathrm{d}^4 x \; \phi^4(x) \right] \right\} | 0 \rangle$, that order by order is composed of disconnected diagrams without external points, given by combinations of vacuum-vacuum diagrams:

$V_i \in \{$



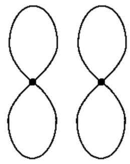
$, \dots \}$ (31)

Take one of these diagrams with n_i pieces of each type V_i. Calling also V_i the value of the piece of type i, it is easy to see that this type of diagrams contribute to the denominator with $\sum_{n_i} \frac{V_i^{n_i}}{n_i!}$, where the $n_i!$ comes from the exchange symmetry of the n_i copies of V_i.

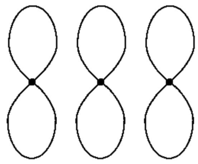
To check this, consider just one type V_i, and take it to be the first of the vacuum-vacuum diagrams listed in (31).

Then

$$= -\mathrm{i}\frac{\lambda}{4!}\int \mathrm{d}^4x\ \overbracket{\phi_x\phi_x}\overbracket{\phi_x\phi_x} \times 3 = \frac{3}{4!}V = \frac{1}{8}V \equiv V_i$$



$$= \frac{1}{2!}\left(-\mathrm{i}\frac{\lambda}{4!}\right)^2 \int \mathrm{d}^4x\ \overbracket{\phi_x\phi_x}\overbracket{\phi_x\phi_x} \int \mathrm{d}^4y\ \overbracket{\phi_y\phi_y}\overbracket{\phi_y\phi_y} \times 3^2 = \frac{1}{2!}V_i^2$$



$$= \frac{1}{3!}V_i^3$$

and so on.

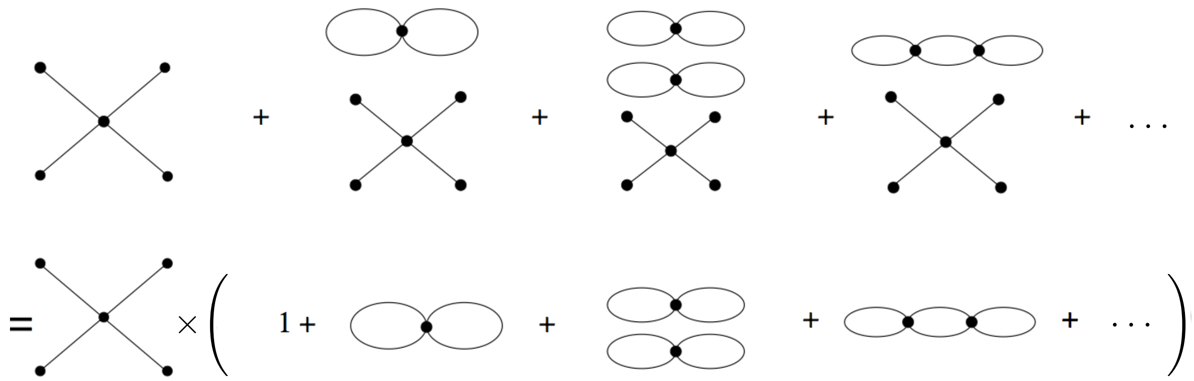
Then the total contribution to the denominator will be

$$\prod_i \left(\sum_{n_i} \frac{V_i^{n_i}}{n_i!}\right) = \prod_i \mathrm{e}^{V_i} = \exp\left\{\sum_i V_i\right\} ,$$

given by the exponential of the sum of all vacuum-vacuum diagrams.

Feynman diagrams $2 \to 2$ $(\lambda\phi^4)$

– Note that in the numerator we have, for each connected diagram, the contribution of an arbitrary number of vacuum-vacuum diagrams.
For example,



Therefore, the general contribution to the numerator can be written as

$$\sum(\text{connected}) \times \exp\left\{\sum_i V_i\right\} .$$

Feynman diagrams $2 \to 2$ $(\lambda\phi^4)$

▷ Thus, the vacuum-vacuum contributions to numerator and denominator in the LSZ formula cancel each other and we conclude that in order to find the $m \to n$ scattering amplitude it suffices to calculate, order by order, the sum of disconnected diagrams with $m+n$ external points.

Using these results and ignoring for the moment the Z factors (we will see soon that $Z = 1 + \mathcal{O}(\lambda^2)$), we can calculate the $2 \to 2$ scattering amplitude at first order, derived from (28) y (30),

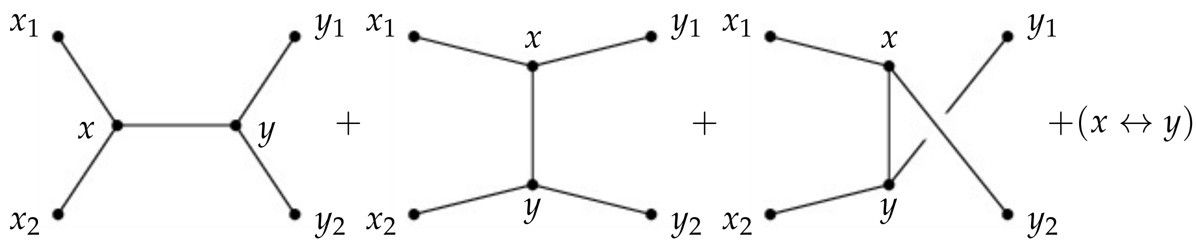
$$\mathcal{H}_{\text{int}} = \frac{\lambda}{4!}\phi^4 : \qquad \langle \vec{p}_1\vec{p}_2|\, \mathrm{i}T \left|\vec{k}_1\vec{k}_2\right\rangle = -\mathrm{i}\lambda(2\pi)^4\delta^4(p_1+p_2-k_1-k_2) + \mathcal{O}(\lambda^2) .$$

▷ We could already write some of the rules allowing us to obtain diagrammatically the scattering amplitude, but we cannot get them all yet because we have not yet encountered the case of diagrams with internal lines nor loops.

Feynman diagrams $2 \to 2$ $(\lambda\phi^3)$

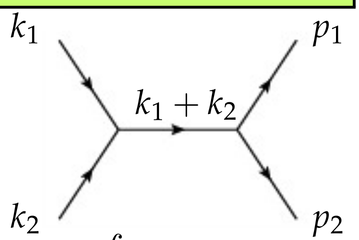
▷ To illustrate the case of diagrams with internal lines, let us suppose that our $2 \to 2$ scattering process is due to a different interaction, $\mathcal{H}_{\text{int}} = \frac{\lambda}{3!}\phi^3(x)$.

Looking for connected diagrams at lowest order yielding a non vanishing contribution we find that it is at $\mathcal{O}(\lambda^2)$ and is given by the following diagrams:



Let us calculate in detail the contribution of the first diagram (including the same with x and y exchanged) that we will represent by the following Feynman diagram in momentum space:

Feynman diagrams $2 \to 2$ $(\lambda\phi^3)$



(32)

$$= \int \mathrm{d}^4x_1\mathrm{d}^4x_2\mathrm{d}^4y_1\mathrm{d}^4y_2\ \mathrm{e}^{\mathrm{i}(p_1x_1+p_2x_2-k_1y_1-k_2y_2)}$$

$$\times \frac{1}{2!}\left(\frac{-\mathrm{i}\lambda}{3!}\right)^2 (3!)^2 2 \int \mathrm{d}^4x\mathrm{d}^4y\ \underbrace{D_F(x_1-x)D_F(x_2-x)D_F(x-y)D_F(y-y_1)D_F(y-y_2)}_{:\phi(x_1)\phi(x_2)\phi(y_1)\phi(y_2)\phi(x)\phi(x)\phi(x)\phi(y)\phi(y)\phi(y):}$$

$$= (-\mathrm{i}\lambda)^2 \int \mathrm{d}^4\tilde{x}_1\mathrm{d}^4\tilde{x}_2\mathrm{d}^4\tilde{y}_1\mathrm{d}^4\tilde{y}_2\mathrm{d}^4x\mathrm{d}^4y\ \mathrm{e}^{\mathrm{i}(p_1+p_2)x-\mathrm{i}(k_1+k_2)y+\mathrm{i}(p_1\tilde{x}_1+p_2\tilde{x}_2-k_1\tilde{y}_1-k_2\tilde{y}_2)}$$
$$\times D_F(\tilde{x}_1)D_F(\tilde{x}_2)D_F(\tilde{y}_1)D_F(\tilde{y}_2)D_F(x-y)$$

$$= (-\mathrm{i}\lambda)^2\widetilde{D}_F(p_1)\widetilde{D}_F(p_2)\widetilde{D}_F(k_1)\widetilde{D}_F(k_2)\int \mathrm{d}^4\tilde{x}\mathrm{d}^4y\ \mathrm{e}^{\mathrm{i}(p_1+p_2)\tilde{x}+\mathrm{i}(p_1+p_2-k_1-k_2)y}D_F(\tilde{x})$$

$$= (-\mathrm{i}\lambda)^2(2\pi)^4\delta^4(p_1+p_2-k_1-k_2)\widetilde{D}_F(k_1+k_2)\widetilde{D}_F(p_1)\widetilde{D}_F(p_2)\widetilde{D}_F(k_1)\widetilde{D}_F(k_2)$$

Feynman diagrams $2 \to 2$ $(\lambda\phi^3)$

where the factor $(3!)^2$ comes from all equivalent Wick contractions and the factor 2 comes from the exchange of x and y. We have also changed the variables $\tilde{x}_i = x_i - x$, $\tilde{y}_i = y_i - y$ and $\tilde{x} = x - y$.

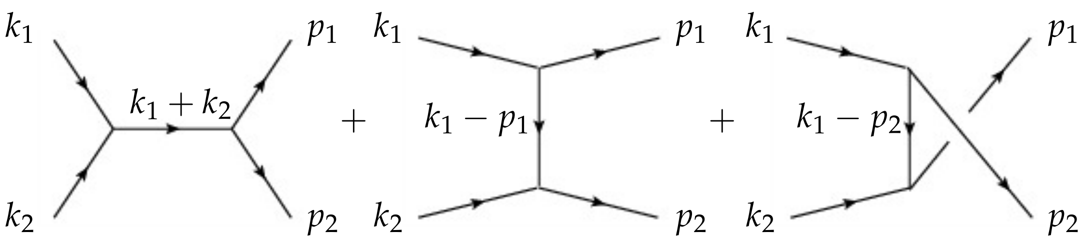
As in (30), we have obtained:

- a factor $(-\mathrm{i}\lambda)$ for each vertex,
- a factor $(2\pi)^4\delta^4(p_1 + p_2 - k_1 - k_2)$ from the four-momentum conservation, and
- the product of the four propagators of the external legs, that will cancel when we solve for the scattering amplitude in the LSZ formula.
- Moreover, we had to introduce the propagator of each internal line.

Note that the factor 3! in the denominator of the coupling constant has canceled when summing over all equivalent Wick contractions.

Feynman diagrams $2 \to 2$ $(\lambda\phi^3)$

Therefore, summing the three diagrams in momentum space



(33)

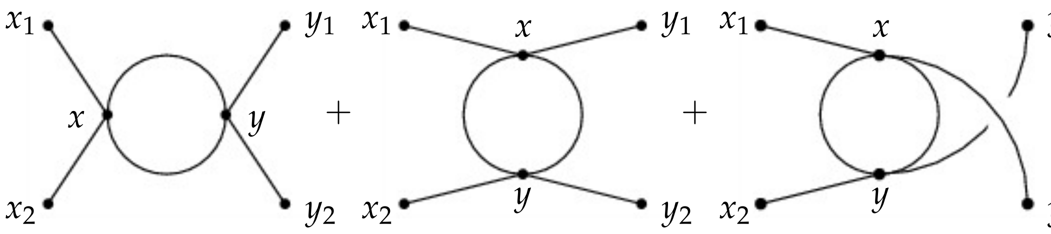
we have

$$\mathcal{H}_{\text{int}} = \frac{\lambda}{3!}\phi^3 : \quad \langle \vec{p}_1 \vec{p}_2 | \, \mathrm{i}T \left| \vec{k}_1 \vec{k}_2 \right\rangle = (-\mathrm{i}\lambda)^2 (2\pi)^4 \delta^4(p_1 + p_2 - k_1 - k_2)$$
$$\times \left[\widetilde{D}_F(k_1 + k_2) + \widetilde{D}_F(k_1 - p_1) + \widetilde{D}_F(k_1 - p_2)\right]$$
$$+ \mathcal{O}(\lambda^4) \,.$$

Note that the integration over the spacetime coordinates of the interaction points gives the four-momentum conservation at every vertex.

▷ Now let us see what happens when there are loops in the diagrams.

Go back to the $\lambda\phi^4$ theory. In order to calculate the contribution at $\mathcal{O}(\lambda^2)$ to the $2 \to 2$ amplitude we need to compute the following connected diagrams:

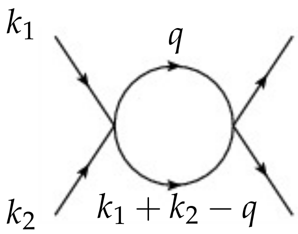


$+(x \leftrightarrow y)$

They all feature a loop made of two internal lines sharing initial and final points.[a]

Focus now on the contribution of the first of these diagrams (including the same with x and y exchanged) represented by the following Feynman diagram in momentum space:

[a] A loop may also come from an internal line starting and ending in the same point. See e.g. the diagram of (34).



$$= \int \mathrm{d}^4x_1\mathrm{d}^4x_2\mathrm{d}^4y_1\mathrm{d}^4y_2\ e^{\mathrm{i}(p_1x_1+p_2x_2-k_1y_1-k_2y_2)}\frac{1}{2!}\left(\frac{-\mathrm{i}\lambda}{4!}\right)^2$$

$$\times (4\times 3)^2 \times 2 \times 2 \int \mathrm{d}^4x\mathrm{d}^4y\ \underbrace{D_F(x_1-x)D_F(x_2-x)D_F^2(x-y)D_F(y-y_1)D_F(y-y_2)}_{:\phi(x_1)\phi(x_2)\phi(y_1)\phi(y_2)\phi(x)\phi(x)\phi(x)\phi(x)\phi(y)\phi(y)\phi(y)\phi(y):}$$

$$= \frac{1}{2}(-\mathrm{i}\lambda)^2 \int \mathrm{d}^4\tilde{x}_1\mathrm{d}^4\tilde{x}_2\mathrm{d}^4\tilde{y}_1\mathrm{d}^4\tilde{y}_2\mathrm{d}^4x\mathrm{d}^4y\ e^{\mathrm{i}(p_1+p_2)x-\mathrm{i}(k_1+k_2)y+\mathrm{i}(p_1\tilde{x}_1+p_2\tilde{x}_2-k_1\tilde{y}_1-k_2\tilde{y}_2)}$$
$$\times D_F(\tilde{x}_1)D_F(\tilde{x}_2)D_F(\tilde{y}_1)D_F(\tilde{y}_2)D_F^2(x-y)$$
$$= \frac{1}{2}(-\mathrm{i}\lambda)^2\widetilde{D}_F(p_1)\widetilde{D}_F(p_2)\widetilde{D}_F(k_1)\widetilde{D}_F(k_2)\int \mathrm{d}^4\tilde{x}\mathrm{d}^4y\ e^{\mathrm{i}(p_1+p_2)\tilde{x}+\mathrm{i}(p_1+p_2-k_1-k_2)y}D_F^2(\tilde{x})$$
$$= \frac{1}{2}(-\mathrm{i}\lambda)^2(2\pi)^4\delta^4(p_1+p_2-k_1-k_2)\widetilde{D}_F(p_1)\widetilde{D}_F(p_2)\widetilde{D}_F(k_1)\widetilde{D}_F(k_2)$$
$$\times \int \frac{\mathrm{d}^4q}{(2\pi)^4}\widetilde{D}_F(q)\widetilde{D}_F(k_1+k_2-q)$$

Feynman diagrams | $2 \to 2$ | $(\lambda\phi^4)$

where we have substituted

$$\int \mathrm{d}^4\tilde{x}\ \mathrm{e}^{\mathrm{i}(k_1+k_2)\tilde{x}} D_F^2(\tilde{x}) = \int \mathrm{d}^4\tilde{x}\ \mathrm{e}^{\mathrm{i}(k_1+k_2)\tilde{x}} D_F(\tilde{x}) \int \frac{\mathrm{d}^4q}{(2\pi)^4}\ \mathrm{e}^{-\mathrm{i}q\tilde{x}} \widetilde{D}_F(q)$$
$$= \int \frac{\mathrm{d}^4q}{(2\pi)^4}\ \widetilde{D}_F(k_1+k_2-q)\widetilde{D}_F(q)\ .$$

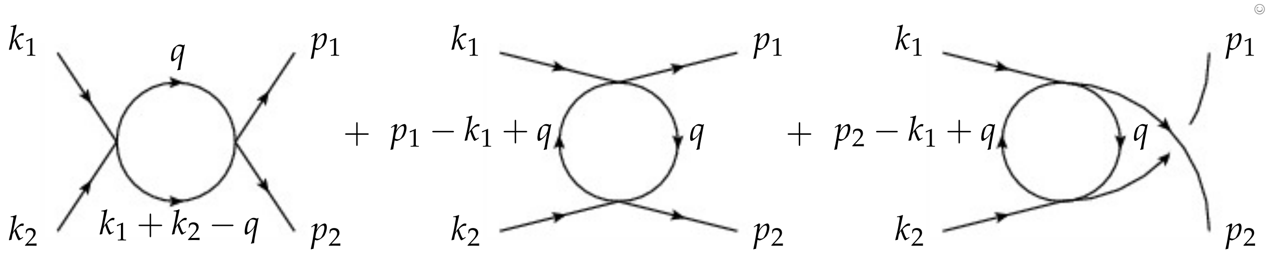
We see that, besides the usual factor $(2\pi)^4\delta^4(p_1+p_2-k_1-k_2)$ from global four-momentum conservation, a factor $(-\mathrm{i}\lambda)$ for every vertex and a propagator for every internal line, there is an integration over the loop four-momentum divided by $(2\pi)^4$.

We also obtain a symmetry factor $\frac{1}{2}$ from the counting of factors of $1/4!$ and equivalent Wick contractions. These are a frequent source of errors in the computations.

And there are propagators of external legs in momentum space whose poles will cancel when solving for the amplitude in the LSZ formula.

Feynman diagrams | $2 \to 2$ | $(\lambda\phi^4)$

▷ Repeating the procedure for the three diagrams in momentum space:



we get ($\mathcal{H}_{\text{int}} = \frac{\lambda}{4!}\phi^4$)

$$\langle \vec{p}_1\vec{p}_2|\,\mathrm{i}T\,\left|\vec{k}_1\vec{k}_2\right\rangle = (2\pi)^4\delta^4(p_1+p_2-k_1-k_2)$$
$$\times \Big\{ -\mathrm{i}\lambda + \frac{1}{2}(-\mathrm{i}\lambda)^2 \int \frac{\mathrm{d}^4q}{(2\pi)^4} [\widetilde{D}_F(q)\widetilde{D}_F(k_1+k_2-q)$$
$$+ \widetilde{D}_F(q)\widetilde{D}_F(k_1-p_1-q)$$
$$+ \widetilde{D}_F(q)\widetilde{D}_F(k_1-p_2-q)] \Big\} + \mathcal{O}(\lambda^3)\ .$$

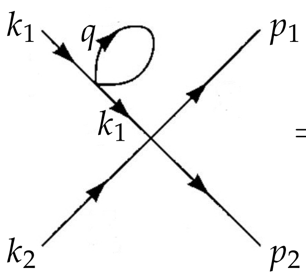
Feynman diagrams $2 \to 2$ $(\lambda\phi^4)$

- The explicit calculation of these loop integrals yields a divergent result in the ultraviolet: they approach infinity when q gets large.

▷ In order to make sense of this infinite correction to the prediction that we had obtained at lower order in PT one has to renormalize the theory.

Feynman diagrams $2 \to 2$ $(\lambda\phi^4)$

- So far we have ignored the Z factors (wave function renormalization) appearing in the LSZ formula. We have also ignored diagrams where the propagator of any of the external legs gets corrected, as for example:[a]



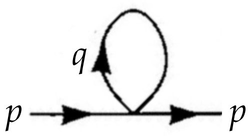
$$
\begin{aligned}
&= (2\pi)^4\delta^4(k_1+k_2-p_1-p_2)(-\mathrm{i}\lambda)\widetilde{D}_F(k_2)\widetilde{D}_F(p_1)\widetilde{D}_F(p_2) \\
&\quad \times\widetilde{D}_F(k_1)\times\frac{1}{2}(-\mathrm{i}\lambda)\int\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{\mathrm{i}}{q^2-m^2}\times\widetilde{D}_F(k_1).
\end{aligned}
\tag{34}
$$

Apart of the loop correction (infinite, by the way) this expression contains a double pole in $\widetilde{D}_F(k_1)$ that cannot be canceled by the simple pole of the LSZ formula, yielding another infinity.

[a] From now on the $\mathrm{i}\varepsilon$ of the Feynman prescription will be omitted, although it is always assumed.

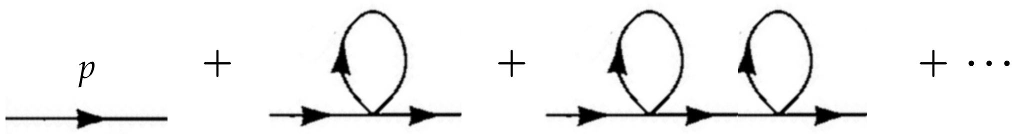
Feynman diagrams | $2 \to 2$ | $(\lambda\phi^4)$

▷ Note that the correction to the external leg factorizes and can be read directly from the following diagram:



$$= \widetilde{D}_F(p)(-\mathrm{i}B)\widetilde{D}_F(p)\ , \quad -\mathrm{i}B = \frac{1}{2}(-\mathrm{i}\lambda)\int \frac{\mathrm{d}^4 q}{(2\pi)^4}\frac{\mathrm{i}}{q^2-m^2}\ . \tag{35}$$

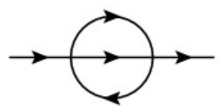
We can resum all these corrections to the propagator,



$$\begin{aligned}
&= \widetilde{D}_F(p) + \widetilde{D}_F(p)(-\mathrm{i}B)\widetilde{D}_F(p) + \widetilde{D}_F(p)(-\mathrm{i}B)\widetilde{D}_F(p)(-\mathrm{i}B)\widetilde{D}_F(p) + \ldots \\
&= \widetilde{D}_F(p)\left[1 + (-\mathrm{i}B\widetilde{D}_F(p)) + (-\mathrm{i}B\widetilde{D}_F(p))^2 + \ldots\right] \\
&= \widetilde{D}_F(p)\frac{1}{1+\mathrm{i}B\widetilde{D}_F(p)} = \frac{\mathrm{i}}{p^2-m^2}\left(\frac{1}{1-\frac{B}{p^2-m^2}}\right) = \frac{\mathrm{i}}{p^2-m^2-B}\ .
\end{aligned}$$

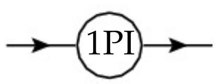
Feynman diagrams | $2 \to 2$ | $(\lambda\phi^4)$

▷ We see that the net effect of these corrections amounts to a mass shift from m^2 to $m^2 + B$.

▷ We could add other corrections, as for example the $\mathcal{O}(\lambda^2)$ 

(that, unlike the previous one, depends on p^2) and everyone else.

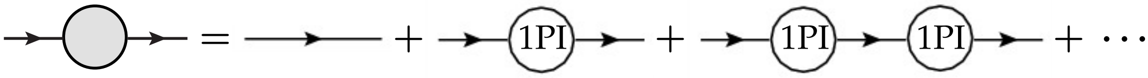
▷ To that end, we sum all two external leg diagrams that are one-particle irreducible (those that cannot be separated into two disconnected parts by cutting a single internal line) and call $-\mathrm{i}M^2(p^2)$ the contribution of all 1PI diagrams (removing the external propagators),



$$= -\mathrm{i}M^2(p^2)\ . \tag{36}$$

Feynman diagrams | $2 \to 2$ | $(\lambda\phi^4)$

▷ Now we can resum all corrections to the propagator using the same procedure as before. We will rename m_0 the mass parameter introduced in the Lagrangian. Then the full propagator is (to any order in PT):



$$
\begin{aligned}
&= \frac{\mathrm{i}}{p^2-m_0^2} + \frac{\mathrm{i}}{p^2-m_0^2}[-\mathrm{i}M^2(p^2)]\frac{\mathrm{i}}{p^2-m_0^2} + \dots \\
&= \frac{\mathrm{i}}{p^2-m_0^2}\left[1 + \frac{M^2(p^2)}{p^2-m_0^2} + \left(\frac{M^2(p^2)}{p^2-m_0^2}\right)^2 + \dots\right] \\
&= \frac{\mathrm{i}}{p^2-m_0^2}\frac{1}{1-\dfrac{M^2(p^2)}{p^2-m_0^2}} = \frac{\mathrm{i}}{p^2-m_0^2-M^2(p^2)} \; .
\end{aligned} \tag{37}
$$

Feynman diagrams | $2 \to 2$ | $(\lambda\phi^4)$

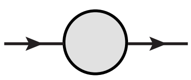
- The physical mass m is defined as the pole of the full propagator,

$$
p^2 - m_0^2 - M^2(p^2)\Big|_{p^2=m^2} = 0 \; . \tag{38}
$$

Series expanding around $p^2 = m^2$ we find

$$
\begin{aligned}
p^2 - m_0^2 - M^2(p^2) &= p^2 - m_0^2 - M^2(m^2) - \left.\frac{\mathrm{d}M^2}{\mathrm{d}p^2}\right|_{p^2=m^2}(p^2-m^2) + \dots \\
&= (p^2-m^2)\left(1 - \left.\frac{\mathrm{d}M^2}{\mathrm{d}p^2}\right|_{p^2=m^2}\right) \qquad (\text{near } p^2 = m^2) \; .
\end{aligned}
$$

Then,



$$
= \frac{\mathrm{i}Z}{p^2-m^2} + \text{regular near } p^2 = m^2 \tag{39}
$$

where

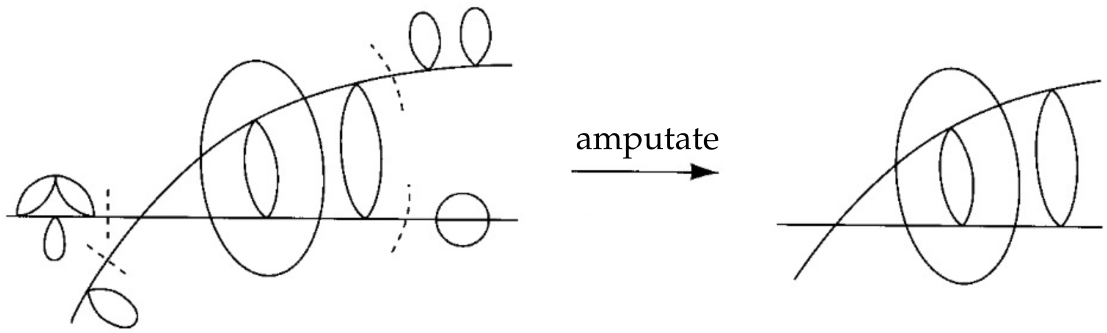
$$
m^2 = m_0^2 + M^2(m^2) \; , \quad Z = \left(1 - \left.\frac{\mathrm{d}M^2}{\mathrm{d}p^2}\right|_{p^2=m^2}\right)^{-1} \; . \tag{40}
$$

This residue Z is the same factor introduced in (2) to account fo the field renormalization due to interactions.

Feynman diagrams | $2 \to 2$ | $(\lambda\phi^4)$

▷ We see, in particular, that the first correction to $Z = 1$ is of order λ^2, as anticipated, because the correction of order λ to the propagator (35) does not depend on p^2.

- In view of what happened to external leg corrections, it is convenient to define amputated diagrams as those obtained by removing every subdiagram associated to external legs that can be separated by cutting a single line. That is, when we eliminate the full external leg propagators. For example,

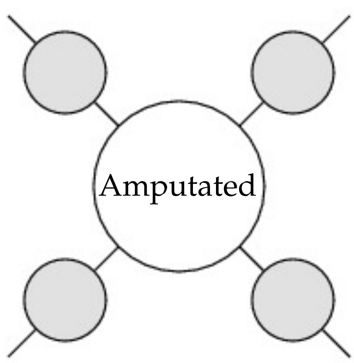


Feynman diagrams | $2 \to 2$ | $(\lambda\phi^4)$

▷ Then the four-point function of interacting fields

$$\int \prod_{i=1}^{2} \mathrm{d}^4 x_i \prod_{j=1}^{2} \mathrm{d}^4 y_j \; \mathrm{e}^{\mathrm{i}(\sum_i p_i x_i - \sum_j k_j y_j)} \langle 0 | \, T\{\phi(x_1)\phi(x_2)\phi(y_1)\phi(y_2)\} \, | 0 \rangle_c$$

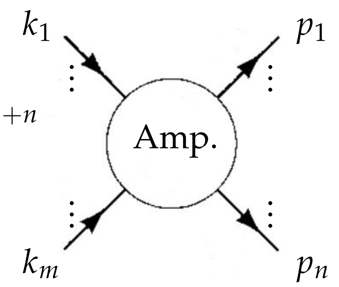
has the following diagrammatic form



Feynman diagrams | $2 \to 2$ $(\lambda\phi^4)$

▷ And in general, using (39) we can rewrite the LSZ formula (9) as

$$\langle \vec{p}_1 \ldots \vec{p}_n |\, \mathrm{i}T \left| \vec{k}_1 \ldots \vec{k}_m \right\rangle = \left(\sqrt{Z}\right)^{m+n} \tag{41}$$



Feynman diagrams | Feynman rules

- We can now list the Feynman rules for real scalar fields in momentum space, assuming interactions of the form $\mathcal{H}_{\text{int}} = \frac{\lambda}{N!}\phi^N$.

  To compute the scattering amplitude for a process of $m \to n$ particles:

  1. Draw every connected and amputated diagrams with m external legs incoming and n external legs outgoing linked in vertices of N legs.
  2. Impose four-momentum conservation in each vertex.
  3. Write a factor $(-\mathrm{i}\lambda)$ for each vertex.
  4. Write a factor $\widetilde{D}_F(p) = \dfrac{\mathrm{i}}{p^2 - m^2 + \mathrm{i}\varepsilon}$ for every internal line of momentum p.
  5. Integrate over every four-momentum q not fixed by external momenta and momentum conservation, one per loop with measure $\dfrac{\mathrm{d}^4 q}{(2\pi)^4}$.
  6. Multiply by the corresponding symmetry factor.

Feynman diagrams | Feynman rules

7. The sum of the contributions of all Feynman diagrams leads to the so called invariant amplitude $\mathrm{i}\mathcal{M}(\vec{k}_1 \cdots \vec{k}_m \to \vec{p}_1 \cdots \vec{p}_n)$ related to the matrix element $S = 1 + \mathrm{i}T$ by

$$\langle \vec{p}_1 \cdots \vec{p}_n | \, \mathrm{i}T \left| \vec{k}_1 \cdots \vec{k}_m \right\rangle = (2\pi)^4 \delta^4 \left(\sum_i p_i - \sum_j k_j \right) \mathrm{i}\mathcal{M} \,, \tag{42}$$

where $\mathrm{i}\mathcal{M}$ include the factors $\left(\sqrt{Z}\right)^{m+n}$, which are irrelevant for calculations at lowest order in perturbation theory, but are important to compute higher order corrections.

On the virtuality of intermediate states

- Consider a diagram with internal lines, as for example (32). The LSZ formula requires incoming and outgoing particles on-shell,

$$k_1^2 = k_2^2 = p_1^2 = p_2^2 = m^2$$

and conservation of the four-momentum at every vertex, so the intermediate particle propagating between two vertices will be off-shell,

$$p_{\text{interm}}^2 = (k_1 + k_2)^2 = k_1^2 + k_2^2 + 2(k_1 k_2) = 2(m^2 + k_1 k_2) \,.$$

▷ Choosing, for example, the center of mass frame for the incoming particles,

$$k_1 = (E, 0, 0, k) \,, \quad k_2 = (E, 0, 0, -k) \Rightarrow k_1 k_2 = E^2 + k^2 = m^2 + 2k^2$$
$$\Rightarrow p_{\text{interm}}^2 = 4(m^2 + k^2) \neq m^2 \,.$$

It is easy to ckeck that p_{interm}^2 of the internal lines of the other two diagrams in (33) is even negative. So in QFT the intermediate states propagating between interaction vertices are virtual particles, i.e. off their mass shell.

Feynman rules for fermions

- Fermionic fields appear in pairs in interaction Hamiltonians, belonging to bilinear covariants.
- Fermionic fields satisfy anticommutation relations, forcing us to define the normal ordering of fermionic operators in a consistent way. As for example

$$: a_{\vec{p},r} a_{\vec{q},s} a^{\dagger}_{\vec{k},t} : = (-1)^2 a^{\dagger}_{\vec{k},t} a_{\vec{p},r} a_{\vec{q},s} = (-1)^3 a^{\dagger}_{\vec{k},t} a_{\vec{q},s} a_{\vec{p},r} \ .$$

  Likewise, we must consistently define the time ordering of fermionic fields,

$$T\{\psi(x_1)\overline{\psi}(x_2)\} = \begin{cases} \psi(x_1)\overline{\psi}(x_2) \ , & x_1^0 > x_2^0 \\ -\overline{\psi}(x_2)\psi(x_1) \ , & x_1^0 < x_2^0 \end{cases}$$

  and similarly for $T\{\psi(x_1)\psi(x_2)\}$ and $T\{\overline{\psi}(x_1)\overline{\psi}(x_2)\}$.
  Thus, for example, if $x_3^0 > x_1^0 > x_4^0 > x_2^0$ then

$$T\{\psi(x_1)\psi(x_2)\psi(x_3)\psi(x_4)\} = -\psi(x_3)\psi(x_1)\psi(x_4)\psi(x_2) \ .$$

Feynman rules for fermions

- We will see now how to define the Wick contraction of two fermionic fields in order to obtain an expression analogous to that of scalar fields,

$$T\{\psi(x)\overline{\psi}(y)\} = :\psi(x)\overline{\psi}(y): + \overbrace{\psi(x)\overline{\psi}}(y) \ .$$

  Separating the components of positive and negative energy,

$$\psi^+(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_{\vec{p}}}} \sum_s a_{\vec{p},s} u^{(s)}(\vec{p}) e^{-ipx}, \ \psi^-(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_{\vec{p}}}} \sum_s b^{\dagger}_{\vec{p},s} v^{(s)}(\vec{p}) e^{ipx}$$

$$\overline{\psi}^+(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_{\vec{p}}}} \sum_s b_{\vec{p},s} \overline{v}^{(s)}(\vec{p}) e^{-ipx}, \ \overline{\psi}^-(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_{\vec{p}}}} \sum_s a^{\dagger}_{\vec{p},s} \overline{u}^{(s)}(\vec{p}) e^{ipx}$$

  we have[a]

[a] If $x^0 = y^0$ then the fields are already time ordered and

$$T\{\psi(x)\overline{\psi}(y)\} = \psi(x)\overline{\psi}(y) =: \psi(x)\overline{\psi}(y) : + \{\psi^+(x), \overline{\psi}^-(y)\} = :\psi(x)\overline{\psi}(y): + \overbrace{\psi(x)\overline{\psi}}(y)$$

because in this case $\{\psi^+(x), \overline{\psi}^-(y)\} = -\{\overline{\psi}^+(x), \psi^-(y)\}$, as can be explicitly checked in (43).

Feynman rules for fermions

$$
\begin{aligned}
T\{\psi(x)\overline{\psi}(y)\} &= \theta(x^0-y^0)\psi(x)\overline{\psi}(y) - \theta(y^0-x^0)\overline{\psi}(y)\psi(x) \\
&= \theta(x^0-y^0)\left[\psi^+(x)\overline{\psi}^+(y) + \psi^+(x)\overline{\psi}^-(y) + \psi^-(x)\overline{\psi}^+(y) + \psi^-(x)\overline{\psi}^-(y)\right] \\
&\quad - \theta(y^0-x^0)\left[\overline{\psi}^+(y)\psi^+(x) + \overline{\psi}^+(y)\psi^-(x) + \overline{\psi}^-(y)\psi^+(x) + \overline{\psi}^-(y)\psi^-(x)\right] \\
&= \theta(x^0-y^0)\Big[:\psi^+(x)\overline{\psi}^+(y): + \{\psi^+(x),\overline{\psi}^-(y)\} + :\psi^+(x)\overline{\psi}^-(y): \\
&\qquad + :\psi^-(x)\overline{\psi}^+(y): + :\psi^-(x)\overline{\psi}^-(y):\Big] \\
&\quad - \theta(y^0-x^0)\Big[:\overline{\psi}^+(y)\psi^+(x): + \{\psi^-(x),\overline{\psi}^+(y)\} + :\overline{\psi}^+(y)\psi^-(x): \\
&\qquad + :\overline{\psi}^-(y)\psi^+(x): + :\overline{\psi}^-(y)\psi^-(x):\Big] \\
&= \theta(x^0-y^0)\left[:\psi(x)\overline{\psi}(y): + \{\psi^+(x),\overline{\psi}^-(y)\}\right] \\
&\quad - \theta(y^0-x^0)\left[:\overline{\psi}(y)\psi(x): + \{\psi^-(x),\overline{\psi}^+(y)\}\right] \\
&= :\psi(x)\overline{\psi}(y): + \wick{\c\psi(x)\c{\overline{\psi}}(y)}
\end{aligned}
$$

Feynman rules for fermions

where we have used that $:\overline{\psi}(y)\psi(x): = -:\psi(x)\overline{\psi}(y):$ and defined

$$
\boxed{\wick{\c\psi(x)\c{\overline{\psi}}(y)} = \theta(x^0-y^0)\{\psi^+(x),\overline{\psi}^-(y)\} - \theta(y^0-x^0)\{\psi^-(x),\overline{\psi}^+(y)\}}
$$

▷ It is easy to check that, as $\{a_{\vec{p},s}, b^\dagger_{\vec{q},r}\} = 0$, the following contractions vanish,

$$
\boxed{\wick{\c\psi(x)\c\psi(y)} = \wick{\c{\overline{\psi}}(x)\c{\overline{\psi}}(y)} = 0}
$$

Feynman rules for fermions

▷ It can be shown that the Wick theorem has the same form for fermionic fields. One has to pay attention because the normal ordering of fermionic Wick contractions can imply a change of sign. For example,

$$
\begin{aligned}
:\overline{\psi}_1\psi_2: \; &= -\psi_2\overline{\psi}_1 \\
:\psi_1\psi_2\overline{\psi}_3\overline{\psi}_4: \; &= -\psi_1\overline{\psi}_3\psi_2\overline{\psi}_4 \,.
\end{aligned}
$$

As the vacuum expectation value of normal ordered operators is zero, we have that, like in the case of scalar fields, the Feynman propagator for fermions is

$$
\langle 0|\, T\{\psi(x)\overline{\psi}(y)\}\, |0\rangle = \psi(x)\overline{\psi}(y) = S_F(x-y)\,,
$$

that can be written, using the completeness relations, as:

Feynman rules for fermions

$$
\begin{aligned}
S_F(x-y) = \;& \theta(x^0-y^0)\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}\sum_s u^{(s)}_{\vec{p}}\overline{u}^{(s)}_{\vec{p}}\,\mathrm{e}^{-\mathrm{i}p(x-y)} \\
& -\theta(y^0-x^0)\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}\sum_s v^{(s)}_{\vec{p}}\overline{v}^{(s)}_{\vec{p}}\,\mathrm{e}^{\mathrm{i}p(x-y)} \\
= \;& \theta(x^0-y^0)\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}(\not{p}+m)\mathrm{e}^{-\mathrm{i}p(x-y)} \\
& -\theta(y^0-x^0)\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}(\not{p}-m)\mathrm{e}^{\mathrm{i}p(x-y)} \\
= \;& \theta(x^0-y^0)(\mathrm{i}\not{\partial}_x+m)\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}\mathrm{e}^{-\mathrm{i}p(x-y)} \\
& -\theta(y^0-x^0)(-\mathrm{i}\not{\partial}_x-m)\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}\mathrm{e}^{\mathrm{i}p(x-y)} \\
= \;& (\mathrm{i}\not{\partial}_x+m)D_F(x-y) \\
= \;& (\mathrm{i}\not{\partial}_x+m)\int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{\mathrm{i}}{p^2-m^2+\mathrm{i}\varepsilon}\mathrm{e}^{-\mathrm{i}p(x-y)}\,.
\end{aligned}
$$

Feynman rules for fermions

▷ Therefore,

$$\boxed{S_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i(\not{p}+m)}{p^2-m^2+i\varepsilon} e^{-ip(x-y)} = \int \frac{d^4p}{(2\pi)^4} \frac{i}{\not{p}-m+i\varepsilon} e^{-ip(x-y)}} \tag{43}$$

and, in momentum space,

$$\widetilde{S}_F(p) = \int d^4x \; S_F(x) e^{ipx} = \frac{i}{\not{p}-m+i\varepsilon}$$

where we have used that $\not{p}\not{p} = p^2$. We see that the fermion propagator is a Green's function of the Dirac operator because

$$(i\not{\partial}_x - m) S_F(x-y) = -(\Box_x + m^2) D_F(x-y) = i\delta^4(x-y) \; .$$

It is very important to note that $\widetilde{S}_F(p) \neq \widetilde{S}_F(-p)$, so one has to pay attention to the sign of the momentum.

Feynman rules for fermions

- To find the S matrix between fermionic states we need to solve for the creation and annihilation operators of the free field ψ_{free},

$$\psi_{\text{free}}(x) = \int \frac{d^3p}{(2\pi)^3 \sqrt{2E_{\vec{p}}}} \sum_s \left[a_{\vec{p},s} u^{(s)}(\vec{p}) e^{-ipx} + b^\dagger_{\vec{p},s} v^{(s)}(\vec{p}) e^{ipx} \right]$$

leading to

$$\begin{aligned}
\sqrt{2E_{\vec{k}}}\; a_{\vec{k},r} &= \int d^3x \; \overline{u}^{(r)}(\vec{k}) e^{ikx} \gamma^0 \psi_{\text{free}}(x) \\
\sqrt{2E_{\vec{k}}}\; b^\dagger_{\vec{k},r} &= \int d^3x \; \overline{v}^{(r)}(\vec{k}) e^{-ikx} \gamma^0 \psi_{\text{free}}(x) \\
\sqrt{2E_{\vec{k}}}\; a^\dagger_{\vec{k},r} &= \int d^3x \; \overline{\psi}_{\text{free}}(x) \gamma^0 e^{-ikx} u^{(r)}(\vec{k}) \\
\sqrt{2E_{\vec{k}}}\; b_{\vec{k},r} &= \int d^3x \; \overline{\psi}_{\text{free}}(x) \gamma^0 e^{ikx} v^{(r)}(\vec{k}) \; .
\end{aligned}$$

Feynman rules for fermions

▷ In fact,

$$
\begin{aligned}
&\int \mathrm{d}^3x\, \overline{u}^{(r)}(\vec{k})\mathrm{e}^{\mathrm{i}kx}\gamma^0\psi_{\text{free}}(x) \\
&\quad = \int \mathrm{d}^3x \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_{\vec{p}}}} \sum_s \left[a_{\vec{p},s}\overline{u}^{(r)}(\vec{k})\gamma^0 u^{(s)}(\vec{p})\mathrm{e}^{\mathrm{i}(k-p)x} + b^\dagger_{\vec{p},s}\overline{u}^{(r)}(\vec{k})\gamma^0 v^{(s)}(\vec{p})\mathrm{e}^{\mathrm{i}(k+p)x}\right] \\
&\quad = \frac{1}{\sqrt{2E_{\vec{k}}}} \sum_s \left[a_{\vec{k},s}\overline{u}^{(r)}(\vec{k})\gamma^0 u^{(s)}(\vec{k}) + b^\dagger_{\vec{k},s}\overline{u}^{(r)}(\vec{k})\gamma^0 v^{(s)}(-\vec{k})\right] = \sqrt{2E_{\vec{k}}}\, a_{\vec{k},r}
\end{aligned}
$$

where we have used that

$$
\overline{u}^{(r)}(\vec{k})\gamma^0 u^{(s)}(\vec{k}) = 2E_{\vec{k}}\delta_{rs}\ , \quad \overline{u}^{(r)}(\vec{k})\gamma^0 v^{(s)}(-\vec{k}) = 0
$$

and likewise for the others.

Feynman rules for fermions

▷ As for scalar fields, we expect that

$$
\psi(x) \xrightarrow[t\to-\infty]{} Z^{1/2}\psi_{\text{in}}(x)\ , \quad \psi(x) \xrightarrow[t\to+\infty]{} Z^{1/2}\psi_{\text{out}}(x)\ ,
$$

where this Z is the renormalization of field ψ. Then,

$$
\begin{aligned}
\sqrt{2E_{\vec{k}}}\, a^{\dagger(\text{in})}_{\vec{k},r} &= \lim_{t\to-\infty} Z^{-1/2} \int \mathrm{d}^3x\, \overline{\psi}(x)\gamma^0\mathrm{e}^{-\mathrm{i}kx}u^{(r)}(\vec{k}) \\
\sqrt{2E_{\vec{k}}}\, a^{\dagger(\text{out})}_{\vec{k},r} &= \lim_{t\to+\infty} Z^{-1/2} \int \mathrm{d}^3x\, \overline{\psi}(x)\gamma^0\mathrm{e}^{-\mathrm{i}kx}u^{(r)}(\vec{k})
\end{aligned}
$$

whose difference yields a covariant expression that we will need as a first step to find the LSZ formula.

Feynman rules for fermions

▷ Using (7),

$$\sqrt{2E_{\vec{k}}}\left(a^{\dagger(\text{in})}_{\vec{k},r} - a^{\dagger(\text{out})}_{\vec{k},r}\right) = Z^{-1/2}\left(\lim_{t\to-\infty} - \lim_{t\to+\infty}\right)\int \mathrm{d}^3x\, \overline{\psi}(x)\gamma^0 \mathrm{e}^{-\mathrm{i}kx} u^{(r)}(\vec{k})$$
$$= -Z^{-1/2}\int \mathrm{d}^4x\, \partial_0\left(\overline{\psi}(x)\gamma^0\mathrm{e}^{-\mathrm{i}kx}\right) u^{(r)}(\vec{k})$$
$$= \text{integrating by parts and using } (\not{k}-m)u(\vec{k}) = 0$$
$$= \mathrm{i}Z^{-1/2}\int \mathrm{d}^4x\, \overline{\psi}(x)(\mathrm{i}\overleftarrow{\not{\partial}}_x + m)\mathrm{e}^{-\mathrm{i}kx}u^{(r)}(\vec{k})\ .$$

▷ Likewise one can get

$$\sqrt{2E_{\vec{k}}}\left(b^{\dagger(\text{in})}_{\vec{k},r} - b^{\dagger(\text{out})}_{\vec{k},r}\right) = \mathrm{i}Z^{-1/2}\int \mathrm{d}^4x\, \overline{v}^{(r)}(\vec{k})\mathrm{e}^{-\mathrm{i}kx}(\mathrm{i}\not{\partial}_x - m)\psi(x)\ ,$$
$$\sqrt{2E_{\vec{k}}}\left(a^{(\text{in})}_{\vec{k},r} - a^{(\text{out})}_{\vec{k},r}\right) = -\mathrm{i}Z^{-1/2}\int \mathrm{d}^4x\, \overline{u}^{(r)}(\vec{k})\mathrm{e}^{\mathrm{i}kx}(\mathrm{i}\not{\partial}_x - m)\psi(x)\ ,$$
$$\sqrt{2E_{\vec{k}}}\left(b^{(\text{in})}_{\vec{k},r} - b^{(\text{out})}_{\vec{k},r}\right) = -\mathrm{i}Z^{-1/2}\int \mathrm{d}^4x\, \overline{\psi}(x)(\mathrm{i}\overleftarrow{\not{\partial}}_x + m)\mathrm{e}^{\mathrm{i}kx}v^{(r)}(\vec{k})\ .$$

[the calculation]

Use integration by parts:

$$\int \mathrm{d}^4x\nabla\cdot(\overline{\psi}\vec{\gamma}\mathrm{e}^{-\mathrm{i}kx})u(\vec{k}) = 0 \Rightarrow \int \mathrm{d}^4x\nabla\overline{\psi}\cdot\vec{\gamma}\mathrm{e}^{-\mathrm{i}kx}u(\vec{k}) = -\mathrm{i}\int \mathrm{d}^4x\overline{\psi}\vec{\gamma}\cdot\vec{k}\mathrm{e}^{-\mathrm{i}kx}u(\vec{k})$$

and the Dirac equation:

$$(\not{k}-m)u(\vec{k}) = (\gamma^0k^0 - \vec{\gamma}\cdot\vec{k} - m)u(\vec{k}) = 0$$

As a consequence:

$$-\int \mathrm{d}^4x\partial_0(\overline{\psi}\gamma^0\mathrm{e}^{-\mathrm{i}kx})u(\vec{k}) = -\int \mathrm{d}^4x(\partial_0\overline{\psi})\gamma^0\mathrm{e}^{-\mathrm{i}kx}u(\vec{k}) - \int \mathrm{d}^4x\overline{\psi}\gamma^0(-\mathrm{i}k^0)\mathrm{e}^{-\mathrm{i}kx}u(\vec{k})$$
$$= -\int \mathrm{d}^4x(\partial_0\overline{\psi})\gamma^0\mathrm{e}^{-\mathrm{i}kx}u(\vec{k}) - \int \mathrm{d}^4x(\nabla\overline{\psi}\cdot\vec{\gamma} - \mathrm{i}\overline{\psi}m)\mathrm{e}^{-\mathrm{i}kx}u(\vec{k})$$
$$= \mathrm{i}\int \mathrm{d}^4x(\mathrm{i}\partial_\mu\overline{\psi}\gamma^\mu + \overline{\psi}m)\mathrm{e}^{-\mathrm{i}kx}u(\vec{k})$$
$$= \mathrm{i}\int \mathrm{d}^4x\overline{\psi}(\mathrm{i}\overleftarrow{\not{\partial}}_x + m)\mathrm{e}^{-\mathrm{i}kx}u(\vec{k})$$

Feynman rules for fermions

▷ Then we can substitute in the S matrix an incoming fermion of momentum $\vec{k}_1$ and spin r by

$$\left\langle \vec{p}_1 \cdots \vec{p}_n; t_f \middle| \vec{k}_1 \vec{k}_2 \cdots \vec{k}_m; t_i \right\rangle = \sqrt{2E_{\vec{k}_1}} \langle \vec{p}_1 \cdots \vec{p}_n; t_f | \left(a^{\dagger(\text{in})}_{\vec{k}_1,r} - a^{\dagger(\text{out})}_{\vec{k}_1,r} \right) \left| \vec{k}_2 \cdots \vec{k}_m; t_i \right\rangle$$
$$= \mathrm{i} Z^{-1/2} \int \mathrm{d}^4 x_1 \, \mathrm{e}^{-\mathrm{i} k_1 x_1} \langle \vec{p}_1 \cdots \vec{p}_n; t_f | \overline{\psi}(x_1) \left| \vec{k}_2 \cdots \vec{k}_m \right\rangle (\mathrm{i} \overleftarrow{\not{\partial}}_{x_1} + m) u^{(r)}(\vec{k}_1) \tag{44}$$

and next substitute, for example, an incoming antifermion of momentum $\vec{k}_2$ and spin s by

$$\langle \vec{p}_1 \cdots \vec{p}_n; t_f | \overline{\psi}(x_1) \left| \vec{k}_2 \cdots \vec{k}_m; t_i \right\rangle$$
$$= \sqrt{2E_{\vec{k}_2}} \langle \vec{p}_1 \cdots \vec{p}_n; t_f | \, T\{ \overline{\psi}(x_1) (b^{\dagger(\text{in})}_{\vec{k}_2,s} - b^{\dagger(\text{out})}_{\vec{k}_2,s}) \} \left| \vec{k}_3 \cdots \vec{k}_m; t_i \right\rangle$$
$$= \mathrm{i} Z^{-1/2} \int \mathrm{d}^4 x_2 \, \mathrm{e}^{-\mathrm{i} k_2 x_2} \, \overline{v}^{(s)}(\vec{k}_2) (\mathrm{i} \not{\partial}_{x_2} - m) \langle \vec{p}_1 \cdots \vec{p}_n; t_f | \, T\{ \overline{\psi}(x_1) \psi(x_2) \} \left| \vec{k}_3 \cdots \vec{k}_m; t_i \right\rangle \tag{45}$$

Feynman rules for fermions

▷ Iterating the procedure one can find the LSZ formula for fermions. Suppose that among the m (n) incoming (outgoing) particles there are m_f (n_f) fermions of spins r_i (r'_j) and the rest are antifermions of spins s_i (s'_j). Then:

$$\boxed{\begin{aligned}
&\left(\prod_{i=1}^{m_f} \frac{\mathrm{i}\sqrt{Z}}{\not{k}_i - m} \right) \left(\prod_{i=m_f+1}^{m} \frac{\mathrm{i}\sqrt{Z}}{\not{k}_i + m} \right) \left(\prod_{j=1}^{n_f} \frac{\mathrm{i}\sqrt{Z}}{\not{p}_j - m} \right) \left(\prod_{j=n_f+1}^{n} \frac{\mathrm{i}\sqrt{Z}}{\not{p}_j + m} \right) \langle \vec{p}_1 \vec{p}_2 \cdots \vec{p}_n | \, \mathrm{i} T \left| \vec{k}_1 \vec{k}_2 \cdots \vec{k}_m \right\rangle \\
&= \int \left(\prod_{i=1}^{m} \mathrm{d}^4 x_i \, \mathrm{e}^{-\mathrm{i} k_i x_i} \right) \int \left(\prod_{j=1}^{n} \mathrm{d}^4 y_j \, \mathrm{e}^{+\mathrm{i} p_j y_j} \right) \prod_{i=m_f+1}^{m} \overline{v}^{(s_i)}(k_i) \prod_{j=1}^{n_f} \overline{u}^{(s'_j)}(p_j) \\
&\quad \times \langle 0 | \, T \left\{ \prod_{i=m_f+1}^{m} \psi(x_i) \prod_{i=1}^{m_f} \overline{\psi}(x_i) \prod_{j=1}^{n_f} \psi(y_j) \prod_{j=n_f+1}^{n} \overline{\psi}(y_j) \right\} |0\rangle \prod_{i=1}^{m_f} u^{(r_i)}(k_i) \prod_{j=n_f+1}^{n} v^{(r'_j)}(p_j)
\end{aligned}}$$

Feynman rules for fermions (Yukawa theory)

- Let us focus on a simple particular case, direct application of the results we have found in (44) and (45). We will study the $2 \to 2$ scattering

$$\text{fermion}(k_1, r) + \text{antifermion}(k_2, s) \to \text{scalar}(p_1) + \text{scalar}(p_2)$$

in the Yukawa theory, whose Lagrangian is

$$\mathcal{L} = \mathcal{L}_{\text{Dirac}} + \mathcal{L}_{\text{KG}} - g\overline{\psi}\psi\phi \Rightarrow \mathcal{H}_{\text{int}} = g\overline{\psi}\psi\phi\ .$$

To alleviate the notation, suppose that all fields have the same mass m.

Feynman rules for fermions (Yukawa theory)

▷ The scattering amplitude is given by

$$\begin{aligned}
&\langle \vec{p}_1\vec{p}_2|\, \mathrm{i}T\, |k_1k_2\rangle \\
&= \left(\mathrm{i}Z_\phi^{-1/2}\right)^2 \left(\mathrm{i}Z_\psi^{-1/2}\right)^2 \int \mathrm{d}^4x_1\mathrm{d}^4x_2\mathrm{d}^4x_3\mathrm{d}^4x_4\ \mathrm{e}^{\mathrm{i}(p_1x_1+p_2x_2-k_1x_3-k_2x_4)}(\Box_{x_1}+m^2)(\Box_{x_2}+m^2) \\
&\quad \times \overline{v}^{(s)}(k_2)(\mathrm{i}\partial\!\!\!/_{x_4}-m)\,\langle 0|\, T\{\phi(x_1)\phi(x_2)\overline{\psi}(x_3)\psi(x_4)\}\,|0\rangle\,(\mathrm{i}\,\overleftarrow{\partial\!\!\!/}_{x_3}+m)u^{(r)}(k_1) \\
&= \left(-\mathrm{i}Z_\phi^{-1/2}\right)^2 \left(\mathrm{i}Z_\psi^{-1/2}\right)^2 (p_1^2-m^2)(p_2^2-m^2)\int \mathrm{d}^4x_1\mathrm{d}^4x_2\mathrm{d}^4x_3\mathrm{d}^4x_4\ \mathrm{e}^{\mathrm{i}(p_1x_1+p_2x_2-k_1x_3-k_2x_4)} \\
&\quad \times \overline{v}^{(s)}(k_2)(-k\!\!\!/_2-m)\,\langle 0|\, T\{\phi(x_1)\phi(x_2)\overline{\psi}(x_3)\psi(x_4)\}\,|0\rangle\,(-k\!\!\!/_1+m)u^{(r)}(k_1)
\end{aligned}$$

where we have used the Fourier transform of the Green's function, as we did to obtain (9).

These fields are not free.

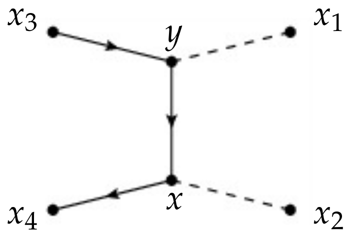
Feynman rules for fermions (Yukawa theory)

- ▷ To write the fields in the interaction picture we replace $\langle 0|\, T\{\phi(x_1)\phi(x_2)\overline{\psi}(x_3)\psi(x_4)\}\, |0\rangle$ by

$$\langle 0|\, T\left\{\phi(x_1)\phi(x_2)\overline{\psi}(x_3)\psi(x_4)\exp\left[-\mathrm{i}\int \mathrm{d}^4x\, \mathcal{H}_I(x)\right]\right\}|0\rangle$$

Then apply the Wick th and focus on connected contributions (arising at $\mathcal{O}(g^2)$)

(A)



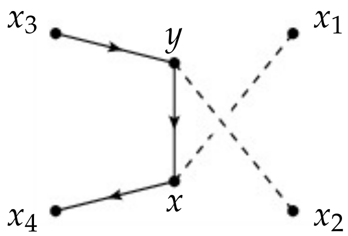
$+ \qquad (x \leftrightarrow y)$

$$:\phi_1\phi_2\overline{\psi}_3\psi_4\;\overline{\psi}_x\psi_x\phi_x\;\overline{\psi}_y\psi_y\phi_y:\; = -\phi_1\phi_y\;\phi_2\phi_x\;\psi_4\overline{\psi}_x\;\psi_x\overline{\psi}_y\psi_y\overline{\psi}_3$$

$$:\phi_1\phi_2\overline{\psi}_3\psi_4\;\overline{\psi}_x\psi_x\phi_x\;\overline{\psi}_y\psi_y\phi_y:\; = -\phi_1\phi_x\;\phi_2\phi_y\;\psi_4\overline{\psi}_y\;\psi_y\overline{\psi}_x\psi_x\overline{\psi}_3$$

Feynman rules for fermions (Yukawa theory)

(B)



$+ \qquad (x \leftrightarrow y)$

$$:\phi_1\phi_2\overline{\psi}_3\psi_4\;\overline{\psi}_x\psi_x\phi_x\;\overline{\psi}_y\psi_y\phi_y:\; = -\phi_1\phi_x\;\phi_2\phi_y\;\psi_4\overline{\psi}_x\;\psi_x\overline{\psi}_y\psi_y\overline{\psi}_3$$

$$:\phi_1\phi_2\overline{\psi}_3\psi_4\;\overline{\psi}_x\psi_x\phi_x\;\overline{\psi}_y\psi_y\phi_y:\; = -\phi_1\phi_y\;\phi_2\phi_x\;\psi_4\overline{\psi}_y\;\psi_y\overline{\psi}_x\psi_x\overline{\psi}_3$$

- ▷ Consider diagram (A), whose contribution gets multiplied by 2 if we include the diagram resulting of exchanging x and y.

  Note that there is a change of sign after normal ordering of the contractions:

Feynman rules for fermions (Yukawa theory)

$$\begin{aligned}
&\langle \vec p_1 \vec p_2 |\, \mathrm{i}T\, | k_1 k_2\rangle_A \\
&= 2\times\left(-\mathrm{i}Z_\phi^{-1/2}\right)^2\left(-\mathrm{i}Z_\psi^{-1/2}\right)^2(p_1^2-m^2)(p_2^2-m^2)\int\prod_{i=1}^{4}\mathrm{d}^4x_i\mathrm{d}^4x\mathrm{d}^4y\ \mathrm{e}^{\mathrm{i}(p_1x_1+p_2x_2-k_1x_3-k_2x_4)}\\
&\quad\times(-\mathrm{i}g)^2\frac{1}{2!}\times(-1)\times D_F(x_1-y)D_F(x_2-x)\\
&\quad\times\overline{v}^{(s)}(\vec k_2)(\not k_2+m)S_F(x_4-x)S_F(x-y)S_F(y-x_3)(\not k_1-m)u^{(r)}(\vec k_1)\\
&= -(-\mathrm{i}g)^2(p_1^2-m^2)(p_2^2-m^2)\int\prod_{i=1}^{4}\mathrm{d}^4\tilde x_i\mathrm{d}^4x\mathrm{d}^4y\ \mathrm{e}^{\mathrm{i}(p_1\tilde x_1+p_2\tilde x_2+k_1\tilde x_3-k_2\tilde x_4)}\mathrm{e}^{\mathrm{i}(p_2-k_2)x+\mathrm{i}(p_1-k_1)y}\\
&\quad\times D_F(\tilde x_1)D_F(\tilde x_2)\overline{v}^{(s)}(\vec k_2)(\not k_2+m)S_F(\tilde x_4)S_F(x-y)S_F(\tilde x_3)(\not k_1-m)u^{(r)}(\vec k_1)\\
&= -(-\mathrm{i}g)^2\mathrm{i}^2\int\mathrm{d}^4x\mathrm{d}^4y\ \mathrm{e}^{\mathrm{i}(p_2-k_2)x+\mathrm{i}(p_1-k_1)y}\\
&\quad\times\overline{v}^{(s)}(\vec k_2)(\not k_2+m)\frac{\mathrm{i}}{-\not k_2-m}S_F(x-y)\frac{\mathrm{i}}{\not k_1-m}(\not k_1-m)u^{(r)}(\vec k_1)\\
&= (-\mathrm{i}g)^2\int\mathrm{d}^4\tilde x\mathrm{d}^4y\ \mathrm{e}^{\mathrm{i}(p_2-k_2)\tilde x}\mathrm{e}^{\mathrm{i}(p_1+p_2-k_1-k_2)y}\overline{v}^{(s)}(\vec k_2)S_F(\tilde x)u^{(r)}(\vec k_1)\\
&= (-\mathrm{i}g)^2(2\pi)^4\delta^4(p_1+p_2-k_1-k_2)\overline{v}^{(s)}(\vec k_2)\frac{\mathrm{i}}{\not p_2-\not k_2-m}u^{(r)}(\vec k_1)
\end{aligned}$$

Feynman rules for fermions (Yukawa theory)

where we have changed $\tilde x_1 = x_1 - y$, $\tilde x_2 = x_2 - x$, $\tilde x_3 = y - x_3$, $\tilde x_4 = x_4 - x$, later $\tilde x = x - y$, and we have taken every $Z = 1$ because their contributions are of higher order in g.

Feynman rules for fermions (Yukawa theory)

▷ The diagram (B), including $x \leftrightarrow y$:

$$\langle \vec{p}_1 \vec{p}_2 |\, \mathrm{i}T \,| k_1 k_2 \rangle_B = -(-\mathrm{i}g)^2 (p_1^2 - m^2)(p_2^2 - m^2) \int \prod_{i=1}^{4} \mathrm{d}^4 x_i \mathrm{d}^4 x \mathrm{d}^4 y \; \mathrm{e}^{\mathrm{i}(p_1 x_1 + p_2 x_2 - k_1 x_3 - k_2 x_4)}$$

$$\times D_F(x_1 - x) D_F(x_2 - y) \overline{v}^{(s)}(\vec{k}_2)(\not{k}_2 + m) S_F(x_4 - x) S_F(x - y) S_F(y - x_3)(\not{k}_1 - m) u^{(r)}(\vec{k}_1)$$

$$= -(-\mathrm{i}g)^2 (p_1^2 - m^2)(p_2^2 - m^2) \int \prod_{i=1}^{4} \mathrm{d}^4 \tilde{x}_i \mathrm{d}^4 x \mathrm{d}^4 y \; \mathrm{e}^{\mathrm{i}(p_1 \tilde{x}_1 + p_2 \tilde{x}_2 + k_1 \tilde{x}_3 - k_2 \tilde{x}_4)} \mathrm{e}^{\mathrm{i}(p_1 - k_2)x + \mathrm{i}(p_2 - k_1) y}$$

$$\times D_F(\tilde{x}_1) D_F(\tilde{x}_2) \overline{v}^{(s)}(\vec{k}_2)(\not{k}_2 + m) S_F(\tilde{x}_4) S_F(x - y) S_F(\tilde{x}_3)(\not{k}_1 - m) u^{(r)}(\vec{k}_1)$$

$$= -(-\mathrm{i}g)^2 \mathrm{i}^2 \int \mathrm{d}^4 x \mathrm{d}^4 y \; \mathrm{e}^{\mathrm{i}(p_1 - k_2)x + \mathrm{i}(p_2 - k_1) y}$$

$$\times \overline{v}^{(s)}(\vec{k}_2)(\not{k}_2 + m) \frac{\mathrm{i}}{-\not{k}_2 - m} S_F(x - y) \frac{\mathrm{i}}{\not{k}_1 - m} (\not{k}_1 - m) u^{(r)}(\vec{k}_1)$$

$$= (-\mathrm{i}g)^2 \int \mathrm{d}^4 \tilde{x} \mathrm{d}^4 y \; \mathrm{e}^{\mathrm{i}(p_1 - k_2)\tilde{x}} \mathrm{e}^{\mathrm{i}(p_1 + p_2 - k_1 - k_2) y} \overline{v}^{(s)}(\vec{k}_2) S_F(\tilde{x}) u^{(r)}(\vec{k}_1)$$

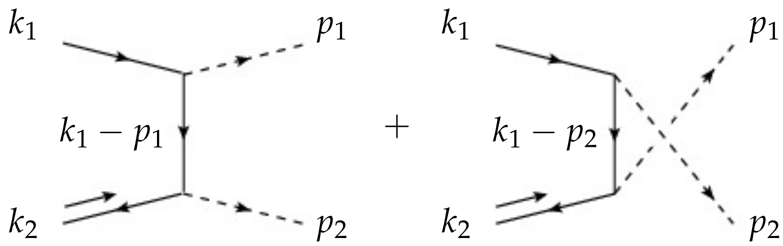
$$= (-\mathrm{i}g)^2 (2\pi)^4 \delta^4(p_1 + p_2 - k_1 - k_2) \overline{v}^{(s)}(\vec{k}_2) \frac{\mathrm{i}}{\not{p}_1 - \not{k}_2 - m} u^{(r)}(\vec{k}_1)$$

Feynman rules for fermions (Yukawa theory)

where we have changed $\tilde{x}_1 = x_1 - x$, $\tilde{x}_2 = x_2 - y$, $\tilde{x}_3 = y - x_3$, $\tilde{x}_4 = x_4 - x$, and later $\tilde{x} = x - y$.

Feynman rules for fermions (Yukawa theory)

▷ We can express both diagrams in momentum space:



▷ The momentum direction in a fermionic line is relevant. It is taken incoming for initial states and outgoing for final states. This direction coincides with that of the fermion number flux for internal lines and for external particle states (for the electron one takes the negative charge flux), but it is opposite to the fermion flux for external antiparticle states.

▷ The fermion lines are represented by solid lines and the fermion flux by an arrow inserted in the line.

Feynman rules for fermions (Yukawa theory)

▷ From now on we will reserve the dashed lines for scalar bosons.

▷ To write the Feynman rules, follow every fermion line in opposite direction to the fermion flux, assigning spinors, vertices and propagators in the order that they appear.

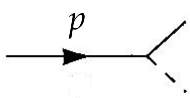
▷ Let us summarize the Feynman rules of the Yukawa theory that follow from previous calculation:

1. The invariant amplitude i$\mathcal{M}$ is defined, as before, from the S matrix element extracting the factor $(2\pi)^4\delta^4(\sum_i p_i - \sum_j k_j)$. To write it, skip external propagators of scalars and fermions, as they cancel in the LSZ formula. Draw all connected amputated diagrams. You may ignore the $\sqrt{Z}$ factors to lowest order in PT, but they must be included in higher order corrections.

Feynman rules for fermions (Yukawa theory)

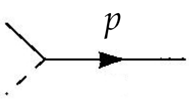
2. Assign spinors to external fermion legs as follows (suppose that time flows from left to right):

incoming fermion:



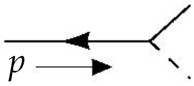
$= u^{(s)}(\vec{p})$

outgoing fermion:



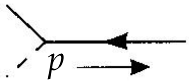
$= \overline{u}^{(s)}(\vec{p})$

incoming antifermion:



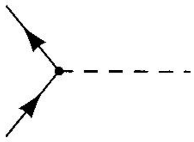
$= \overline{v}^{(s)}(\vec{p})$

outgoing antifermion



$= v^{(s)}(\vec{p})$

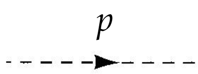
3. Vertex (impose conservation of four-momentum at every vertex):



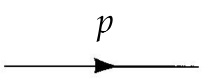
$= -\mathrm{i}g$

Feynman rules for fermions (Yukawa theory)

4. Propagators:



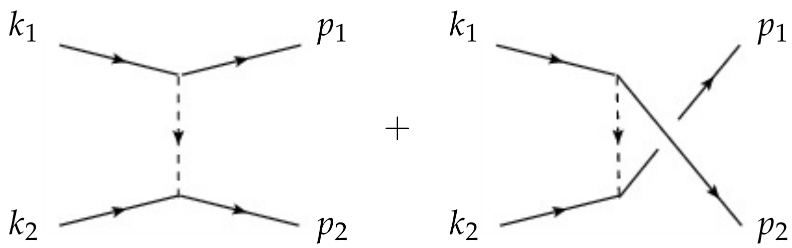
$$= \frac{\mathrm{i}}{p^2 - m^2 + \mathrm{i}\varepsilon}$$



$$= \frac{\mathrm{i}(\not{p} + m)}{p^2 - m^2 + \mathrm{i}\varepsilon}$$

Feynman rules for fermions (Yukawa theory)

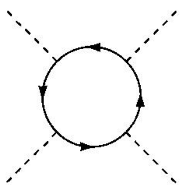
5. Pay attention to the relative signs of diagrams involving several fermion lines, coming from Wick contractions. For example, the following diagrams contribute with opposite signs to the amplitude because they differ in an odd permutation of fermionic fields:



$$\mathrm{i}\mathcal{M} = (-\mathrm{i}g)^2 \left[\overline{u}(p_2)u(k_2)\frac{\mathrm{i}}{(k_1-p_1)^2-m^2}\overline{u}(p_1)u(k_1) - \overline{u}(p_2)u(k_1)\frac{\mathrm{i}}{(k_1-p_2)^2-m^2}\overline{u}(p_1)u(k_2) \right]$$

Feynman rules for fermions (Yukawa theory)

6. Assign a factor (-1) to every closed fermion loop, because



$$= \overline{\psi}\psi\overline{\psi}\psi\overline{\psi}\psi\overline{\psi}\psi = -\mathrm{Tr}\left[\psi\overline{\psi}\ \psi\overline{\psi}\ \psi\overline{\psi}\ \psi\overline{\psi}\right] = -\mathrm{Tr}(\widetilde{S}_F\widetilde{S}_F\widetilde{S}_F\widetilde{S}_F)$$

5. Observables in field and particle theory

Observables — Relativistic and non-relativistic state normalization

- In non-relativistic quantum mechanics the wave function of a particle of momentum $\vec{p}$ freely moving in a box of volume $V = L^3$ is

$$\psi_{\vec{p}}(\vec{x}) = \langle \vec{x} | \vec{p} \rangle = C\, \mathrm{e}^{\mathrm{i}\vec{p}\cdot\vec{x}} \quad \text{con} \quad \int_V \mathrm{d}^3x\, |\psi_{\vec{p}}(\vec{x})|^2 = 1 \Rightarrow C = \frac{1}{\sqrt{V}}$$

and the possible momenta p^i are quantized, $p_i = (2\pi/L)n_i$ with $n_i = 0, \pm 1, \pm 2, \ldots$. Then, in momentum space,

$$\langle \vec{p} | \vec{q} \rangle^{(\mathrm{NR})} = \int \mathrm{d}^3x\; \langle \vec{p} | x \rangle \langle x | \vec{q} \rangle = \int \mathrm{d}^3x\; \psi^*_{\vec{p}}(\vec{x}) \psi_{\vec{q}}(x) = \delta_{\vec{p}\vec{q}}\, .$$

Observables — Relativistic and non-relativistic state normalization

- In QFT, which is a quantum relativistic theory, previous normalization is not Lorentz invariant. That is why we had introduced the normalization

$$\langle \vec{p} | \vec{q} \rangle = 2E_{\vec{p}} (2\pi)^3 \delta^3(\vec{p} - \vec{q})$$

which is the limit when the volume approaches infinity of

$$\langle \vec{p} | \vec{q} \rangle = 2E_{\vec{p}} V \delta_{\vec{p}\vec{q}}$$

because remember that

$$\lim_{\vec{p}\to\vec{q}} (2\pi)^3 \delta^3(\vec{p} - \vec{q}) = \lim_{\vec{p}\to\vec{q}} \int \mathrm{d}x^3\ \mathrm{e}^{-\mathrm{i}(\vec{p}-\vec{q})\cdot\vec{x}} = V(\to \infty) \ .$$

Comparing both normalizations, we see that

$$|\vec{p}\rangle = (2E_{\vec{p}} V)^{1/2} |\vec{p}\rangle^{(\mathrm{NR})}$$

$$|\vec{p}_1 \vec{p}_2 \cdots \vec{p}_n\rangle = \prod_{i=1}^{n} (2E_{\vec{p}_i} V)^{1/2} |\vec{p}_1 \vec{p}_2 \cdots \vec{p}_n\rangle^{(\mathrm{NR})} \ .$$

Observables — Relativistic and non-relativistic state normalization

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- In non-relativistic QM the $S = 1 + \mathrm{i}T$ matrix element between $|i\rangle$ and $|f\rangle$ is

$$S_{fi} = \delta_{fi} + (2\pi)^4 \delta^4(P_f - P_i)\ \mathrm{i}\mathcal{T}_{fi}$$

where for convenience we have factored out $(2\pi)^4\delta^4(P_f - P_i)$ that expresses the energy and momentum conservation.

- Likewise, in QFT we have introduced the invariant matrix element $\mathcal{M}$ between two relativistic states $|i\rangle = \left|\vec{k}_1 \vec{k}_2 \cdots \vec{k}_m\right\rangle$ and $|f\rangle = |\vec{p}_1 \vec{p}_2 \cdots \vec{p}_n\rangle$, so then

$$\mathcal{M}_{fi} = \prod_{\ell=1}^{m} (2E_{\vec{k}_\ell} V)^{1/2} \prod_{j=1}^{n} (2E_{\vec{p}_j} V)^{1/2} \mathcal{T}_{fi} \ . \tag{1}$$

Observables | Decay width

- Take as initial state $|i\rangle = |\vec{k}\rangle$, a particle of mass M and momentum $\vec{k}$, and as final state $|f\rangle = |\vec{p}_1\vec{p}_2\cdots\vec{p}_n\rangle$, n particles of masses m_j and momenta $\vec{p}_j$.
- The probability that the initial particle decays to n particles ($1\to n$), $|f\rangle \neq |i\rangle$, is

$$\mathrm{d}w = |(2\pi)^4\delta^4(P_f-P_i)\mathrm{i}\mathcal{T}_{fi}|^2\mathrm{d}N_f = (2\pi)^4\delta^4(P_f-P_i)VT|\mathcal{T}_{fi}|^2\mathrm{d}N_f$$

where we have symbolically substituted $(2\pi)^4\delta^4(0) = VT$ from

$$\lim_{p\to q}(2\pi)^4\delta^4(p-q) = \lim_{p\to q}\int \mathrm{d}^4x\ \mathrm{e}^{\mathrm{i}(p-q)x} = VT(\to\infty)$$

and $\mathrm{d}N_f$ is the number of n particle states with momenta between $\vec{p}_i$ and $\vec{p}_i+\mathrm{d}\vec{p}_i$.

Observables | Decay width

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▷ Let us find first the number of one-particle states of momentum between $\vec{p}$ and $\vec{p}+\mathrm{d}\vec{p}$. For that we need the completeness relation

$$1 = \int \frac{\mathrm{d}^3p}{(2\pi)^3 2E_{\vec{p}}}\,|\vec{p}\rangle\langle\vec{p}|$$

that is easy to check, because

$$|\vec{q}\rangle = \int \frac{\mathrm{d}^3p}{(2\pi)^3 2E_{\vec{p}}}\,|\vec{p}\rangle\langle\vec{p}|\vec{q}\rangle = \int \frac{\mathrm{d}^3p}{(2\pi)^3 2E_{\vec{p}}}\,|\vec{p}\rangle\,2E_{\vec{p}}(2\pi)^3\delta^3(\vec{p}-\vec{q}) = |\vec{q}\rangle$$

which allows us to write $\mathrm{d}N$ as the product of the probability that a particle has a momentum between $\vec{p}$ and $\vec{p}+\mathrm{d}\vec{p}$, namely $\langle\vec{p}|\vec{p}\rangle = 2E_{\vec{p}}V$, and the density of states in that interval,

$$\mathrm{d}N = 2E_{\vec{p}}V\frac{\mathrm{d}^3p}{(2\pi)^3 2E_{\vec{p}}} = \frac{V\mathrm{d}^3p}{(2\pi)^3}\ .$$

▷ Therefore, in the case of n particles in the final state, we have

$$\mathrm{d}N_f = \prod_{j=1}^{n} \frac{V\mathrm{d}^3 p_j}{(2\pi)^3} \,.$$

▷ Thus the decay probability per unit of time, that we call decay width, is given by

$$\mathrm{d}\Gamma = \frac{\mathrm{d}w}{T} = (2\pi)^4 \delta^4(P_f - P_i) V |\mathcal{T}_{fi}|^2 \prod_{j=1}^{n} \frac{V\mathrm{d}^3 p_j}{(2\pi)^3} \tag{2}$$

which, using (1), can be written as

$$\boxed{\mathrm{d}\Gamma = \frac{1}{2E_{\vec{k}}} |\mathcal{M}_{fi}|^2 \mathrm{d}\Phi_n}$$

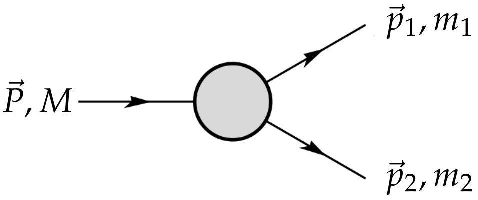
where we have introduced the volume element in the n-body phase space,

$$\boxed{\mathrm{d}\Phi_n = (2\pi)^4 \delta^4 \left(\sum_{j=1}^{n} p_j - P_i \right) \prod_{j=1}^{n} \frac{\mathrm{d}^3 p_j}{(2\pi)^3 2E_j}}$$

- The decay width has the dimensions of energy or inverse of time in our natural system of units.
- If we work in the decaying particle rest frame the energy is the particle mass, $E_{\vec{k}} = M$.
- The total width is the sum of the partial widths of every decay channel.
- The inverse of the total width is the particle lifetime,

$$\boxed{\tau = \Gamma^{-1}}$$

- Consider now the decay $1 \to 2$:



The Lorentz invariant phase space (LIPS) integration for $n = 2$ final particles in the center of mass (CM) frame gets reduced to an integral over the solid angle of one of them (the other has opposite direction):

$$\begin{aligned}\int \mathrm{d}\Phi_2 &= (2\pi)^4 \int \delta^4(p_1 + p_2 - P) \frac{\mathrm{d}^3 p_1}{(2\pi)^3 2E_1} \frac{\mathrm{d}^3 p_2}{(2\pi)^3 2E_2} \\ &= \int \delta(E_1 + E_2 - E_{\mathrm{CM}}) \frac{\mathrm{d}^3 p_1}{(2\pi)^2 2E_1 2E_2} \\ &= \int \frac{|\vec{p}|^2 \mathrm{d}\Omega}{(2\pi)^2 4E_1E_2} \frac{E_1E_2}{|\vec{p}|(E_1 + E_2)} = \int \frac{|\vec{p}|\mathrm{d}\Omega}{16\pi^2 E_{\mathrm{CM}}} ,\end{aligned} \tag{3}$$

Observables | Decay width | $1 \to 2$

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where we have used

$$\mathrm{d}^3 p_1 = |\vec{p}_1|^2 \mathrm{d}|\vec{p}_1| \mathrm{d}\Omega$$

$$\delta(E_1 + E_2 - E_{\mathrm{CM}}) = \delta(f(|\vec{p}_1|)) = \frac{\delta(|\vec{p}_1| - |\vec{p}|)}{|f'(|\vec{p}_1| = |\vec{p}|)|}$$

$$E_1 = \sqrt{m_1^2 + |\vec{p}_1|^2} , \quad E_2 = \sqrt{m_2^2 + |-\vec{p}_1|^2}$$

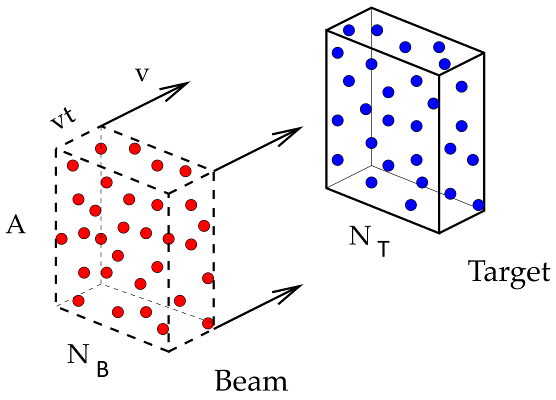
$$f'(|\vec{p}_1|) = \frac{\partial f}{\partial E_1} \frac{\partial E_1}{\partial |\vec{p}_1|} + \frac{\partial f}{\partial E_2} \frac{\partial E_2}{\partial |\vec{p}_1|} = \frac{|\vec{p}_1|}{E_1} + \frac{|\vec{p}_1|}{E_2} = |\vec{p}_1| \left(\frac{E_1 + E_2}{E_1 E_2} \right) .$$

▷ Note that $E_{\mathrm{CM}} = M$ and, for this particular $1 \to 2$ case, the masses M, m_1 and m_2 fully determine the final energies and momenta:

$$E_1 = \frac{M^2 - m_2^2 + m_1^2}{2M} , \quad E_2 = \frac{M^2 - m_1^2 + m_2^2}{2M} ,$$

$$|\vec{p}| \equiv |\vec{p}_1| = |\vec{p}_2| = \frac{\{[M^2 - (m_1 + m_2)^2][M^2 - (m_1 - m_2)^2]\}^{1/2}}{2M} .$$

Observables **Cross section**



- The cross section σ is the effective area of a particle (target) as seen by a projectile (in the incoming beam).

▷ Suppose that in the target there are N_T particles and the collision area is A. Then the collision probability is

$$\mathrm{P} = \frac{N_T \sigma}{A} \ .$$

Observables **Cross section**

▷ It ther are N_B particles in the beam, then the number of events is $N_B \mathrm{P}$,

$$(\text{\# events}) = N_B \frac{N_T \sigma}{A} \Rightarrow \sigma = \frac{(\text{\# events})}{N_B N_T} A \ .$$

▷ The beam is made of a cloud of particles of density ρ moving at speed v, so

$$\begin{aligned} N_B = \rho v t A \Rightarrow \sigma &= \frac{(\text{\# events})}{\rho v t A N_T} A = \frac{(\text{\# events})}{\rho v \ t N_T} \\ &= \frac{\text{transition probability per unit of time}}{\text{incoming flux}} \\ \Rightarrow \mathrm{d}\sigma &= \frac{(2\pi)^4 \delta^4(P_f - P_i) V |\mathcal{T}_{fi}|^2 \prod_{j=1}^{n} \frac{V \mathrm{d}^3 p_j}{(2\pi)^3}}{\rho v} \ , \end{aligned} \tag{4}$$

where we have substituted the transition probability per unit of time (number of events per scatterer) by the analogous expression for the decay width in (2) (equivalent to the number of decays if the initial state was a single particle).

▷ We will find now the incoming flux ρv corresponding to one particle per unit of volume,

$$\rho v = \frac{1}{V}|\vec{v}_1 - \vec{v}_2| = \frac{1}{V}\left|\frac{\vec{k}_1}{E_B} - \frac{\vec{k}_2}{E_T}\right| = \frac{|E_T\vec{k}_1 - E_B\vec{k}_2|}{VE_BE_T} = \frac{\{(k_1k_2)^2 - M_1^2M_2^2\}^{1/2}}{VE_BE_T} \quad (5)$$

where $(\vec{v}_1 - \vec{v}_2)$ is the relative velocity of a particle in the beam and a particle in the target, with masses M_1 and M_2 respectively, that we will assume collinear $(\vec{k}_1||\vec{k}_2)$,[a] so, in fact, we find an expression for the flux that is invariant under boosts in the collinear direction,

$$k_1 = (E_B, \vec{k}_1)\ , \quad k_2 = (E_T, \vec{k}_2)$$
$$\Rightarrow (k_1k_2)^2 - M_1^2M_2^2 = (E_BE_T + |\vec{k}_1||\vec{k}_2|)^2 - M_1^2M_2^2 = |E_T\vec{k}_1 - E_B\vec{k}_2|^2\ ,$$

where we have used $\vec{k}_1 \cdot \vec{k}_2 = -|\vec{k}_1||\vec{k}_2|$ because $\vec{k}_1$ and $\vec{k}_2$ are parallel.

[a] This is valid both for fix-target collisions, as in the figure, and for particle colliders.

▷ Therefore, from (4) and (5), the cross section $|i\rangle = \left|\vec{k}_1\vec{k}_2\right\rangle \to |f\rangle = |\vec{p}_1\vec{p}_2\cdots\vec{p}_n\rangle$ reads

$$\mathrm{d}\sigma = \frac{(2\pi)^4\delta^4(P_f - P_i)|\mathcal{T}_{fi}|^2}{4\{(k_1k_2)^2 - M_1^2M_2^2\}^{1/2}} 2E_H 2E_B V^2 \prod_{j=1}^{n} \frac{V\mathrm{d}^3p_j}{(2\pi)^3}\ .$$

and substituting (1), we have finally

$$\boxed{\mathrm{d}\sigma = \frac{1}{4\{(k_1k_2)^2 - M_1^2M_2^2\}^{1/2}}|\mathcal{M}_{fi}|^2\mathrm{d}\Phi_n}$$

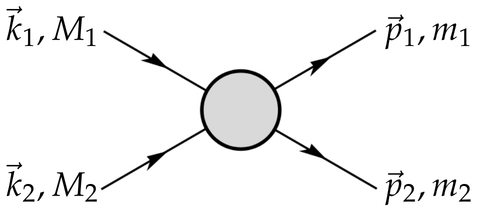
▷ If there are n_r identical particles of species r in the final state then the total cross section (integration over phase phase) must be divided by the symmetry factor

$$S = \prod_r n_r!\ .$$

▷ If the initial state is unpolarized and/or the polarization of the final state is not measured one has to average over initial polarizations and/or sum over final ones.

Observables | Cross section | $2 \to 2$

- Consider now the particular case of the $2 \to 2$ scattering in the CM frame:



▷ The LIPS integration can be read from (3). The flux factor is obtained from

$$k_1 \equiv (E_1, \vec{k}) \ , \ k_2 = (E_2, -\vec{k}) \ , \ E_{\text{CM}} = E_1 + E_2 \ , \ 4\{(k_1 k_2)^2 - M_1^2 M_2^2\}^{1/2} = 4E_{\text{CM}}|\vec{k}| \ .$$

Then

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2 E_{\text{CM}}^2} \frac{|\vec{p}|}{|\vec{k}|} |\mathcal{M}_{fi}|^2 \ .$$

Observables | Cross section

- A comment on the dimensions of the physical magnitudes we have introduced:

$$\mathcal{S}_{fi} = \delta_{fi} + (2\pi)^4 \delta^4(P_f - P_i) \mathrm{i} \mathcal{T}_{fi} \Rightarrow [\mathcal{S}_{fi}] = [\text{energy}]^0 \ , \quad [\mathcal{T}_{fi}] = [\text{energy}]^4$$

$$\mathcal{M}_{fi} = \prod_{j=1}^{n_i+n_f} (2E_j V)^{1/2} \mathcal{T}_{fi} \Rightarrow [\mathcal{M}_{fi}] = [\text{energy}]^{4-n_i-n_f}$$

$$\mathrm{d}\Phi_n = (2\pi)^4 \delta^4(P_f - P_i) \prod_{j=1}^{n} \frac{\mathrm{d}^3 p_j}{(2\pi)^3 2E_j} \Rightarrow [\mathrm{d}\Phi_n] = [\text{energy}]^{-4+2n}$$

$$\mathrm{d}\sigma(n_i = 2 \to n) = \frac{|\mathcal{M}_{fi}|^2 \mathrm{d}\Phi_n}{4\{(k_1 k_2)^2 - M_1^2 M_2^2\}^{1/2}} \Rightarrow [\sigma] = [\text{energy}]^{-2} = [\text{length}]^2$$

$$\mathrm{d}\Gamma(n_i = 1 \to n) = \frac{1}{2M} |\mathcal{M}_{fi}|^2 \mathrm{d}\Phi_n \Rightarrow [\Gamma] = [\text{energy}] = [\text{time}]^{-1}$$

The following conversion factors are very useful:

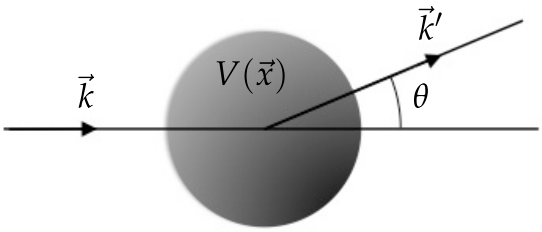
$$\hbar \approx 6.582 \times 10^{-22} \text{ MeV s} \ , \quad (\hbar c)^2 \approx 0.389 \text{ GeV}^2 \text{ mbarn}$$

They are easy to remember from:

$$1 \equiv \hbar c \approx 200 \text{ MeV fm} \ , \quad c \approx 3 \times 10^8 \text{ m/s} \ , \quad 1 \text{ fm} = 10^{-15} \text{ m} \ , \quad 1 \text{ barn} \equiv 10^{-24} \text{ cm}^2.$$

Non-relativistic limit: interaction potentials

- Taking the non-relativistic limit (NR) the calculations done using Feynman diagrams (QFT) must reproduce the results from the non-relativistic quantum mechanics, where the particle interaction is described in terms of a potential $V(\vec{x})$.



To find the form of the potential remember that the elastic scattering cross section of a mass m particle off a potential $V(\vec{x})$ is

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 \tag{6}$$

where θ is the scattering angle and $f(\theta)$ is the non-relativistic scattering amplitude, that can be calculated perturbatively.

Non-relativistic limit: interaction potentials

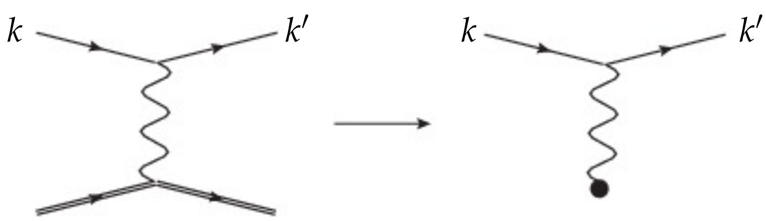
▷ At first order (Born approximation),

$$f(\theta) = -\frac{m}{2\pi}\int d^3x\, e^{-i\vec{q}\cdot\vec{x}} V(\vec{x}) \,, \quad \vec{q} = \vec{k}' - \vec{k} \,, \quad k = |\vec{k}| = |\vec{k}'| \,. \tag{7}$$

For a central potential, $V = V(r)$, previous expression can be written as

$$f(\theta) = -\frac{2m}{q}\int_0^\infty dr\, rV(r)\sin qr \,, \quad q = |\vec{q}| = 2k\sin\frac{\theta}{2} \,.$$

Consider $k \ll m$ (as corresponds to the NR limit) and let the target generating the potential be a very heavy particle of mass $M_A \gg m$. This is, for example, the typical situation when an electron is scattered by a nucleus, so the nucleus recoil can be neglected. Diagrammatically:



Non-relativistic limit: interaction potentials

▷ The elactic cross section ($k = |\vec{k}| = |\vec{k}'|$, $E_{\rm CM} \approx M_A$) reads

$$d\sigma_{\rm elast} = \frac{1}{64\pi^2 E_{CM}^2}\frac{|\vec{k}'|}{|\vec{k}|}|\mathcal{M}_{fi}|^2 d\Omega \approx \frac{1}{64\pi^2 M_A^2}|\mathcal{M}_{fi}|^2 d\Omega \,.$$

To find the corresponding non-relativistic expression, remember that

$$\mathcal{M}_{fi} = (2E_{\vec{k}}V)^{1/2}(2E_{\vec{k}'}V)^{1/2}(2E_A V)^{1/2}(2E_A V)^{1/2}\mathcal{T}_{fi} \approx 4mM_A V^2 \mathcal{T}_{fi} \,,$$

so in the NR limit, neglecting the factors of V that we know will cancel with the right normalization,

$$d\sigma_{\rm elast} \approx \frac{m^2}{4\pi^2}|\mathcal{T}_{fi}|^2 d\Omega$$

which, comparing with (6), tells us that

$$f(\theta) = \frac{m}{2\pi}\mathcal{T}_{fi} \,,$$

where the global sign is the correct one as we will show in an example.

Non-relativistic limit: interaction potentials

▷ From (7),

$$\mathcal{T}_{fi}(\vec{q}) = -\int d^3x\, e^{-i\vec{q}\cdot\vec{x}} V(\vec{x}) \Rightarrow V(\vec{x}) = -\int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{x}} \mathcal{T}_{fi}(\vec{q})$$

$$\Rightarrow V(\vec{x}) = -\frac{1}{4mM_A}\int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{x}} \mathcal{M}_{fi}(\vec{q}). \qquad (8)$$

Note that the interaction potential is a non-relativistic concept, describing an instantaneous interaction. However, the more precise QFT description is based on the exchange and propagation of particles, as shown along this course.

6. Elementary processes in QED

Lagrangian and Feynman rules of QED

- The quantum electrodynamics (QED) describes the interaction of electrons (or any other charged particle of spin 1/2) and photons.
- It is most convenient to perform a covariant quantization of Maxwell field. And it is customary to generalize the non gauge-invariant term of the Lagrangian by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 \,, \tag{1}$$

  where ξ is a free parameter. In a preceding chapter we used $\xi = 1$ but one can show that, regardless of the value of ξ, by imposing that $\partial_\mu A^\mu$ vanishes between physical states, the physical spectrum of the theory is given only by transverse photon polarization states.

Lagrangian and Feynman rules of QED

▷ The net effect of the second term in (1), called *gauge fixing* term, is breaking the gauge invariance of the Lagrangian, but the matrix elements between physical states will be independent of the choice of ξ.[a]
However, the field commutation rules and the progagator will depend on ξ.

▷ It is recommended to use a generic ξ and check that the calculation is correct by verifying the cancelation of ξ in the matrix elements between physical states. Nevertheless, depending on the type of problem, the calculations get simplified if from the beginning an appropriate value of the so called R_ξ gauge is chosen.
In particular,

- $\xi = 1$ is the 't Hooft-Feynman gauge,
- $\xi = 0$ is the Landau gauge, and
- $\xi \to \infty$ is the unitary gauge (with only physical degrees of freedom involved).

[a]In the presence of interactions the non-dependence on ξ is achieved as long as A_μ couples to matter respecting the gauge invariance, that is, when it couples to a conserved current.

Lagrangian and Feynman rules of QED

▷ Let us find the photon propagator. In order to write the Euler-Lagrange equations for this Lagrangian note that

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu) - \frac{1}{2\xi} g^{\mu\nu} \partial_\mu A_\nu \partial_\alpha A^\alpha$$

and hence

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial A_\nu} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = 0 &\Rightarrow \partial_\mu F^{\mu\nu} + \frac{1}{\xi}\partial^\nu \partial^\mu A_\mu = 0 \\ &\Rightarrow \Box A^\nu - \left(1 - \frac{1}{\xi}\right) \partial^\mu \partial^\nu A_\mu = 0 \\ &\Rightarrow \left[g^{\mu\nu} \Box - \left(1 - \frac{1}{\xi}\right) \partial^\mu \partial^\nu \right] A_\mu = 0\end{aligned}$$

▷ We already know that the propagator is a Green's function of the operator acting on the field in the equation of motion. In momentum space, the photon propagator is then, up to some phase factor, the inverse of

$$-k^2 g^{\mu\nu} + \left(1 - \frac{1}{\xi}\right) k^\mu k^\nu \,. \tag{2}$$

Note that this operator is invertible thanks to the introduction of the gauge fixing term, because $-k^2 g^{\mu\nu} + k^\mu k^\nu$ is singular (it has a null eigenvalue for the eigenvector k^μ). This has to do with the gauge invariance: the vector field $A'_\mu = A_\mu + \partial_\mu \Lambda$ is also a solution of $(g^{\mu\nu}\Box - \partial^\mu\partial^\nu)A_\mu = 0$ but the introduction of the gauge fixing term has broken the gauge invariance.

The inverse of (2), including the Feynman prescription as earlier discussed, is the photon propagator

$$\boxed{\widetilde{D}_F^{\mu\nu}(k) = \frac{\mathrm{i}}{k^2 + \mathrm{i}\varepsilon}\left[-g^{\mu\nu} + (1-\xi)\frac{k^\mu k^\nu}{k^2}\right]} \tag{3}$$

Lagrangian and Feynman rules of QED

▷ In fact,

$$\widetilde{D}_F^{\mu\nu}(k)\left[-k^2 g_{\nu\rho} + \left(1 - \frac{1}{\xi}\right) k_\nu k_\rho\right] = \mathrm{i}\delta^\mu_\rho \,.$$

The choice of global sign is the right one, opposite to that for scalar fields, because the commutation relation of creation and annihilation operators of the scalar field, $[a_{\vec{p}}, a^\dagger_{\vec{q}}] = (2\pi)^3\delta^3(\vec{p}-\vec{q})$, differ from those of the photon Maxwell field with $\xi = 1$ in a minus sign, apart from the $g_{\lambda\lambda'}$ factor from the polarization vectors:

$$[a_{\vec{p},\lambda}, a^\dagger_{\vec{q},\lambda'}] = \zeta_\lambda \delta_{\lambda\lambda'}(2\pi)^3\delta^3(\vec{p}-\vec{q}) = -g_{\lambda\lambda'}(2\pi)^3\delta^3(\vec{p}-\vec{q}).$$

- Remember that Maxwell equations in presence of sources are given by the U(1) gauge-invariant Lagrangian obtained by the introduction of the covariant derivative, leading to the minimal coupling of the electromagnetic field to charges and currents $j^\mu = (\rho, \vec{j})$. This provides the classical electrodynamics Lagrangian.

  In quantum electrodynamics, $j^\mu = eQ_f\overline{\psi}\gamma^\mu\psi$, where Q_f is the electric charge in units of e of the fermion f annihilated by the field ψ.

Lagrangian and Feynman rules of QED

- To describe the electromagnetic interaction at the quantum level (QED) one has to fix the gauge, as in (1), and interpret the interactions between quantum fields as the exchange of particles (photons, electrons and antielectrons or positrons).

  The full Lagrangian is

$$\mathcal{L}_{\text{QED}} = \overline{\psi}(\mathrm{i}\not{D} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 , \quad D_\mu = \partial_\mu + \mathrm{i}eQ_f A_\mu$$

  that contains just one type of interactions of the form

$$\mathcal{L}_{\text{int}} = -eQ_f A_\mu \overline{\psi}\gamma^\mu \psi \; . \tag{4}$$

Lagrangian and Feynman rules of QED

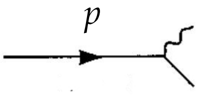
- To compute perturbatively the scattering matrix of a QED process one has just to apply the corresponding Feynman rules. Compared to the cases studied in the preceeding chapters, what is new is: the photon propagator in (3), the interaction vertex that can be readily derived from (4) and a factor accounting for the polarization of the photon when it appears as an external leg.

  Let us summarize next the Feynman rules of QED:

Lagrangian and Feynman rules of QED

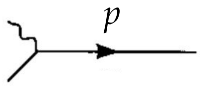
– External legs:

incoming fermion:



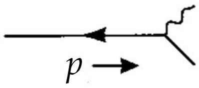
$= u^{(s)}(\vec{p})$

outgoing fermion:



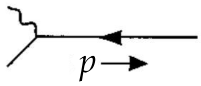
$= \overline{u}^{(s)}(\vec{p})$

incoming antifermion:



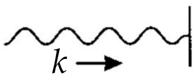
$= \overline{v}^{(s)}(\vec{p})$

outgoing antifermion:



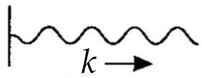
$= v^{(s)}(\vec{p})$

incoming photon:



$= \epsilon_\mu(\vec{k},\lambda)$

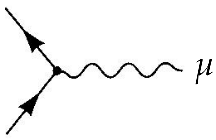
outgoing photon:



$= \epsilon^*_\mu(\vec{k},\lambda)$

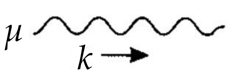
Lagrangian and Feynman rules of QED

– Vertex:

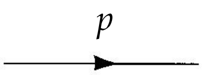


$$= -\mathrm{i}eQ_f\gamma^\mu$$

– Propagators:



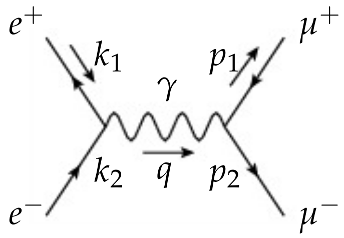
$$= -\frac{\mathrm{i}}{k^2+\mathrm{i}\varepsilon}\left[g^{\mu\nu}-(1-\xi)\frac{k^\mu k^\nu}{k^2}\right]$$



$$= \frac{\mathrm{i}(\not{p}+m)}{p^2-m^2+\mathrm{i}\varepsilon}$$

A simple process: $e^+e^- \to \mu^+\mu^-$

- Consider the annihilation of an electron and a positron to give a muon and an antimuon. In QED this process is described at lowest order in PT (tree level) by the diagram in the figure:



▷ The muon has the same charge as the electron, $Q_\mu = Q_e = -1$, and a mass M about 200 times greater than the electron mass m.

▷ Let us calculate step by step and in full detail the cross section of this process.

A simple process: $e^+e^- \to \mu^+\mu^-$

- First, assign momenta to all particles in the diagram, using the four-momentum conservation at every vertex. This fixes the four-momentum of the virtual photon momentum propagating between the two interaction vertices,

$$q = k_1 + k_2 = p_1 + p_2 \, .$$

▷ The external legs are fermions, whose spinors are labeled by indices r_1, r_2, s_1, s_2 taking any possible values in $\{1, 2\}$.

- Applying the Feynman rules, following every fermion line in opposite direction to the fermion flux, the invariant amplitude is then given by

$$\mathrm{i}\mathcal{M} = \overline{u}^{(s_2)}(\vec{p}_2)(\mathrm{i}e\gamma^\beta)v^{(s_1)}(\vec{p}_1)\frac{(-\mathrm{i})}{q^2}\left[g_{\alpha\beta} - (1-\xi)\frac{q_\alpha q_\beta}{q^2}\right]\overline{v}^{(r_1)}(\vec{k}_1)(\mathrm{i}e\gamma^\alpha)u^{(r_2)}(\vec{k}_2) \, .$$

A simple process: $e^+e^- \to \mu^+\mu^-$

▷ Note that because external fermions are on-shell they satisfy the corresponding Dirac equations,

$$\not{k}_1 v^{(r_1)}(\vec{k}_1) = -m v^{(r_1)}(\vec{k}_1) \,, \quad \not{k}_2 u^{(r_2)}(\vec{k}_2) = m u^{(r_2)}(\vec{k}_2) \,,$$

so the amplitude will not depend on the parameter ξ, as it must be, because

$$q_\alpha \overline{v}^{(r_1)}(\vec{k}_1)\gamma^\alpha u^{(r_2)}(\vec{k}_2) = \overline{v}^{(r_1)}(\vec{k}_1)(\not{k}_1 + \not{k}_2) u^{(r_2)}(\vec{k}_2) = 0 \,.$$

▷ We could have worked from the beginning in the 't Hooft-Feynman gauge ($\xi = 1$) and obtain directly the same result. Therefore

$$\mathcal{M} = \frac{e^2}{q^2} \overline{u}^{(s_2)}(\vec{p}_2)\gamma^\alpha v^{(s_1)}(\vec{p}_1)\; \overline{v}^{(r_1)}(\vec{k}_1)\gamma_\alpha u^{(r_2)}(\vec{k}_2) \,.$$

A simple process: $e^+e^- \to \mu^+\mu^-$

- To compute $|\mathcal{M}|^2$, note that

$$(\overline{u}\gamma^\alpha v)^* = v^\dagger \gamma^{\alpha\dagger}\gamma^{0\dagger} u = v^\dagger \gamma^0 \gamma^0 \gamma_\alpha \gamma^0 u = \overline{v}\gamma^\alpha u \,,$$

where we hav used

$$\overline{u} = u^\dagger \gamma^0 \,, \quad \gamma^{\alpha\dagger} = \gamma_\alpha \,, \quad \gamma^0\gamma_\alpha\gamma^0 = \gamma^\alpha \,.$$

This is a complex number that can be multiplied in any ordering. The same happens to the other fermion line.

▷ It is then convenient to write

$$|\mathcal{M}|^2 = \frac{e^4}{q^4} \overline{u}^{(s_2)}(\vec{p}_2)\gamma^\alpha v^{(s_1)}(\vec{p}_1)\; \overline{v}^{(s_1)}(\vec{p}_1)\gamma^\beta u^{(s_2)}(\vec{p}_2)$$
$$\times\, \overline{v}^{(r_1)}(\vec{k}_1)\gamma_\alpha u^{(r_2)}(\vec{k}_2)\; \overline{u}^{(r_2)}(\vec{k}_2)\gamma_\beta v^{(r_1)}(\vec{k}_1) \qquad (5)$$

We can now use the properties of spinors and Dirac matrices, leading to a plethora of identities (Diracology).

A simple process: $e^+e^- \to \mu^+\mu^-$

▷ In particular, one can see that the two spin states along the $\hat{z}$ axis satisfy

$$u^{(1)}(\vec{p})\overline{u}^{(1)}(\vec{p}) = (\not{p}+m)\frac{1+\gamma_5\not{n}}{2},$$
$$u^{(2)}(\vec{p})\overline{u}^{(2)}(\vec{p}) = (\not{p}+m)\frac{1-\gamma_5\not{n}}{2},$$
$$v^{(1)}(\vec{p})\overline{v}^{(1)}(\vec{p}) = (\not{p}-m)\frac{1+\gamma_5\not{n}}{2},$$
$$v^{(2)}(\vec{p})\overline{v}^{(2)}(\vec{p}) = (\not{p}-m)\frac{1-\gamma_5\not{n}}{2},$$

where $n^\mu = (0,0,0,1)$ in the frame where $p^\mu = (m,0,0,0)$.
In general,

$$u(\vec{p},n)\overline{u}(\vec{p},n) = (\not{p}+m)\frac{1+\gamma_5\not{n}}{2}, \quad v(\vec{p},n)\overline{v}(\vec{p},n) = (\not{p}-m)\frac{1+\gamma_5\not{n}}{2}$$

project on well defined polarizations along a given direction n^μ, with $n^2 = -1$ and $p_\mu n^\mu = 0$.

A simple process: $e^+e^- \to \mu^+\mu^-$

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▷ If we choose, for simplicity, the $\hat{z}$ axis as the direction of motion, $p^\mu = (E,0,0,|\vec{p}|)$, previous operators project on the two helicity states of particle and antiparticle, respectively, taking $n^\mu = \pm(|\vec{p}|/m,0,0,E/m)$.

▷ In particular, in the ultrarelativistic limit ($E \gg m$) the left and right chirality projectors of particle and antiparticle are:

$$u^{(1)}(\vec{p})\overline{u}^{(1)}(\vec{p}) = (\not{p}+m)\frac{1+\gamma_5\not{n}}{2} \to u_R(p)\overline{u}_R(\vec{p}) = \not{p}\frac{1-\gamma_5}{2},$$
$$u^{(2)}(\vec{p})\overline{u}^{(2)}(\vec{p}) = (\not{p}+m)\frac{1-\gamma_5\not{n}}{2} \to u_L(p)\overline{u}_L(\vec{p}) = \not{p}\frac{1+\gamma_5}{2},$$
$$v^{(1)}(\vec{p})\overline{v}^{(1)}(\vec{p}) = (\not{p}-m)\frac{1+\gamma_5\not{n}}{2} \to v_L(p)\overline{v}_L(\vec{p}) = \not{p}\frac{1+\gamma_5}{2},$$
$$v^{(2)}(\vec{p})\overline{v}^{(2)}(\vec{p}) = (\not{p}-m)\frac{1-\gamma_5\not{n}}{2} \to v_R(\vec{p})\overline{v}_R(\vec{p}) = \not{p}\frac{1-\gamma_5}{2}.$$

A simple process: $e^+e^- \to \mu^+\mu^-$

▷ Another property that can be easily derived from the above is

$$\overline{u}(\vec{p},n)\Gamma u(\vec{p},n) = \mathrm{Tr}\left[\Gamma(\not{p}+m)\frac{1+\gamma_5\not{n}}{2}\right] , \quad \overline{v}(\vec{p},n)\Gamma v(\vec{p},n) = \mathrm{Tr}\left[\Gamma(\not{p}-m)\frac{1+\gamma_5\not{n}}{2}\right]$$

where Γ in an arbitrary 4×4 matrix.

▷ On the other hand, if fermions are not polarized the calculation becomes much simpler because we can directly apply the completeness relations,

$$\sum_s u^{(s)}(\vec{p})\overline{u}^{(s)}(\vec{p}) = \not{p}+m , \quad \sum_s v^{(s)}(\vec{p})\overline{v}^{(s)}(\vec{p}) = \not{p}-m ,$$

leading to

$$\sum_s \overline{u}^{(s)}(\vec{p})\Gamma u^{(s)}(\vec{p}) = \mathrm{Tr}\left[\Gamma(\not{p}+m)\right] , \quad \sum_s \overline{v}^{(s)}(\vec{p})\Gamma v^{(s)}(\vec{p}) = \mathrm{Tr}\left[\Gamma(\not{p}-m)\right] .$$

A simple process: $e^+e^- \to \mu^+\mu^-$

- Coming back to our calculation (5) suppose for simplicity that initial and final fermions are not polarized. Then we have to average over initial polarizations and sum over the final ones:

$$\widetilde{\sum_{r_i}}\sum_{s_i}|\mathcal{M}|^2 = \frac{1}{4}\sum_{r_i}\sum_{s_i}|\mathcal{M}|^2$$
$$= \frac{e^4}{4q^4}\mathrm{Tr}[\gamma^\alpha(\not{p}_1-M)\gamma^\beta(\not{p}_2+M)]\mathrm{Tr}[\gamma_\alpha(\not{k}_2+m)\gamma_\beta(\not{k}_1-m)] ,$$

which is the product of the traces of two fermion chains.

A simple process: $e^+e^- \to \mu^+\mu^-$

- To calculate the traces we resort again on the Diracology. We need in particular,

$$\begin{aligned}
\mathrm{Tr}[\#\text{ impar } \gamma'\mathrm{s}] &= 0 \\
\mathrm{Tr}[\gamma^\mu\gamma^\nu] &= 4g^{\mu\nu} \\
\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] &= 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})
\end{aligned}$$

where

$$\begin{aligned}
\mathrm{Tr}[\gamma^\alpha(\not{p}_1 - M)\gamma^\beta(\not{p}_2 + M)] &= \mathrm{Tr}[\gamma^\alpha\not{p}_1\gamma^\beta\not{p}_2] - M^2\mathrm{Tr}[\gamma^\alpha\gamma^\beta] \\
&= 4(p_1^\alpha p_2^\beta - (p_1p_2)g^{\alpha\beta} + p_1^\beta p_2^\alpha) - 4M^2g^{\alpha\beta} \\
\mathrm{Tr}[\gamma_\alpha(\not{k}_2 + m)\gamma_\beta(\not{k}_1 - m)] &= \mathrm{Tr}[\gamma_\alpha\not{k}_1\gamma_\beta\not{k}_2] - m^2\mathrm{Tr}[\gamma_\alpha\gamma_\beta] \\
&= 4(k_{1\alpha}k_{2\beta} - (k_1k_2)g_{\alpha\beta} + k_{1\beta}k_{2\alpha}) - 4m^2g_{\alpha\beta}
\end{aligned}$$

A simple process: $e^+e^- \to \mu^+\mu^-$

and then,

$$\begin{aligned}
\widetilde{\sum_{r_i}}\sum_{s_i}|\mathcal{M}|^2 = \frac{16e^4}{4q^4}&[(p_1k_1)(p_2k_2) - (p_1p_2)(k_1k_2) + (p_1k_2)(p_2k_1) - m^2(p_1p_2) \\
&- (p_1p_2)(k_1k_2) + 4(p_1p_2)(k_1k_2) - (p_1p_2)(k_1k_2) + 4m^2(p_1p_2) \\
&+ (p_1k_2)(p_2k_1) - (p_1p_2)(k_1k_2) + (p_1k_1)(p_2k_2) - m^2(p_1p_2) \\
&- M^2(k_1k_2) + 4M^2(k_1k_2) - M^2(k_1k_2) + 4M^2m^2] \\
= \frac{8e^4}{q^4}&[(p_1k_1)(p_2k_2) + (p_1k_2)(p_2k_1) + m^2(p_1p_2) + M^2(k_1k_2) + 2M^2m^2]\,. \qquad (6)
\end{aligned}$$

A simple process: $e^+e^- \to \mu^+\mu^-$

- The following step is choosing a reference frame. Assume the center of mass frame and let θ be the angle of the outgoing μ^+ with the incoming e^+,

$$\begin{aligned}
k_1^\mu &= E(1,0,0,\beta_i) \ , \\
k_2^\mu &= E(1,0,0,-\beta_i) \ , \quad \beta_i = \sqrt{1-m^2/E^2} \ , \\
p_1^\mu &= E(1,\beta_f \sin\theta, 0, \beta_f \cos\theta) \ , \\
p_2^\mu &= E(1,-\beta_f \sin\theta, 0, -\beta_f \cos\theta) \ , \quad \beta_f = \sqrt{1-M^2/E^2} \ .
\end{aligned}$$

Then,

$$\begin{aligned}
q^2 = (k_1+k_2)^2 = (p_1+p_2)^2 = E_{\rm CM}^2 &= 4E^2 \ , \\
(p_1k_1) = (p_2k_2) &= E^2(1-\beta_i\beta_f\cos\theta) \ , \\
(p_1k_2) = (p_2k_1) &= E^2(1+\beta_i\beta_f\cos\theta) \ , \\
(p_1p_2) = E^2(1+\beta_f^2) &= E^2(2-M^2/E^2) \ , \\
(k_1k_2) = E^2(1+\beta_i^2) &= E^2(2-m^2/E^2)
\end{aligned}$$

A simple process: $e^+e^- \to \mu^+\mu^-$

and expression (6) reads

$$\begin{aligned}
\widetilde{\sum_{r_i}}\sum_{s_i}|\mathcal{M}|^2 &= \frac{e^4}{2E^4}[2E^4(1+\beta_i^2\beta_f^2\cos^2\theta)+2E^2(m^2+M^2)] \\
&= e^4\left[1+4\frac{m^2+M^2}{E_{\rm CM}^2}+\left(1-\frac{4m^2}{E_{\rm CM}^2}\right)\left(1-\frac{4M^2}{E_{\rm CM}^2}\right)\cos^2\theta\right] \ .
\end{aligned}$$

- The differential cross section of the process is obtained from

$$\begin{aligned}
\frac{d\sigma}{d\Omega} &= \frac{1}{64\pi^2E_{\rm CM}^2}\frac{|\vec{p}|}{|\vec{k}|}|\mathcal{M}|^2 \\
\frac{d\sigma}{d\Omega} &= \frac{\alpha^2}{4E_{\rm CM}^2}\sqrt{\frac{E_{\rm CM}^2-4M^2}{E_{\rm CM}^2-4m^2}}\left[1+4\frac{m^2+M^2}{E_{\rm CM}^2}+\left(1-\frac{4m^2}{E_{\rm CM}^2}\right)\left(1-\frac{4M^2}{E_{\rm CM}^2}\right)\cos^2\theta\right]
\end{aligned}$$

where we have substituted the fine structure constant $\alpha = e^2/(4\pi)$.

A simple process: $e^+e^- \to \mu^+\mu^-$

▷ Note that $E_{\text{CM}} > 2M > 2m$, the threshold energy of the process. The total cross section is

$$\sigma = \int \mathrm{d}\Omega \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = 2\pi \int \mathrm{d}\cos\theta \, \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \, .$$

▷ In the ultrarelativistic limit ($E_{\text{CM}} \gg M > m$),

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \to \frac{\alpha^2}{4E_{\text{CM}}^2}(1+\cos^2\theta)$$
$$\sigma \to \frac{4\pi\alpha^2}{3E_{\text{CM}}^2} \, .$$

Comments | About the propagator and the polarization states

- For the covariant quantization the Maxwell field we had to introduce four polarization vectors $\epsilon^\mu(\vec{k},\lambda)$ that satisfy the following orthonormality and completeness relations:

$$\epsilon^*_\mu(\vec{k},\lambda)\epsilon^\mu(\vec{k},\lambda') = -\zeta_\lambda \delta_{\lambda\lambda'} \, , \quad \zeta_0 = -1 \, , \quad \zeta_1 = \zeta_2 = \zeta_3 = 1 \, ,$$
$$\sum_{\lambda=0}^{3} \zeta_\lambda \epsilon^{\mu *}(\vec{k},\lambda)\epsilon^\nu(\vec{k},\lambda) = -g^{\mu\nu} \, . \tag{7}$$

Comments | About the propagator and the polarization states

▷ Let n^μ be a spacelike vector that satisfies $n_\mu n^\mu = 1$ and $n^0 > 0$.
We say that $\epsilon^\mu(\vec{k},0) = n^\mu$ is the scalar polarization vector.
We will call $\epsilon^\mu(\vec{k},3)$ the longitudinal polarization in the $n-k$ plane if $\epsilon_\mu(\vec{k},3)n^\mu = 0$ and $\epsilon_\mu(\vec{k},3)\epsilon^\mu(\vec{k},3) = -1$, namely,

$$\epsilon^\mu(\vec{k},3) = \frac{k^\mu - (kn)n^\mu}{\sqrt{(kn)^2 - k^2}} \,.$$

The remaining two are the so called tranverse polarizations $\epsilon^\mu(\vec{k},1)$ and $\epsilon^\mu(\vec{k},2)$, that we take orthogonal to each other and perpendicular to the $n-k$ plane, so

$$\epsilon^*_\mu(\vec{k},\lambda)\epsilon^\mu(\vec{k},\lambda') = -\delta_{\lambda\lambda'} \,, \quad \lambda,\lambda' = 1,2 \,.$$

Comments | About the propagator and the polarization states

▷ For example, in the reference frame where the propagation vector $\vec{k}$ defines the $\hat{z}$ axis, i.e. $k^\mu = (\omega,0,0,\omega)$, and $n^\mu = (1,0,0,0)$, from the above we can derive that

$$\epsilon^\mu(\vec{k},0) = (1,0,0,0) \,, \quad \epsilon^\mu(\vec{k},3) = (0,0,0,1) \,,$$

and we can choose several bases for the tranverse polarizations, as

$$\epsilon^\mu(\vec{k},1) = (0,1,0,0) \,, \quad \epsilon^\mu(\vec{k},2) = (0,0,1,0) \quad \text{(linear)}$$

or

$$\epsilon^\mu(\vec{k},L) = (0,\cos\theta,\mathrm{i}\sin\theta,0) \,, \quad \epsilon^\mu(\vec{k},R) = (0,\cos\theta,-\mathrm{i}\sin\theta,0) \quad \text{(elliptical)}$$

(which are called circular polarizations when $\theta = \pi/4$).

Comments **About the propagator and the polarization states**

- ▷ We have already seen that the scalar polarization state $\left|\vec{k},0\right\rangle$ has a negative norm.
- ▷ On the other hand, the classical electromagnetic field in absence of sources (radiation) has just two polarization states, while its quantum version seems to have four.
- ▷ Both problems are related and are solved, as we have seen, by imposing that $\partial_\mu A^\mu$ vanishes when evaluated between physical states (Gupta-Bleuler quantization): only the two transvere polarization states contribute to physical observables, so we don't need to care about scalar and longitudinal polarizations.
- ▷ However, all of them contribute to the propagator, that is not an observable. Let us see this.

Comments **About the propagator and the polarization states**

- ▷ Take the 't Hooft-Feynman gauge ($\xi = 1$), in which the propagator is

$$\widetilde{D}_F^{\mu\nu}(k) = -\frac{\mathrm{i}g^{\mu\nu}}{k^2+\mathrm{i}\varepsilon}$$

that using the completeness relations (7) can be written as

$$\widetilde{D}_F^{\mu\nu}(k) = \frac{\mathrm{i}}{k^2+\mathrm{i}\varepsilon}\sum_{\lambda=0}^{3}\zeta_\lambda \epsilon^\mu(\vec{k},\lambda)\epsilon^{\nu*}(\vec{k},\lambda)\ .$$

Separating the tranverse, longitudinal and scalar contributions we have

$$\widetilde{D}_F^{\mu\nu}(k) = \frac{\mathrm{i}}{k^2+\mathrm{i}\varepsilon}\left[\sum_{\lambda=1,2}\epsilon^\mu(\vec{k},\lambda)\epsilon^{\nu*}(\vec{k},\lambda) + \frac{[k^\mu-(kn)n^\mu][k^\nu-(kn)n^\nu]}{(kn)^2-k^2} - n^\mu n^\nu\right]\ .$$

The last two terms can be rewritten as the sum of

$$\widetilde{D}_C^{\mu\nu}(k) = \frac{\mathrm{i}n^\mu n^\nu}{(kn)^2-k^2} \quad \text{and} \quad \widetilde{D}_R^{\mu\nu}(k) = \frac{\mathrm{i}}{k^2[(kn)^2-k^2]}[k^\mu k^\nu-(kn)(k^\mu n^\nu+k^\nu n^\mu)]\ .$$

Comments | About the propagator and the polarization states

▷ If we use the reference frame where $n^\mu = (1,0,0,0)$, i.e. $n^\mu = g^{\mu 0}$, $kn = k^0$ and $(kn)^2 - k^2 = \vec{k}^2$, we see that the first term reads

$$
\begin{aligned}
D_C^{\mu\nu}(x-x') &= \int \frac{\mathrm{d}^4k}{(2\pi)^4} \widetilde{D}_C^{\mu\nu}(k) \mathrm{e}^{-\mathrm{i}k(x-x')} \\
&= \int \frac{\mathrm{d}^3k}{(2\pi)^3} \frac{g^{\mu 0} g^{\nu 0}}{\vec{k}^2} \mathrm{e}^{\mathrm{i}\vec{k}\cdot(\vec{x}-\vec{x}')} \int \frac{\mathrm{d}k^0}{(2\pi)} \mathrm{e}^{-\mathrm{i}k^0(x^0-x'^0)} \\
&= \frac{1}{4\pi} \frac{g^{\mu 0} g^{\nu 0}}{|\vec{x}-\vec{x}'|} \delta(x^0 - x'^0) \ .
\end{aligned}
$$

To understand the meaning of the different contributions, consider a process mediated by the exchange of a photon (e.g. as the one before).
In this situation we are in the presence of sources.

▷ The matrix element of this process can be written as

$$
\int \mathrm{d}^4x \int \mathrm{d}^4x' \ j_{1\mu}(x) D_F^{\mu\nu}(x-x') j_{2\nu}(x') \tag{8}
$$

where $j_1^\mu(x)$ y $j_2^\mu(x')$ are currents interacting by the mediation of the photon field.

Comments | About the propagator and the polarization states

▷ The piece $D_C^{\mu\nu}$ of the propagator, in the reference frame we have chosen, contributes with

$$
\int \mathrm{d}^4x \int \mathrm{d}^4x' \ \frac{j_1^0(x) j_2^0(x')}{4\pi|\vec{x}-\vec{x}'|} \delta(x^0 - x'^0) \ ,
$$

which is nothing but the instantaneous Coulomb interaction of both charge densities $j_1^0(x)$ and $j_2^0(x')$ in the same instant of time!

▷ As for the Remaining piece $D_R^{\mu\nu}$, note that (8) in momentum space is

$$
\int \frac{\mathrm{d}^4k}{(2\pi)^4} \ j_{1\mu}(k) \widetilde{D}_F^{\mu\nu}(k) j_{2\nu}(k)
$$

and the currents are conserved, namely, $\partial_\mu j^\mu(x) = 0$ or $k_\mu j^\mu(k) = 0$. Therefore the terms proportional to k^μ or k^ν in $\widetilde{D}_F^{\mu\nu}$ are irrelevant and $\widetilde{D}_R^{\mu\nu}(k)$ does not contribute to the interaction. This is the reason why the 't Hooft-Feynman works: if we had not taken $\xi = 1$ there would be extra terms in the propagator proportional to $k^\mu k^\nu$, that would be irrelevant because they couple to a conserved current.

Comments	About the propagator and the polarization states

▷ We see that field of the electromagnetic interaction has a triple nature:[a]

1. One part is completely arbitrary (due to the gauge invariance);
2. Another one, the so called constrained field (the coulomb interaction, in the reference frame we have chosen), is completely determined by the sources;
3. And the third part is dynamical, consisting of independent degrees of freedom, is the pure electromagnetic radiation, that in quantum electrodynamics corresponds to the photons.

[a] This also happens with the rest of fundamental interactions.

Comments	About the propagator and the polarization states

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▷ The dynamical contribution, present even in absence of sources, can be subdominant with respect to the constrained contribution.

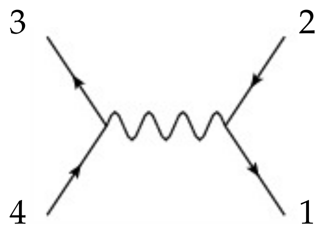
▷ It is the Coulomb interaction that explains in first approximation the atomic structure of hydrogen: the Coulomb potential is given by the position of the electron.

▷ The quantization of the atomic energy levels has to do with the fact that position is an operator in quantum mechanics, not commuting with the momentum operator, and hence the Hamiltonian has a discrete spectrum. However notice that the quantum degrees of freedom of the electromagnetic field do not play any role at this point. The truely quantum effects of the electromagnetic field manifest as a small correction (the Lamb effect, for instance).

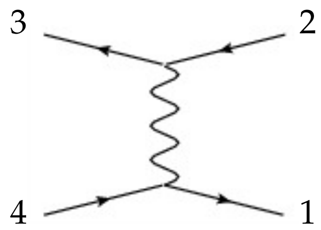
Comments | About the relative signs of diagrams

- In QED we work with fermion fields and we have seen already that one has to be careful because the Wick contractions of these fields can give rise to relative signs between several diagrams contributing to the amplitude of a given process. Remember that one has to see whether the reordering of spinors corresponds to an even or an odd permutation. We show a few examples below.

– Bhabha scatttering: $e^+e^- \to e^+e^-$



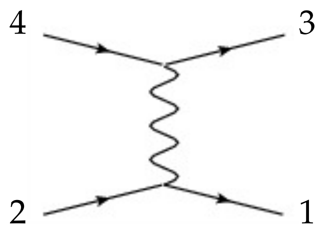
$\overline{u}_1 v_2 \overline{v}_3 u_4$



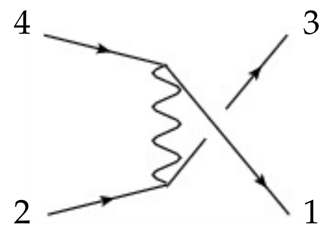
$-\overline{u}_1 u_4 \overline{v}_3 v_2$

Comments | About the relative signs of diagrams

– Møller scattering: $e^-e^- \to e^-e^-$

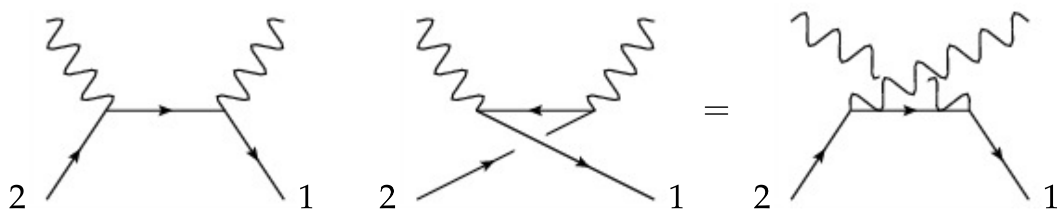


$\overline{u}_1 u_2 \overline{u}_3 u_4$



$-\overline{u}_1 u_4 \overline{u}_3 u_2$

– Compton scattering: $e\gamma \to e\gamma$ (no change of sign!)



$\overline{u}_1 u_2$ $+\overline{u}_1 u_2$

Comments | About identical particles

- Remember that when there are identical particles in the final state (for example, $\gamma\gamma$, e^+e^+, e^-e^-) the total cross section is

$$\sigma = \frac{1}{2}\int \mathrm{d}\Omega\,\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\,.$$

Comments | About vector boson polarizations

– The case of a photon.

The photon has two (transverse) polarization states. Take the reference frame where $k^\mu = (\omega, 0, 0, \omega)$ (Lorentz covariance ensures that our conclusions will be independent of this choice). Then, they can be

$$\begin{aligned}
\text{linear:}\quad & \epsilon^\mu(\vec{k},1) = (0,1,0,0)\ , && \epsilon^\mu(\vec{k},2) = (0,0,1,0) \\
\text{elliptical:}\quad & \epsilon^\mu(\vec{k},L) = (0,\cos\theta,\mathrm{i}\sin\theta,0)\ , && \epsilon^\mu(\vec{k},R) = (0,\cos\theta,-\mathrm{i}\sin\theta,0)\ .
\end{aligned}$$

In any case, summing over polarization states,

$$\sum_\lambda \epsilon^*_\mu(\vec{k},\lambda)\epsilon_\nu(\vec{k},\lambda) = -g_{\mu\nu} + Q_{\mu\nu}\ ,\quad Q_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\,.$$

Let us see that, due to gauge invariance, one can ignore in practice the term $Q_{\mu\nu}$.

Comments	About vector boson polarizations

– The case of a photon.

In fact, the amplitude of an arbitrary process in QED involving an external photon of momentum k (take an outgoing photon) can be written without loss of generality as

$$\mathcal{M}(\vec{k},\lambda) = \epsilon^*_\mu(\vec{k},\lambda)\mathcal{M}^\mu(\vec{k})$$

and any observable, in this reference frame, will be proportional to

$$\begin{aligned}\sum_\lambda |\mathcal{M}(\vec{k},\lambda)|^2 &= \sum_{\lambda=1,2} \epsilon^*_\mu(\vec{k},\lambda)\epsilon_\nu(\vec{k},\lambda)\mathcal{M}^\mu(\vec{k})\mathcal{M}^{\nu *}(\vec{k}) \\ &= |\mathcal{M}^1(\vec{k})|^2 + |\mathcal{M}^2(\vec{k})|^2 \; . \end{aligned} \tag{9}$$

Comments	About vector boson polarizations

– The case of a photon.

We know that the photon field couples to a conserved current by an interaction $\int d^4x \; j^\mu(x)A_\mu(x)$, with $\partial_\mu j^\mu(x) = 0$, so

$$\mathcal{M}^\mu(\vec{k}) = \int d^4x \; e^{ikx} \langle f|\, j^\mu(x)\, |i\rangle$$

where initial and final states include all external particles except for the concerned photon.

Because the gauge invariance must be preserved also at the quantum level, from the current conservation and the expression above we have[a]

$$k_\mu \mathcal{M}^\mu(\vec{k}) = i \int d^4x \; e^{ikx} \langle f|\, \partial_\mu j^\mu(x)\, |i\rangle = 0 \quad \text{(Ward identity)}$$

$$\Rightarrow k_\mu \mathcal{M}^\mu(\vec{k}) = \omega \mathcal{M}^0(\vec{k}) - \omega \mathcal{M}^3(\vec{k}) = 0 \Rightarrow \mathcal{M}^0(\vec{k}) = \mathcal{M}^3(\vec{k}) \; .$$

[a] $0 = \int d^4x \; \partial_\mu \left[e^{ikx} \langle f|\, j^\mu(x)\, |i\rangle \right] = ik_\mu \int d^4x \; e^{ikx} \langle f|\, j^\mu(x)\, |i\rangle + \int d^4x \; e^{ikx} \langle f|\, \partial_\mu j^\mu(x)\, |i\rangle$.

Comments | About vector boson polarizations

– The case of a photon.

Therefore we can rewrite (9) as

$$\sum_{\lambda=1,2} \epsilon^*_\mu(\vec{k},\lambda)\epsilon_\nu(\vec{k},\lambda)\mathcal{M}^\mu(\vec{k})\mathcal{M}^{\nu*}(\vec{k})$$
$$= |\mathcal{M}^1(\vec{k})|^2 + |\mathcal{M}^2(\vec{k})|^2 + |\mathcal{M}^3(\vec{k})|^2 - |\mathcal{M}^0(\vec{k})|^2$$

that corresponds to replacing

$$\sum_\lambda \epsilon^*_\mu(\vec{k},\lambda)\epsilon_\nu(\vec{k},\lambda) \to -g_{\mu\nu}$$

(not an equality!)

Comments | About vector boson polarizations

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– The case of a massive vector boson.

A massive vector bosons has three degrees of polarization (one longitudinal and two transverse). In this case we can choose the rest frame, $k^\mu = (M,0,0,0)$ and the polarization states

$$\epsilon^\mu(\vec{k},1) = (0,1,0,0)\ , \quad \epsilon^\mu(\vec{k},2) = (0,0,1,0)\ , \quad \epsilon^\mu(\vec{k},3) = (0,0,0,1)\ .$$

Summing over polarizations,

$$\sum_\lambda \epsilon^*_\mu(\vec{k},\lambda)\epsilon_\nu(\vec{k},\lambda) = -g_{\mu\nu} + Q_{\mu\nu}\ , \quad Q_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\ ,$$

valid in the rest frame.

Comments | About vector boson polarizations

– The case of a massive vector boson.
We can obtain the expression valid for $k^\mu = (k^0, \vec{k})$ with $M^2 = (k^0)^2 - \vec{k}^2$ performing a boost with $\gamma = k^0/M$, $\gamma\vec{\beta} = \vec{k}/M$,

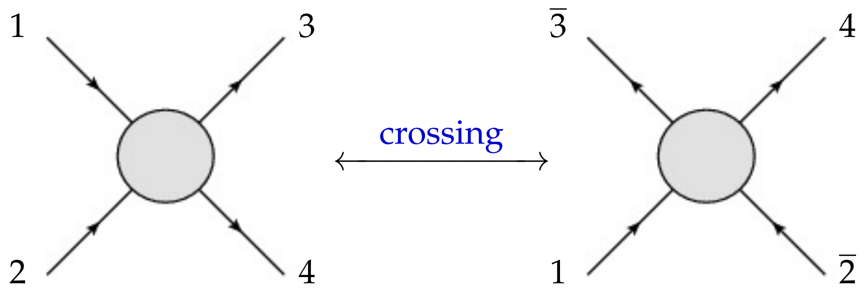
$$\Lambda^\mu{}_{\mu'} = \begin{pmatrix} \gamma & \gamma\beta_1 & \gamma\beta_2 & \gamma\beta_3 \\ \gamma\beta_1 & & & \\ \gamma\beta_2 & & \delta_{ij} + (\gamma - 1)\dfrac{\beta_i\beta_j}{\beta^2} & \\ \gamma\beta_3 & & & \end{pmatrix}$$

leading to

$$\sum_\lambda \epsilon^*_\mu(\vec{k},\lambda)\epsilon_\nu(\vec{k},\lambda) = -g_{\mu\nu} + \Lambda^0{}_\mu\Lambda^0{}_\nu = -g_{\mu\nu} + \frac{k_\mu k_\nu}{M^2} .$$

Comments | Crossing symmetry and Mandelstam variables

- The matrix elements of processes like $1 + 2 \to 3 + 4$ and $1 + \bar{3} \to \bar{2} + 4$ are related by the so called crossing symmetry: the S matrix is the same replacing the momenta accordingly.



$1 + 2 \to 3 + 4$ $\qquad$ $1 + \bar{3} \to \bar{2} + 4$

▷ In this case,

$$k_1,\ k_2 \to p_1,\ p_2 \xleftrightarrow{\text{crossing}} k_1,\ -p_1 \to -k_2,\ p_2$$

Comments | Crossing symmetry and Mandelstam variables

- Before giving some examples of processes whose amplitudes are related by the crossing symmetry, it is convenient to introduce the Mandelstam variables that result very useful to describe the kinematics enabling a simple application of this symmetry.

  For the same process of previous example,

$$\begin{aligned} s &= (k_1+k_2)^2 = (p_1+p_2)^2 \\ t &= (k_1-p_1)^2 = (p_2-k_2)^2 \\ u &= (k_1-p_2)^2 = (p_1-k_2)^2 \end{aligned} \qquad (s,\ t,\ u) \xleftrightarrow[k_2 \leftrightarrow -p_1]{\text{crossing}} (t,\ s,\ u)$$

  It is easy to check that that

$$s+t+u = \sum_i m_i^2$$

  (the sum of squared masses of the four external particles)

Comments | Crossing symmetry and Mandelstam variables

▷ Then, in terms of Mandelstam variables, the kinematics of the process previously studied in detail, $e^+e^- \to \mu^+\mu^-$, reads

$$\begin{aligned} q^2 &= s\,, \\ (p_1k_1) = (p_2k_2) &= (m^2+M^2-t)/2\,, \\ (p_1k_2) = (p_2k_1) &= (m^2+M^2-u)/2\,, \\ (p_1p_2) &= (s-2M^2)/2\,, \\ (k_1k_2) &= (s-2m^2)/2 \end{aligned}$$

leading to

$$\widetilde{\sum_{r_i}}\sum_{s_i} |\mathcal{M}(e^+e^- \to \mu^+\mu^-)|^2 = \frac{8e^4}{s^2}\left[\left(\frac{t}{2}\right)^2 + \left(\frac{u}{2}\right)^2\right].$$

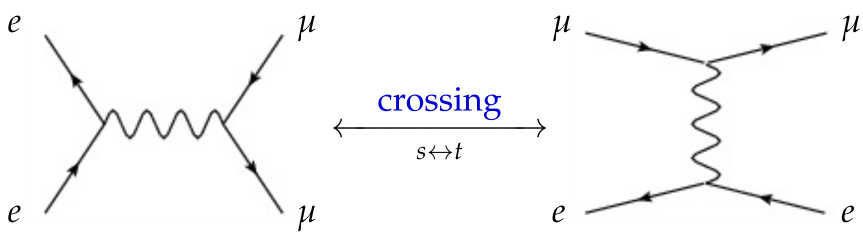
We say that this reaction proceeds through the s channel.

Comments | Crossing symmetry and Mandelstam variables

- The crossing symmetry allows us to write the amplitude of the "crossed" process $e^+\mu^- \to e^+\mu^-$ exchanging s with t in previous expression,

$$\widetilde{\sum_{r_i}}\sum_{s_i} |\mathcal{M}(e^+\mu^- \to e^+\mu^-)|^2 = \frac{8e^4}{t^2}\left[\left(\frac{s}{2}\right)^2 + \left(\frac{u}{2}\right)^2\right] .$$

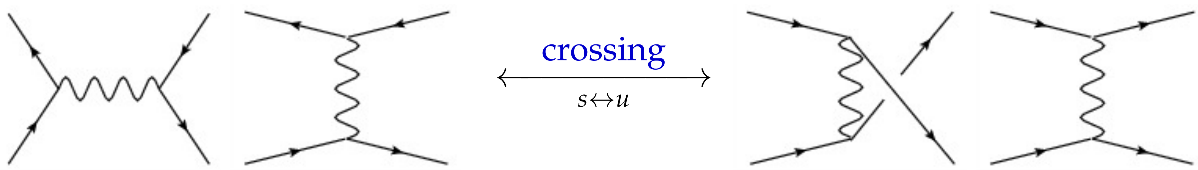
This proceeds through the t channel:



$e^+e^- \to \mu^+\mu^-$ $\qquad$ $e^+\mu^- \to e^+\mu^-$

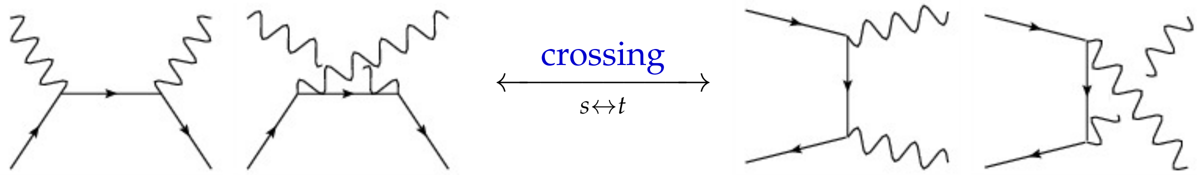
Comments | Crossing symmetry and Mandelstam variables

- Other examples, where two channels contribute, are:



Bhabha $\qquad$ Møller

$e^+e^- \to e^+e^-$ $\qquad$ $e^-e^- \to e^-e^-$



Compton $\qquad$ Annihilation

$\gamma e^- \to \gamma e^-$ $\qquad$ $e^+e^- \to \gamma\gamma$

7. The Standard Model

- Gauge Theories
  - ▷ The gauge symmetry principle
  - ▷ Quantization of gauge theories
- Spontaneous Symmetry Breaking
  - ▷ Discrete symmetry
  - ▷ Continuous symmetry: global *vs* gauge
- **The Standard Model**
  - ▷ Gauge group and particle representations
- Electroweak interactions
  - ▷ Case of one family
  - ▷ Electroweak SSB: Higgs sector, electroweak gauge boson and fermion masses
  - ▷ Additional generations: fermion mixings (quark sector)
  - ▷ Complete Lagrangian and Feynman rules
  - ▷ Electroweak phenomenology
- Strong interactions

Gauge Theories

The symmetry principle | free Lagrangian

- In addition to **spacetime** (Poincaré) symmetries, the free Lagrangian

$$\text{(Dirac)} \qquad \boxed{\mathcal{L}_0 = \overline{\psi}(\mathrm{i}\not{\partial} - m)\psi} \quad \not{\partial} \equiv \gamma^\mu \partial_\mu \ , \quad \overline{\psi} \equiv \psi^\dagger \gamma^0$$

⇒ Invariant under **internal** global U(1) phase transformations:

$$\psi(x) \mapsto \psi'(x) = \mathrm{e}^{-\mathrm{i}Q\theta}\psi(x) \ , \quad Q,\ \theta \text{ (constants)} \in \mathbb{R}$$

⇒ By Noether's theorem, divergentless current:

$$\mathcal{J}^\mu = Q\, \overline{\psi}\gamma^\mu\psi \ , \quad \partial_\mu \mathcal{J}^\mu = 0$$

and a conserved «charge»

$$\mathcal{Q} = \int \mathrm{d}^3x\ \mathcal{J}^0, \quad \partial_t \mathcal{Q} = 0$$

- A $\boxed{\text{quantized}}$ free fermion field:

$$\psi(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_{\vec p}}} \sum_{s=1,2} \left(a_{\vec p,s} u^{(s)}(\vec p) \mathrm{e}^{-\mathrm{i}px} + b^\dagger_{\vec p,s} v^{(s)}(\vec p) \mathrm{e}^{\mathrm{i}px} \right)$$

– is a solution of the Dirac equation (Euler-Lagrange):

$$(\mathrm{i}\partial\!\!\!/ - m)\psi(x) = 0\,, \quad (p\!\!\!/ - m)u(\vec p) = 0\,, \quad (p\!\!\!/ + m)v(\vec p) = 0\,,$$

– is an operator from the canonical quantization rules (anticommutation):

$$\{a_{\vec p,r}, a^\dagger_{\vec k,s}\} = \{b_{\vec p,r}, b^\dagger_{\vec k,s}\} = (2\pi)^3\delta^3(\vec p - \vec k)\delta_{rs}\,, \quad \{a_{\vec p,r}, a_{\vec k,s}\} = \cdots = 0\,,$$

that annihilates/creates particles/antiparticles on the Fock space of fermions

The symmetry principle | free Lagrangian

- For a $\boxed{\text{quantized}}$ free fermion field:

⇒ Normal ordering for fermionic operators (H spectrum bounded from below):

$$:a_{\vec p,r}a^\dagger_{\vec q,s}: \equiv -a^\dagger_{\vec q,s}a_{\vec p,r}\,, \quad :b_{\vec p,r}b^\dagger_{\vec q,s}: \equiv -b^\dagger_{\vec q,s}b_{\vec p,r}$$

⇒ The Noether charge is an operator:*

$$\mathcal{Q} = Q\int \mathrm{d}^3x\ :\bar\psi\gamma^0\psi: = Q\int \frac{\mathrm{d}^3p}{(2\pi)^3} \sum_{s=1,2}\left(a^\dagger_{\vec p,s}a_{\vec p,s} - b^\dagger_{\vec p,s}b_{\vec p,s}\right)$$

$$\mathcal{Q}\ a^\dagger_{\vec k,s}\,|0\rangle = +Q\ a^\dagger_{\vec k,s}\,|0\rangle\ \text{(particle)}\,, \quad \mathcal{Q}\ b^\dagger_{\vec k,s}\,|0\rangle = -Q\ b^\dagger_{\vec k,s}\,|0\rangle\ \text{(antiparticle)}$$

The symmetry principle | gauge invariance dictates interactions

- To make $\mathcal{L}_0$ invariant under local $\equiv$ gauge transformations of U(1):

$$\psi(x) \mapsto \psi'(x) = \mathrm{e}^{-\mathrm{i}Q\theta(x)}\psi(x)\,, \quad \theta = \theta(x) \in \mathbb{R}$$

perform the minimal substitution:

$$\partial_\mu \to D_\mu = \partial_\mu + \mathrm{i}eQA_\mu \quad \text{(covariant derivative)}$$

where a gauge field $A_\mu(x)$ is introduced transforming as:

$$A_\mu(x) \mapsto A'_\mu(x) = A_\mu(x) + \frac{1}{e}\partial_\mu\theta(x) \quad \Leftarrow \quad \boxed{D_\mu\psi \mapsto \mathrm{e}^{-\mathrm{i}Q\theta(x)}D_\mu\psi} \quad \overline{\psi}\not{D}\psi \text{ inv.}$$

$\Rightarrow$ The new Lagrangian contains interactions between ψ and A_μ:

$$\boxed{\mathcal{L}_{\text{int}} = -e\,Q\,\overline{\psi}\gamma^\mu\psi A_\mu} \quad \propto \begin{cases} \text{coupling} & e \\ \text{charge} & Q \end{cases}$$

$$(= -e\,\mathcal{J}^\mu A_\mu)$$

The symmetry principle | gauge invariance dictates interactions

- Dynamics for the gauge field $\Rightarrow$ add gauge invariant kinetic term:

$$\text{(Maxwell)} \qquad \boxed{\mathcal{L}_1 = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}} \qquad \Leftarrow \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \mapsto F_{\mu\nu}$$

- The full U(1) gauge invariant Lagrangian for a fermion field $\psi(x)$ reads:

$$\mathcal{L}_{\text{sym}} = \overline{\psi}(\mathrm{i}\not{D} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \qquad (= \mathcal{L}_0 + \mathcal{L}_{\text{int}} + \mathcal{L}_1)$$

- The same applies to a complex scalar field $\phi(x)$:

$$\mathcal{L}_{\text{sym}} = (D_\mu\phi)^\dagger D^\mu\phi - m^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

The symmetry principle | non-Abelian gauge theories

- A general gauge symmetry group G is an N-dimensional compact Lie group

$$\mathrm{g} \in G\,, \quad \mathrm{g}(\vec{\theta}) = \mathrm{e}^{-\mathrm{i}T^a\theta^a}\,, \quad a = 1,\ldots,N$$

$$\theta^a = \theta^a(x) \in \mathbb{R}\,, \quad T^a = \text{Hermitian generators}\,, \quad [T^a, T^b] = \mathrm{i}f^{abc}T^c \quad \text{(Lie algebra)}$$

$$\mathrm{Tr}\{T^aT^b\} \equiv \tfrac{1}{2}\delta_{ab}\,, \quad \text{structure constants:} \quad \begin{array}{ll} f^{abc} = 0 & \text{Abelian} \\ f^{abc} \neq 0 & \text{non-Abelian} \end{array}$$

$\Rightarrow$ Finite-dimensional irreducible representations are unitary:

$$d\text{-multiplet}: \quad \Psi(x) \mapsto \Psi'(x) = U(\vec{\theta})\Psi(x)\,, \quad \Psi = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_d \end{pmatrix}$$

$$d \times d \text{ matrices}: \quad U(\vec{\theta}) \text{ [given by } \{T^a\} \text{ algebra representation]}$$

The symmetry principle | non-Abelian gauge theories

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- Examples:

G	N	Abelian	
U(1)	1	Yes	
SU(n)	n^2-1	No	($n \times n$ unitary matrices with det $= 1$)

- U(1): 1 generator (Q), one-dimensional irreps only

- SU(2): 3 generators
  $f^{abc} = \epsilon^{abc}$ (Levi-Civita symbol)
  - Fundamental irrep ($d = 2$): $T^a = \frac{1}{2}\sigma^a$ (3 Pauli matrices)
  - Adjoint irrep ($d = N = 3$): $(T^a_{\mathrm{adj}})^{bc} = -\mathrm{i}f^{abc}$

- SU(3): 8 generators
  $f^{123} = 1$, $f^{458} = f^{678} = \frac{\sqrt{3}}{2}$, $f^{147} = f^{156} = f^{246} = f^{247} = f^{345} = -f^{367} = \frac{1}{2}$
  - Fundamental irrep ($d = 3$): $T^a = \frac{1}{2}\lambda^a$ (8 Gell-Mann matrices)
  - Adjoint irrep ($d = N = 8$): $(T^a_{\mathrm{adj}})^{bc} = -\mathrm{i}f^{abc}$

(for SU(n): f^{abc} totally antisymmetric)

- To make $\mathcal{L}_0$ invariant under local $\equiv$ gauge transformations of G:

$$\mathcal{L}_0 = \overline{\Psi}(\mathrm{i}\partial\!\!\!/ - m)\Psi\,, \qquad \Psi(x) \mapsto \Psi'(x) = U(\vec{\theta})\Psi(x)\,, \quad \vec{\theta} = \vec{\theta}(x) \in \mathbb{R}$$

substitute the covariant derivative:

$$\partial_\mu \to D_\mu = \partial_\mu - \mathrm{i}g\widetilde{W}_\mu\,, \quad \widetilde{W}_\mu \equiv T_a W_\mu^a$$

where a gauge field $W_\mu^a(x)$ per generator is introduced, transforming as:

$$\widetilde{W}_\mu(x) \mapsto \widetilde{W}'_\mu(x) = \underbrace{U\widetilde{W}_\mu(x)U^\dagger}_{\text{adjoint irrep}} - \frac{\mathrm{i}}{g}(\partial_\mu U)U^\dagger \quad \Leftarrow \quad \boxed{D_\mu\Psi \mapsto UD_\mu\Psi} \quad \overline{\Psi}D\!\!\!\!/\,\Psi \text{ inv.}$$

$\Rightarrow$ The new Lagrangian contains interactions between Ψ and W_μ^a:

$$\boxed{\mathcal{L}_{\text{int}} = g\,\overline{\Psi}\gamma^\mu T^a\Psi W_\mu^a} \quad \propto \begin{cases} \text{coupling} & g \\ \text{charge} & T^a \end{cases}$$

$$(= g\,\mathcal{J}_a^\mu W_\mu^a)$$

- Dynamics for the gauge fields $\Rightarrow$ add gauge invariant kinetic terms:

$$\text{(Yang-Mills)} \quad \boxed{\mathcal{L}_{\text{YM}} = -\frac{1}{2}\mathrm{Tr}\left\{\widetilde{W}_{\mu\nu}\widetilde{W}^{\mu\nu}\right\} = -\frac{1}{4}W_{\mu\nu}^a W^{a,\mu\nu}}$$

$$\widetilde{W}_{\mu\nu} \equiv T_a W_{\mu\nu}^a \equiv D_\mu\widetilde{W}_\nu - D_\nu\widetilde{W}_\mu = \partial_\mu\widetilde{W}_\nu - \partial_\nu\widetilde{W}_\mu - \mathrm{i}g[\widetilde{W}_\mu, \widetilde{W}_\nu] \;\Leftrightarrow\; \widetilde{W}_{\mu\nu} \mapsto U\widetilde{W}_{\mu\nu}U^\dagger$$

$$\Rightarrow \quad W_{\mu\nu}^a = \partial_\mu W_\nu^a - \partial_\nu W_\mu^a + gf^{abc}W_\mu^b W_\nu^c$$

$\Rightarrow \mathcal{L}_{\text{YM}}$ contains cubic and quartic self-interactions of the gauge fields W_μ^a:

$$\mathcal{L}_{\text{kin}} = -\frac{1}{4}(\partial_\mu W_\nu^a - \partial_\nu W_\mu^a)(\partial^\mu W^{a,\nu} - \partial^\nu W^{a,\mu})$$

$$\mathcal{L}_{\text{cubic}} = -\frac{1}{2}gf^{abc}\,(\partial_\mu W_\nu^a - \partial_\nu W_\mu^a)W^{b,\mu}W^{c,\nu}$$

$$\mathcal{L}_{\text{quartic}} = -\frac{1}{4}g^2 f^{abe}f^{cde}\,W_\mu^a W_\nu^b W^{c,\mu}W^{d,\nu}$$

Quantization of gauge theories | propagators

(particle interpretation of field correlators)

- The (Feynman) propagator of a scalar field:

$$D_F(x-y) = \langle 0| T\{\phi(x)\phi^\dagger(y)\} |0\rangle = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\varepsilon} e^{-ip\cdot(x-y)}$$

(Feynman *prescription* $\varepsilon \to 0^+$)

is a Green's function of the Klein-Gordon operator:

$$(\Box_x + m^2) D_F(x-y) = -i\delta^4(x-y) \quad \Leftrightarrow \quad \widetilde{D}_F(p) = \frac{i}{p^2 - m^2 + i\varepsilon}$$

- The propagator of a fermion field:

$$S_F(x-y) = \langle 0| T\{\psi(x)\overline{\psi}(y)\} |0\rangle = (i\not{\partial}_x + m) \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\varepsilon} e^{-ip\cdot(x-y)}$$

is a Green's function of the Dirac operator:

$$(i\not{\partial}_x - m) S_F(x-y) = i\delta^4(x-y) \quad \Leftrightarrow \quad \widetilde{S}_F(p) = \frac{i}{\not{p} - m + i\varepsilon}$$

Quantization of gauge theories | propagators

- HOWEVER the propagator of a gauge field cannot be defined unless $\mathcal{L}$ is modified:

(e.g. modified Maxwell) $\quad \mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial^\mu A_\mu)^2$

Euler-Lagrange: $\quad \frac{\partial \mathcal{L}}{\partial A_\nu} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = 0 \quad \Rightarrow \quad \left[g^{\mu\nu}\Box - \left(1 - \frac{1}{\xi}\right)\partial^\mu\partial^\nu \right] A_\mu = 0$

– In momentum space the propagator is the inverse of:

$$-k^2 g^{\mu\nu} + \left(1 - \frac{1}{\xi}\right) k^\mu k^\nu \quad \Rightarrow \quad \widetilde{D}_{\mu\nu}(k) = \frac{i}{k^2 + i\epsilon}\left[-g_{\mu\nu} + (1-\xi)\frac{k_\mu k_\nu}{k^2} \right]$$

$\Rightarrow$ Note that $(-k^2 g^{\mu\nu} + k^\mu k^\nu)$ is singular!

$\Rightarrow$ One may argue that $\mathcal{L}$ above will not lead to Maxwell equations ... unless we fix a (Lorenz) gauge where:

$$\partial^\mu A_\mu = 0 \quad \Leftarrow \quad A_\mu \mapsto A'_\mu = A_\mu + \partial_\mu \Lambda \ \text{ with } \ \partial^\mu \partial_\mu \Lambda \equiv -\partial^\mu A_\mu$$

Quantization of gauge theories | gauge fixing (Abelian case)

- The extra term is called Gauge Fixing:

$$\boxed{\mathcal{L}_{\mathrm{GF}} = -\frac{1}{2\xi}(\partial^\mu A_\mu)^2}$$

$\Rightarrow$ modified $\mathcal{L}$ equivalent to Maxwell Lagrangian just in the gauge $\partial^\mu A_\mu = 0$

$\Rightarrow$ the ξ-dependence always cancels out in physical amplitudes

- Several choices for the gauge fixing term (simplify calculations): R_ξ gauges

$$(\text{'t Hooft-Feynman gauge}) \quad \xi = 1: \quad \widetilde{D}_{\mu\nu}(k) = -\frac{\mathrm{i}g_{\mu\nu}}{k^2 + \mathrm{i}\varepsilon}$$

$$(\text{Landau gauge}) \quad \xi = 0: \quad \widetilde{D}_{\mu\nu}(k) = \frac{\mathrm{i}}{k^2 + \mathrm{i}\varepsilon}\left[-g_{\mu\nu} + \frac{k_\mu k_\nu}{k^2}\right]$$

Quantization of gauge theories | gauge fixing (non-Abelian case)

- For a non-Abelian gauge theory, the gauge fixing terms:

$$\boxed{\mathcal{L}_{\mathrm{GF}} = -\sum_a \frac{1}{2\xi_a}(\partial^\mu W^a_\mu)^2}$$

allow to define the propagators:

$$\widetilde{D}^{ab}_{\mu\nu}(k) = \frac{\mathrm{i}\delta_{ab}}{k^2 + \mathrm{i}\epsilon}\left[-g_{\mu\nu} + (1-\xi_a)\frac{k_\mu k_\nu}{k^2}\right]$$

HOWEVER, unlike the Abelian case, this is not the end of the story ...

Quantization of gauge theories — Faddeev-Popov ghosts (⋆)

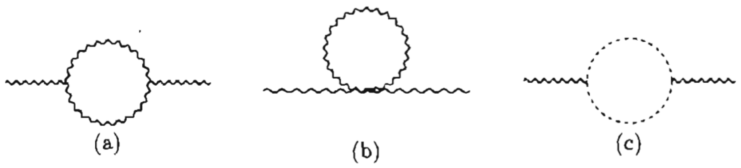
- Add Faddeev-Popov ghost fields $c_a(x)$, $a = 1, \dots, N$:

$$\boxed{\mathcal{L}_{\rm FP} = (\partial^\mu \bar{c}^a)(D_\mu^{\rm adj})^{ab} c^b = (\partial^\mu \bar{c}^a)(\partial_\mu c^a - g f^{abc} c^b W_\mu^c)} \quad \Leftarrow \quad D_\mu^{\rm adj} = \partial_\mu - \mathrm{i} g T^c_{\rm adj} W_\mu^c$$

Computational trick: anticommuting scalar fields, just in loops as virtual particles

$$\widetilde{D}_{ab}(k) = \frac{\mathrm{i}\delta_{ab}}{k^2 + \mathrm{i}\epsilon} \qquad [(-1) \text{ sign for closed loops! (like fermions)}]$$

⇒ Faddeev-Popov ghosts needed to preserve gauge symmetry:



$$= \mathrm{i}(g_{\mu\nu} k^2 - k_\mu k_\nu)\Pi(k^2)$$

⇒ Faddeev-Popov ghosts needed to preserve unitarity at the loop level:

$$q\bar{q} \to q\bar{q}$$

Quantization of gauge theories — complete Lagrangian

- Then the complete quantum Lagrangian is

$$\boxed{\mathcal{L}_{\rm sym} + \mathcal{L}_{\rm GF} + \mathcal{L}_{\rm FP}}$$

⇒ Note that in the case of a massive vector field

$$\text{(Proca)} \qquad \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} M^2 A_\mu A^\mu$$

is not gauge invariant. What about the gauge principle???

– The propagator is:

$$\widetilde{D}_{\mu\nu}(k) = \frac{\mathrm{i}}{k^2 - M^2 + \mathrm{i}\epsilon} \left(-g_{\mu\nu} + \frac{k^\mu k^\nu}{M^2} \right)$$

Spontaneous Symmetry Breaking

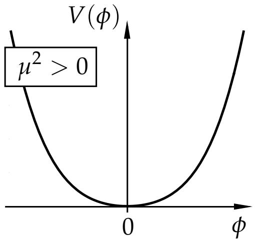
Spontaneous Symmetry Breaking | discrete symmetry

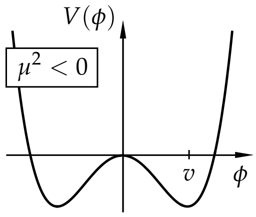
- Consider a real scalar field $\phi(x)$ with Lagrangian:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}\mu^2\phi^2 - \frac{\lambda}{4}\phi^4 \quad \text{invariant under} \quad \phi \mapsto -\phi$$

$$\Rightarrow \quad \mathcal{H} = \frac{1}{2}(\dot{\phi}^2 + (\nabla\phi)^2) + V(\phi)$$

$$V = \frac{1}{2}\mu^2\phi^2 + \frac{1}{4}\lambda\phi^4$$





μ^2, $\lambda \in \mathbb{R}$ (Real/Hermitian Hamiltonian) and $\lambda > 0$ (existence of a ground state)

(a) $\mu^2 > 0$: min of $V(\phi)$ at $\phi_{\text{cl}} = 0$

(b) $\mu^2 < 0$: min of $V(\phi)$ at $\phi_{\text{cl}} = v \equiv \pm\sqrt{\dfrac{-\mu^2}{\lambda}}$, in QFT $\langle 0|\,\phi\,|0\rangle = v \neq 0$ (VEV)

– A quantum field must have $v = 0$

$$a\,|0\rangle = 0$$

$$\Rightarrow \qquad \phi(x) \equiv v + \eta(x)\,, \quad \langle 0|\,\eta\,|0\rangle = 0$$

Spontaneous Symmetry Breaking — discrete symmetry

- At the quantum level, the same system is described by $\eta(x)$ with Lagrangian:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \eta)(\partial^\mu \eta) - \lambda v^2 \eta^2 - \lambda v \eta^3 - \frac{\lambda}{4}\eta^4 + \frac{1}{4}\lambda v^4 \quad \text{not invariant under} \quad \eta \mapsto -\eta$$

$$(m_\eta = \sqrt{2\lambda}\, v)$$

⇒ Lesson:

$\mathcal{L}(\phi)$ had the symmetry but the parameters can be such that the ground state of the Hamiltonian is not symmetric (Spontaneous Symmetry Breaking)

⇒ Note:

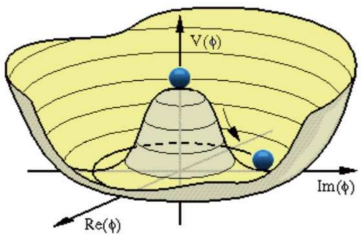
One may argue that $\mathcal{L}(\eta)$ exhibits an explicit breaking of the symmetry. However this is not the case since the coefficients of terms η^2, η^3 and η^4 are determined by just two parameters, λ and v (remnant of the original symmetry)

Spontaneous Symmetry Breaking — continuous symmetry

- Consider a complex scalar field $\phi(x)$ with Lagrangian:

$$\mathcal{L} = (\partial_\mu \phi^\dagger)(\partial^\mu \phi) - \mu^2 \phi^\dagger \phi - \lambda (\phi^\dagger \phi)^2 \quad \text{invariant under U(1):} \quad \phi \mapsto \mathrm{e}^{-\mathrm{i}q\theta}\phi$$

$$\lambda > 0,\ \mu^2 < 0: \quad \langle 0|\,\phi\,|0\rangle \equiv \frac{v}{\sqrt{2}}, \quad |v| = \sqrt{\frac{-\mu^2}{\lambda}}$$



Take $v \in \mathbb{R}^+$. In terms of quantum fields:

$$\phi(x) \equiv \frac{1}{\sqrt{2}}[v + \eta(x) + \mathrm{i}\chi(x)], \quad \langle 0|\,\eta\,|0\rangle = \langle 0|\,\chi\,|0\rangle = 0$$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \eta)(\partial^\mu \eta) + \frac{1}{2}(\partial_\mu \chi)(\partial^\mu \chi) - \lambda v^2 \eta^2 - \lambda v \eta(\eta^2 + \chi^2) - \frac{\lambda}{4}(\eta^2 + \chi^2)^2 + \frac{1}{4}\lambda v^4$$

Note: if $v\mathrm{e}^{\mathrm{i}\alpha}$ (complex) replace η by $(\eta \cos\alpha - \chi \sin\alpha)$ and χ by $(\eta \sin\alpha + \chi \cos\alpha)$

⇒ The actual quantum Lagrangian $\mathcal{L}(\eta, \chi)$ is not invariant under U(1)
U(1) broken ⇒ one scalar field remains massless: $m_\chi = 0$, $m_\eta = \sqrt{2\lambda}\, v$

- Another example: consider a real scalar SU(2) triplet $\Phi(x)$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \Phi^\mathsf{T})(\partial^\mu \Phi) - \frac{1}{2}\mu^2 \Phi^\mathsf{T}\Phi - \frac{\lambda}{4}(\Phi^\mathsf{T}\Phi)^2 \quad \text{inv. under SU(2):} \quad \Phi \mapsto \mathrm{e}^{-\mathrm{i}T^a\theta^a}\Phi$$

that for $\lambda > 0,\ \mu^2 < 0$ acquires a VEV $\langle 0|\,\Phi^\mathsf{T}\Phi\,|0\rangle = v^2 \qquad (\mu^2 = -\lambda v^2)$

Assume $\Phi(x) = \begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \\ v + \varphi_3(x) \end{pmatrix}$ and define $\varphi \equiv \frac{1}{\sqrt{2}}(\varphi_1 + \mathrm{i}\varphi_2)$

$$\mathcal{L} = (\partial_\mu \varphi^\dagger)(\partial^\mu \varphi) + \frac{1}{2}(\partial_\mu \varphi_3)(\partial^\mu \varphi_3) - \lambda v^2 \varphi_3^2 - \lambda v(2\varphi^\dagger\varphi + \varphi_3^2)\varphi_3 - \frac{\lambda}{4}(2\varphi^\dagger\varphi + \varphi_3^2)^2 + \frac{1}{4}\lambda v^4$$

⇒ Not symmetric under SU(2) but invariant under U(1):

$$\varphi \mapsto \mathrm{e}^{-\mathrm{i}q\theta}\varphi \quad (q = \text{arbitrary}) \qquad\qquad \varphi_3 \mapsto \varphi_3 \quad (q = 0)$$

SU(2) broken to U(1) ⇒ $3 - 1 = 2$ broken generators

⇒ 2 (real) scalar fields (= 1 complex) remain massless: $m_\varphi = 0$, $m_{\varphi_3} = \sqrt{2\lambda}\, v$

⇒ Goldstone's theorem: [Nambu '60; Goldstone '61]

The number of massless particles (Nambu-Goldstone bosons) is equal to the number of spontaneously broken generators of the symmetry

$$\text{Hamiltonian symmetric under group } G \quad \Rightarrow \quad [T^a, H] = 0\,, \quad a = 1, \ldots, N$$

$$\text{By definition: } H\,|0\rangle = 0 \quad \Rightarrow \quad H(T^a\,|0\rangle) = T^a H\,|0\rangle = 0$$

– If $|0\rangle$ is such that $T^a\,|0\rangle = 0$ for all generators

⇒ non-degenerate minimum: *the* vacuum

– If $|0\rangle$ is such that $T^{a'}\,|0\rangle \neq 0$ for some (broken) generators a'

⇒ degenerate minimum: chose one (*true* vacuum) and $\mathrm{e}^{-\mathrm{i}T^{a'}\theta^{a'}}\,|0\rangle \neq |0\rangle$

⇒ excitations (particles) from $|0\rangle$ to $\mathrm{e}^{-\mathrm{i}T^{a'}\theta^{a'}}\,|0\rangle$ cost no energy: massless!

Spontaneous Symmetry Breaking | gauge symmetry

- Consider a U(1) gauge invariant Lagrangian for a complex scalar field $\phi(x)$:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^\dagger(D^\mu\phi) - \mu^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2\,, \quad D_\mu = \partial_\mu + \mathrm{i}eQA_\mu$$

inv. under $\phi(x) \mapsto \phi'(x) = \mathrm{e}^{-\mathrm{i}Q\theta(x)}\phi(x)\,, \quad A_\mu(x) \mapsto A'_\mu(x) = A_\mu(x) + \frac{1}{e}\partial_\mu\theta(x)$

If $\lambda > 0$, $\mu^2 < 0$, the $\mathcal{L}$ in terms of quantum fields η and χ with null VEVs:

$$\phi(x) \equiv \frac{1}{\sqrt{2}}[v + \eta(x) + \mathrm{i}\chi(x)]\,, \quad \mu^2 = -\lambda v^2$$

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(\partial_\mu\eta)(\partial^\mu\eta) + \frac{1}{2}(\partial_\mu\chi)(\partial^\mu\chi)\\ &\boxed{-\lambda v^2\eta^2} - \lambda v\eta(\eta^2+\chi^2) - \frac{\lambda}{4}(\eta^2+\chi^2)^2 + \frac{1}{4}\lambda v^4\\ &\boxed{+\,eQvA_\mu\partial^\mu\chi} + eQA_\mu(\eta\partial^\mu\chi - \chi\partial^\mu\eta)\\ &\boxed{+\,\frac{1}{2}(eQv)^2A_\mu A^\mu} + \frac{1}{2}(eQ)^2A_\mu A^\mu(\eta^2 + 2v\eta + \chi^2)\end{aligned}$$

Comments:

(i) $m_\eta = \sqrt{2\lambda}\,v$

$m_\chi = 0$

(ii) $M_A = |eQv|$ (!)

(iii) Term $A_\mu\partial^\mu\chi$ (?)

(iv) Add $\mathcal{L}_{\mathrm{GF}}$

Spontaneous Symmetry Breaking | gauge symmetry

- Removing the cross term and the (new) gauge fixing Lagrangian:

$$\mathcal{L}_{\mathrm{GF}} = -\frac{1}{2\xi}(\partial_\mu A^\mu - \xi M_A\chi)^2$$

$$M_A[\partial_\mu A^\mu\chi + A_\mu\partial^\mu\chi]$$

$$\begin{aligned}\Rightarrow \quad \mathcal{L} + \mathcal{L}_{\mathrm{GF}} = &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}M_A^2A_\mu A^\mu - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 + \overbrace{M_A\partial_\mu(A^\mu\chi)}^{\text{total deriv.}}\\ &+\frac{1}{2}(\partial_\mu\chi)(\partial^\mu\chi) - \frac{1}{2}\xi M_A^2\chi^2 + \ldots\end{aligned}$$

and the propagators of A_μ and χ are:

$$\widetilde{D}_{\mu\nu}(k) = \frac{\mathrm{i}}{k^2 - M_A^2 + \mathrm{i}\varepsilon}\left[-g_{\mu\nu} + (1-\xi)\frac{k_\mu k_\nu}{k^2 - \xi M_A^2}\right]$$

$$\widetilde{D}(k) = \frac{\mathrm{i}}{k^2 - \xi M_A^2 + \mathrm{i}\varepsilon}$$

$\Rightarrow$ χ has a gauge-dependent mass: actually it is not a physical field!

Spontaneous Symmetry Breaking | gauge symmetry

- A more transparent parameterization of the quantum field ϕ is

$$\phi(x) \equiv \mathrm{e}^{\mathrm{i}Q\zeta(x)/v}\frac{1}{\sqrt{2}}[v+\eta(x)]\,, \quad \langle 0|\,\eta\,|0\rangle = \langle 0|\,\zeta\,|0\rangle = 0$$

$$\phi(x) \mapsto \mathrm{e}^{-\mathrm{i}Q\zeta(x)/v}\phi(x) = \frac{1}{\sqrt{2}}[v+\eta(x)] \quad \Rightarrow \quad \zeta \text{ gauged away!}$$

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(\partial_\mu\eta)(\partial^\mu\eta) \\ &-\lambda v^2\eta^2 - \lambda v\eta^3 - \frac{\lambda}{4}\eta^4 + \frac{1}{4}\lambda v^4 \\ &+\frac{1}{2}(eQv)^2 A_\mu A^\mu + \frac{1}{2}(eQ)^2 A_\mu A^\mu (2v\eta + \eta^2)\end{aligned}$$

Comments:

(i) $m_\eta = \sqrt{2\lambda}\, v$

(ii) $M_A = |eQv|$

(iii) No need for $\mathcal{L}_{\mathrm{GF}}$

$\Rightarrow$ This is the unitary gauge ($\xi \to \infty$): just physical fields

$$\widetilde{D}_{\mu\nu}(k) \to \frac{\mathrm{i}}{k^2 - M_A^2 + \mathrm{i}\varepsilon}\left[-g_{\mu\nu} + \frac{k_\mu k_\nu}{M_A^2}\right] \quad \text{and} \quad \widetilde{D}(k) \to 0$$

Spontaneous Symmetry Breaking | gauge symmetry

$\Rightarrow$ Brout-Englert-Higgs mechanism:

[Anderson '62]
[Higgs '64; Englert, Brout '64; Guralnik, Hagen, Kibble '64]

The gauge bosons associated with the spontaneously broken generators become massive, the corresponding would-be Goldstone bosons are unphysical and can be absorbed, the remaining massive scalars (Higgs bosons) are physical (the smoking gun!)

- The would-be Goldstone bosons are 'eaten up' by the gauge bosons ('get fat') and disappear (gauge away) in the unitary gauge ($\xi \to \infty$)
  $\Rightarrow$ Degrees of freedom are preserved

Before SSB:	2 (massless gauge boson)	+ 1 (Goldstone boson)
After SSB:	3 (massive gauge boson)	+ 0 (absorbed would-be Goldstone)

- For loops calculations, 't Hooft-Feynman gauge ($\xi = 1$) is more convenient:
  $\Rightarrow$ Gauge boson propagators are simpler, but
  $\Rightarrow$ Goldstone bosons must be included in internal lines

Spontaneous Symmetry Breaking | gauge symmetry

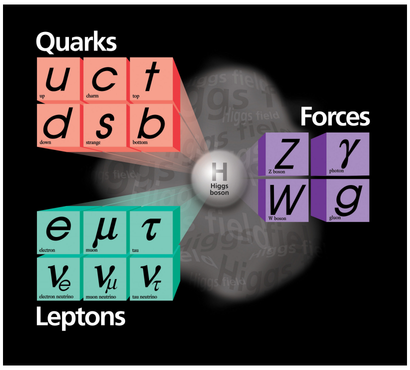
- Comments:

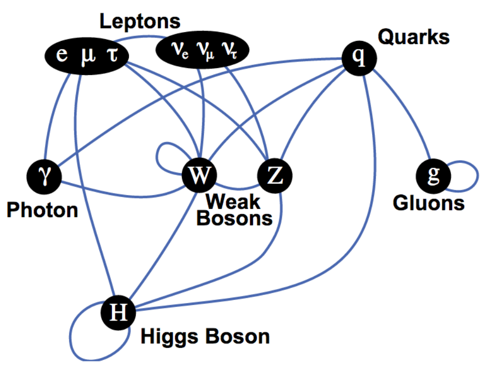
  - After SSB the FP ghost fields (unphysical) acquire a gauge-dependent mass, due to interactions with the scalar field(s):

$$\widetilde{D}_{ab}(k) = \frac{\mathrm{i}\delta_{ab}}{k^2 - \xi_a M^2_{W^a} + \mathrm{i}\varepsilon}$$

  - Gauge theories with SSB are renormalizable ['t Hooft, Veltman '72]

    UV divergences appearing at loop level can be removed by renormalization of parameters and fields of the classical Lagrangian ⇒ predictive!





The Standard Model

SM: Gauge group and particle reps

[Glashow '61; Weinberg '67; Salam '68]
[D. Gross, F. Wilczek; D. Politzer '73]

- The Standard Model is a gauge theory based on the local symmetry group:

$$\underbrace{\mathrm{SU}(3)_c}_{\text{strong}} \otimes \underbrace{\mathrm{SU}(2)_L \otimes \mathrm{U}(1)_Y}_{\text{electroweak}} \to \mathrm{SU}(3)_c \otimes \underbrace{\mathrm{U}(1)_Q}_{\text{em}}$$

with the electroweak symmetry spontaneously broken to the electromagnetic $\mathrm{U}(1)_Q$ symmetry by the Brout-Englert-Higgs mechanism

- The particle (field) content: (ingredients: 12 *flavors* + 12 gauge bosons + H)

		Fermions	I	II	III	Q
spin $\frac{1}{2}$	Quarks	f	uuu	ccc	ttt	$\frac{2}{3}$
		f'	ddd	sss	bbb	$-\frac{1}{3}$
	Leptons	f	ν_e	ν_μ	ν_τ	0
		f'	e	μ	τ	-1

$Q_f = Q_{f'} + 1$

	Bosons	
spin 1	8 gluons	strong interaction
	$W^\pm$, Z	weak interaction
	γ	em interaction
spin 0	Higgs	origin of mass

SM: Gauge group and particle representations

- The fields lay in the following representations (color, weak isospin, hypercharge):

Multiplets	$\mathrm{SU}(3)_c \otimes \mathrm{SU}(2)_L \otimes \mathrm{U}(1)_Y$	I	II	III	$Q = T^3 + Y$
Quarks	$(\mathbf{3}, \mathbf{2}, \frac{1}{6})$	$\begin{pmatrix} u_L \\ d_L \end{pmatrix}$	$\begin{pmatrix} c_L \\ s_L \end{pmatrix}$	$\begin{pmatrix} t_L \\ b_L \end{pmatrix}$	$\frac{2}{3} = \frac{1}{2} + \frac{1}{6}$ $-\frac{1}{3} = -\frac{1}{2} + \frac{1}{6}$
	$(\mathbf{3}, \mathbf{1}, \frac{2}{3})$	u_R	c_R	t_R	$\frac{2}{3} = 0 + \frac{2}{3}$
	$(\mathbf{3}, \mathbf{1}, -\frac{1}{3})$	d_R	s_R	b_R	$-\frac{1}{3} = 0 - \frac{1}{3}$
Leptons	$(\mathbf{1}, \mathbf{2}, -\frac{1}{2})$	$\begin{pmatrix} \nu_{e_L} \\ e_L \end{pmatrix}$	$\begin{pmatrix} \nu_{\mu_L} \\ \mu_L \end{pmatrix}$	$\begin{pmatrix} \nu_{\tau_L} \\ \tau_L \end{pmatrix}$	$0 = \frac{1}{2} - \frac{1}{2}$ $-1 = -\frac{1}{2} - \frac{1}{2}$
	$(\mathbf{1}, \mathbf{1}, -1)$	e_R	μ_R	τ_R	$-1 = 0 - 1$
	$(\mathbf{1}, \mathbf{1}, 0)$	ν_{e_R}	ν_{μ_R}	ν_{τ_R}	$0 = 0 + 0$
Higgs	$(\mathbf{1}, \mathbf{2}, \frac{1}{2})$	(3 families of quarks & leptons)			UNIVERSAL

$\Rightarrow$ Electroweak (QFD): $\mathrm{SU}(2)_L \otimes \mathrm{U}(1)_Y$ Strong (QCD): $\mathrm{SU}(3)_c$

Electroweak interactions

The EWSM with one family (of quarks or leptons)

- Consider two massless fermion fields $f(x)$ and $f'(x)$ with electric charges $Q_f = Q_{f'} + 1$ in three irreps of $\mathrm{SU(2)}_L \otimes \mathrm{U(1)}_Y$:

$$\mathcal{L}_F^0 = \mathrm{i}\overline{f}\partial\!\!\!/ f + \mathrm{i}\overline{f}'\partial\!\!\!/ f' \qquad f_{R,L} = \frac{1}{2}(1 \pm \gamma_5)f\ , \quad f'_{R,L} = \frac{1}{2}(1 \pm \gamma_5)f'$$

$$= \mathrm{i}\overline{\Psi}_1\partial\!\!\!/\Psi_1 + \mathrm{i}\overline{\psi}_2\partial\!\!\!/\psi_2 + \mathrm{i}\overline{\psi}_3\partial\!\!\!/\psi_3 \quad ; \quad \Psi_1 = \underbrace{\begin{pmatrix} f_L \\ f'_L \end{pmatrix}}_{(\mathbf{2}, y_1)}\ , \quad \psi_2 = \underbrace{f_R}_{(\mathbf{1}, y_2)}\ , \quad \psi_3 = \underbrace{f'_R}_{(\mathbf{1}, y_3)}$$

- To get a Langrangian invariant under gauge transformations:

$$\Psi_1(x) \mapsto U_L(x)\mathrm{e}^{-\mathrm{i}y_1\beta(x)}\Psi_1(x), \quad U_L(x) = \mathrm{e}^{-\mathrm{i}T^i\alpha^i(x)}, \quad T^i = \frac{\sigma^i}{2} \quad \text{(weak isospin gen.)}$$

$$\psi_2(x) \mapsto \mathrm{e}^{-\mathrm{i}y_2\beta(x)}\psi_2(x)$$

$$\psi_3(x) \mapsto \mathrm{e}^{-\mathrm{i}y_3\beta(x)}\psi_3(x)$$

The EWSM with one family | covariant derivatives

$\Rightarrow$ Introduce gauge fields $W_\mu^i(x)$ $(i=1,2,3)$ and $B_\mu(x)$ through covariant derivatives:

$$\left.\begin{aligned} D_\mu\Psi_1 &= (\partial_\mu - \mathrm{i}g\widetilde{W}_\mu + \mathrm{i}g'y_1B_\mu)\Psi_1\,, \quad \widetilde{W}_\mu \equiv \frac{\sigma^i}{2}W_\mu^i \\ D_\mu\psi_2 &= (\partial_\mu + \mathrm{i}g'y_2B_\mu)\psi_2 \\ D_\mu\psi_3 &= (\partial_\mu + \mathrm{i}g'y_3B_\mu)\psi_3 \end{aligned}\right\} \quad \Rightarrow \quad \boxed{\mathcal{L}_F} \quad (\not{P}, \not{C})$$

where two couplings g and g' have been introduced and

$$\begin{aligned} \widetilde{W}_\mu(x) &\mapsto U_L(x)\widetilde{W}_\mu(x)U_L^\dagger(x) - \frac{\mathrm{i}}{g}(\partial_\mu U_L(x))U_L^\dagger(x) \\ B_\mu(x) &\mapsto B_\mu(x) + \frac{1}{g'}\partial_\mu\beta(x) \end{aligned}$$

$\Rightarrow$ Add Yang-Mills: gauge invariant kinetic terms for the gauge fields

$$\boxed{\mathcal{L}_{\mathrm{YM}}} = -\frac{1}{4}W_{\mu\nu}^iW^{i,\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu}\,, \quad W_{\mu\nu}^i = \partial_\mu W_\nu^i - \partial_\nu W_\mu^i + g\epsilon^{ijk}W_\mu^jW_\nu^k$$

(include self-interactions of the SU(2) gauge fields) and $B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$

The EWSM with one family | mass terms forbidden

$\Rightarrow$ Note that mass terms are not invariant under SU(2)$_L$⊗U(1)$_Y$, since LH and RH components do not transform the same:

$$m\bar{f}f = m(\overline{f_L}f_R + \overline{f_R}f_L)$$

$\Rightarrow$ Mass terms for the gauge bosons are not allowed either

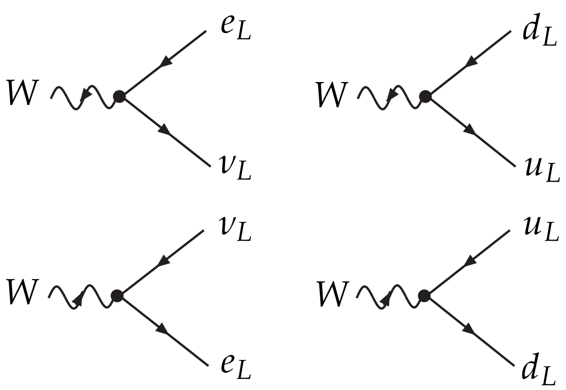
$\Rightarrow$ Next the different types of interactions are analyzed,
and later the EWSB will be discussed

The EWSM with one family | charged current interactions

- $$\mathcal{L}_F \supset g\overline{\Psi}_1\gamma^\mu \widetilde{W}_\mu \Psi_1\,, \quad \widetilde{W}_\mu = \frac{1}{2}\begin{pmatrix} W^3_\mu & \sqrt{2}W^\dagger_\mu \\ \sqrt{2}W_\mu & -W^3_\mu \end{pmatrix}$$

$\Rightarrow$ charged current interactions of LH fermions with complex vector boson field W_μ:

$$\begin{aligned}\mathcal{L}_{\rm CC} &= \frac{g}{\sqrt{2}}\overline{f}_L\gamma^\mu f'_L W^\dagger_\mu + \text{h.c.} \\ &= \frac{g}{2\sqrt{2}}\overline{f}\gamma^\mu(1-\gamma_5)f' W^\dagger_\mu + \text{h.c.}\,, \quad W_\mu \equiv \frac{1}{\sqrt{2}}(W^1_\mu + \mathrm{i}W^2_\mu)\end{aligned}$$



The EWSM with one family | neutral current interactions

- The diagonal part of

$$\mathcal{L}_F \supset g\overline{\Psi}_1\gamma^\mu \widetilde{W}_\mu \Psi_1 - g' B_\mu (y_1\overline{\Psi}_1\gamma^\mu\Psi_1 + y_2\overline{\psi}_2\gamma^\mu\psi_2 + y_3\overline{\psi}_3\gamma^\mu\psi_3)$$

$\Rightarrow$ neutral current interactions with neutral vector boson fields W^3_μ and B_μ

We would like to identify B_μ with the photon field A_μ but that requires:

$$y_1 = y_2 = y_3 \quad \text{and} \quad g' y_j = eQ_j \quad \Rightarrow \quad \text{impossible!}$$

$\Rightarrow$ Since they are both neutral, try a combination:

$$\begin{pmatrix} W^3_\mu \\ B_\mu \end{pmatrix} \equiv \begin{pmatrix} c_W & -s_W \\ s_W & c_W \end{pmatrix}\begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix} \qquad \begin{aligned} s_W \equiv \sin\theta_W\,, \quad c_W \equiv \cos\theta_W \\ \theta_W = \text{weak mixing angle}\end{aligned}$$

$$\mathcal{L}_{\rm NC} = \sum_{j=1}^{3}\overline{\psi}_j\gamma^\mu\left\{-\left[g s_W T^3 + g' c_W y_j\right]A_\mu + \left[g c_W T^3 - g' s_W y_j\right]Z_\mu\right\}\psi_j$$

with $T^3 = \dfrac{\sigma^3}{2}$ (0) the third weak isospin component of the doublet (singlet)

- To make A_μ the photon field:

$$(1)\ \boxed{e = g s_W = g' c_W} \qquad (2)\ \boxed{Q = T^3 + Y}$$

where the electric charge operator is: $Q_1 = \begin{pmatrix} Q_f & 0 \\ 0 & Q_{f'} \end{pmatrix}, \quad Q_2 = Q_f, \quad Q_3 = Q_{f'}$

$\Rightarrow$ (1) Electroweak unification: g of SU(2) and g' of U(1) related to $e = \dfrac{gg'}{\sqrt{g^2 + g'^2}}$

$\Rightarrow$ (2) The hyperchages are fixed in terms of electric charges and weak isospin:

$$y_1 = Q_f - \frac{1}{2} = Q_{f'} + \frac{1}{2}, \qquad y_2 = Q_f, \qquad y_3 = Q_{f'}$$

$$\mathcal{L}_{\text{QED}} = -e\, Q_f \bar{f} \gamma^\mu f\, A_\mu \quad + (f \to f')$$

$\Rightarrow$ RH neutrinos are sterile: $y_2 = Q_f = 0$

The EWSM with one family | neutral current interactions

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- The Z_μ is the neutral weak boson field:

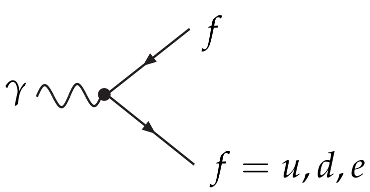
$$\mathcal{L}_{\text{NC}}^Z = e\, \bar{f} \gamma^\mu (v_f - a_f \gamma_5) f\, Z_\mu \quad + (f \to f')$$

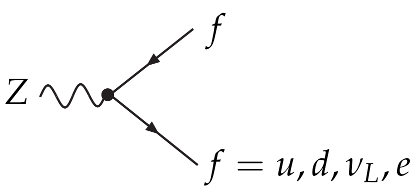
with

$$v_f = \frac{T^3_{f_L} - 2 Q_f s_W^2}{2 s_W c_W}, \qquad a_f = \frac{T^3_{f_L}}{2 s_W c_W}$$

- The complete neutral current Lagrangian reads:

$$\mathcal{L}_{\text{NC}} = \mathcal{L}_{\text{QED}} + \mathcal{L}_{\text{NC}}^Z$$





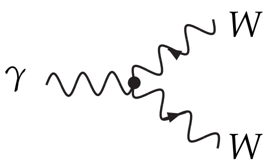
The EWSM with one family — gauge boson self-interactions

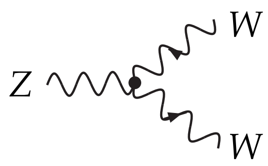
- Cubic:

$$
\begin{aligned}
\mathcal{L}_{\rm YM} \supset \mathcal{L}_3 = & -\frac{\mathrm{i}ec_W}{s_W}\left\{W^{\mu\nu}W^\dagger_\mu Z_\nu - W^\dagger_{\mu\nu}W^\mu Z^\nu - W^\dagger_\mu W_\nu Z^{\mu\nu}\right\} \\
& + \mathrm{i}e\left\{W^{\mu\nu}W^\dagger_\mu A_\nu - W^\dagger_{\mu\nu}W^\mu A^\nu - W^\dagger_\mu W_\nu F^{\mu\nu}\right\}
\end{aligned}
$$

with

$$
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \qquad Z_{\mu\nu} = \partial_\mu Z_\nu - \partial_\nu Z_\mu \qquad W_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu
$$

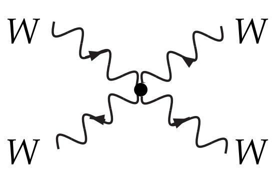


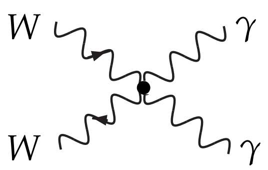


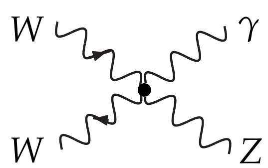
The EWSM with one family — gauge boson self-interactions

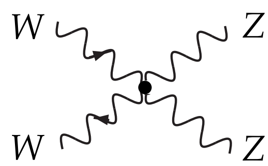
- Quartic:

$$
\begin{aligned}
\mathcal{L}_{\rm YM} \supset \mathcal{L}_4 = & -\frac{e^2}{2s_W^2}\left\{\left(W^\dagger_\mu W^\mu\right)^2 - W^\dagger_\mu W^{\mu\dagger} W_\nu W^\nu\right\} \\
& -\frac{e^2c_W^2}{s_W^2}\left\{W^\dagger_\mu W^\mu Z_\nu Z^\nu - W^\dagger_\mu Z^\mu W_\nu Z^\nu\right\} \\
& +\frac{e^2c_W}{s_W}\left\{2W^\dagger_\mu W^\mu Z_\nu A^\nu - W^\dagger_\mu Z^\mu W_\nu A^\nu - W^\dagger_\mu A^\mu W_\nu Z^\nu\right\} \\
& - e^2\left\{W^\dagger_\mu W^\mu A_\nu A^\nu - W^\dagger_\mu A^\mu W_\nu A^\nu\right\}
\end{aligned}
$$









Note: even number of W and no vertex with just γ or Z

Electroweak symmetry breaking | setup

- Out of the 4 gauge bosons of $\mathrm{SU(2)}_L \otimes \mathrm{U(1)}_Y$ with generators T^1, T^2, T^3, Y we need all to be broken except the combination $Q = T^3 + Y$ so that A_μ remains massless and the other three gauge bosons get massive after SSB

  $\Rightarrow$ Introduce a complex SU(2) Higgs doublet

$$\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} , \qquad \langle 0 | \, \Phi \, | 0 \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}$$

with gauge invariant Lagrangian ($\mu^2 = -\lambda v^2$):

$$\boxed{\mathcal{L}_\Phi} = (D_\mu \Phi)^\dagger D^\mu \Phi - \mu^2 \Phi^\dagger \Phi - \lambda (\Phi^\dagger \Phi)^2 , \qquad D_\mu \Phi = (\partial_\mu - \mathrm{i} g \widetilde{W}_\mu + \mathrm{i} g' y_\Phi B_\mu) \Phi$$

$$\text{take } y_\Phi = \frac{1}{2} \quad \Rightarrow \quad (T^3 + Y) \begin{pmatrix} 0 \\ v \end{pmatrix} = Q \begin{pmatrix} 0 \\ v \end{pmatrix} = 0$$

$$\{T^1, T^2, T^3 - Y\} \begin{pmatrix} 0 \\ v \end{pmatrix} \neq 0$$

Electroweak symmetry breaking | gauge boson masses

- Quantum fields in the unitary gauge:

$$\Phi(x) \equiv \exp\left\{ \mathrm{i} \frac{\sigma^i}{2v} \theta^i(x) \right\} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + H(x) \end{pmatrix}$$

$$\Phi(x) \mapsto \exp\left\{ -\mathrm{i} \frac{\sigma^i}{2v} \theta^i(x) \right\} \Phi(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + H(x) \end{pmatrix} \quad \Rightarrow$$

1 physical Higgs field $H(x)$

3 would-be Goldstones $\theta^i(x)$ gauged away

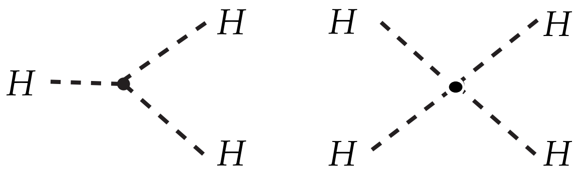
– The 3 dof apparently lost become the longitudinal polarizations of $\mathrm{W}^\pm$ and Z that get massive after SSB:

$$\mathcal{L}_\Phi \supset \mathcal{L}_M = \underbrace{\frac{g^2 v^2}{4}}_{M_W^2} W_\mu^\dagger W^\mu + \underbrace{\frac{g^2 v^2}{8 c_W^2}}_{\frac{1}{2} M_Z^2} Z_\mu Z^\mu \quad \Rightarrow \quad \underbrace{M_W = M_Z c_W}_{\text{custodial symmetry}} = \frac{1}{2} g v$$

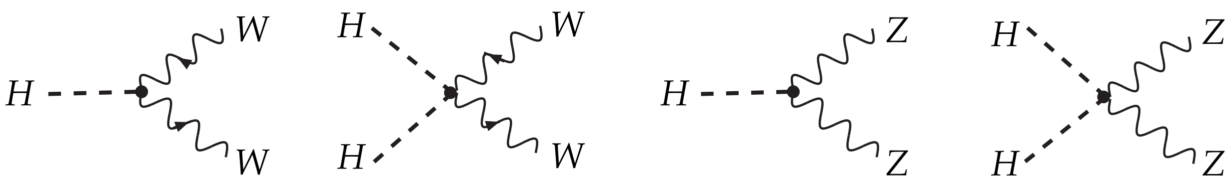
Electroweak symmetry breaking | Higgs sector

⇒ In the unitary gauge (just physical fields): $\mathcal{L}_\Phi = \mathcal{L}_H + \mathcal{L}_M + \mathcal{L}_{HV^2} + \frac{1}{4}\lambda v^4$

$$\mathcal{L}_H = \frac{1}{2}\partial_\mu H \partial^\mu H - \frac{1}{2}M_H^2 H^2 - \frac{M_H^2}{2v}H^3 - \frac{M_H^2}{8v^2}H^4 \,, \quad M_H = \sqrt{-2\mu^2} = \sqrt{2\lambda}\, v$$



$$\mathcal{L}_M + \mathcal{L}_{HV^2} = M_W^2 W_\mu^\dagger W^\mu \left\{1 + \frac{2}{v}H + \frac{H^2}{v^2}\right\} + \frac{1}{2}M_Z^2 Z_\mu Z^\mu \left\{1 + \frac{2}{v}H + \frac{H^2}{v^2}\right\}$$



Electroweak symmetry breaking | Higgs sector

- Quantum fields in the R_ξ gauges:

$$\Phi(x) = \begin{pmatrix} \phi^+(x) \\ \frac{1}{\sqrt{2}}[v + H(x) + \mathrm{i}\chi(x)] \end{pmatrix} \,, \quad \phi^-(x) = [\phi^+(x)]^*$$

$$\begin{aligned} \mathcal{L}_\Phi &= \mathcal{L}_H + \mathcal{L}_M + \mathcal{L}_{HV^2} + \frac{1}{4}\lambda v^4 \\ &+ (\partial_\mu \phi^+)(\partial^\mu \phi^-) + \frac{1}{2}(\partial_\mu \chi)(\partial^\mu \chi) \\ &+ \mathrm{i}M_W\,(W_\mu \partial^\mu \phi^+ - W_\mu^\dagger \partial^\mu \phi^-) + M_Z\, Z_\mu \partial^\mu \chi \\ &+ \text{trilinear interactions [SSS, SSV, SVV]} \\ &+ \text{quadrilinear interactions [SSSS, SSVV]} \end{aligned}$$

Electroweak symmetry breaking | gauge fixing

- To remove the cross terms $W_\mu \partial^\mu \phi^+$, $W^\dagger_\mu \partial^\mu \phi^-$, $Z_\mu \partial^\mu \chi$ and define propagators add:

$$\boxed{\mathcal{L}_{\rm GF}} = -\frac{1}{2\xi_\gamma}(\partial_\mu A^\mu)^2 - \frac{1}{2\xi_Z}(\partial_\mu Z^\mu - \xi_Z M_Z \chi)^2 - \frac{1}{\xi_W}|\partial_\mu W^\mu + \mathrm{i}\xi_W M_W \phi^-|^2$$

$\Rightarrow$ Massive propagators for gauge and (unphysical) would-be Goldstone fields:

$$\widetilde{D}^\gamma_{\mu\nu}(k) = \frac{\mathrm{i}}{k^2+\mathrm{i}\varepsilon}\left[-g_{\mu\nu} + (1-\xi_\gamma)\frac{k_\mu k_\nu}{k^2}\right]$$

$$\widetilde{D}^Z_{\mu\nu}(k) = \frac{\mathrm{i}}{k^2 - M_Z^2+\mathrm{i}\varepsilon}\left[-g_{\mu\nu} + (1-\xi_Z)\frac{k_\mu k_\nu}{k^2-\xi_Z M_Z^2}\right] \quad ; \quad \widetilde{D}^\chi(k) = \frac{\mathrm{i}}{k^2 - \xi_Z M_Z^2 + \mathrm{i}\varepsilon}$$

$$\widetilde{D}^W_{\mu\nu}(k) = \frac{\mathrm{i}}{k^2 - M_W^2+\mathrm{i}\varepsilon}\left[-g_{\mu\nu} + (1-\xi_W)\frac{k_\mu k_\nu}{k^2-\xi_W M_W^2}\right] \quad ; \quad \widetilde{D}^\phi(k) = \frac{\mathrm{i}}{k^2 - \xi_W M_W^2 + \mathrm{i}\varepsilon}$$

('t Hooft-Feynman gauge: $\xi_\gamma = \xi_Z = \xi_W = 1$)

Electroweak symmetry breaking | Faddeev-Popov ghosts (⋆)

- The SM is a non-Abelian theory $\Rightarrow$ add Faddeev-Popov ghosts $c^i(x)$ $(i=1,2,3)$

$$c^1 \equiv \frac{1}{\sqrt{2}}(u_+ + u_-)\,, \quad c^2 \equiv \frac{\mathrm{i}}{\sqrt{2}}(u_+ - u_-)\,, \quad c^3 \equiv c_W\, u_Z - s_W\, u_\gamma$$

$$\mathcal{L}_{\rm FP} = \underbrace{(\partial^\mu \bar{c}^i)(\partial_\mu c^i - g\epsilon^{ijk}c^j W^k_\mu)}_{\text{U kinetic + [UUV]}} + \underbrace{\text{interactions with }\Phi}_{\text{U masses + [SUU]}}$$

$\Rightarrow$ Massive propagators for (unphysical) FP ghost fields:

$$\widetilde{D}^{u_\gamma}(k) = \frac{\mathrm{i}}{k^2+\mathrm{i}\epsilon}\,, \quad \widetilde{D}^{u_Z}(k) = \frac{\mathrm{i}}{k^2 - \xi_Z M_Z^2 + \mathrm{i}\epsilon}\,, \quad \widetilde{D}^{u_\pm}(k) = \frac{\mathrm{i}}{k^2 - \xi_W M_W^2 + \mathrm{i}\epsilon}$$

('t Hooft-Feynman gauge: $\xi_Z = \xi_W = 1$)

Electroweak symmetry breaking | Faddeev-Popov ghosts ($\star$)

$$\mathcal{L}_{\rm FP} = (\partial_\mu \overline{u}_\gamma)(\partial^\mu u_\gamma) + (\partial_\mu \overline{u}_Z)(\partial^\mu u_Z) + (\partial_\mu \overline{u}_+)(\partial^\mu u_+) + (\partial_\mu \overline{u}_-)(\partial^\mu u_-)$$

$$[\text{UUV}] \left\{ \begin{aligned} &+ \mathrm{i}e[(\partial^\mu \overline{u}_+)u_+ - (\partial^\mu \overline{u}_-)u_-]A_\mu - \frac{\mathrm{i}ec_W}{s_W}[(\partial^\mu \overline{u}_+)u_+ - (\partial^\mu \overline{u}_-)u_-]Z_\mu \\ &- \mathrm{i}e[(\partial^\mu \overline{u}_+)u_\gamma - (\partial^\mu \overline{u}_\gamma)u_-]W^\dagger_\mu + \frac{\mathrm{i}ec_W}{s_W}[(\partial^\mu \overline{u}_+)u_Z - (\partial^\mu \overline{u}_Z)u_-]W^\dagger_\mu \\ &+ \mathrm{i}e[(\partial^\mu \overline{u}_-)u_\gamma - (\partial^\mu \overline{u}_\gamma)u_+]W_\mu - \frac{\mathrm{i}ec_W}{s_W}[(\partial^\mu \overline{u}_-)u_Z - (\partial^\mu \overline{u}_Z)u_+]W_\mu \end{aligned} \right.$$

$$- \xi_Z M_Z^2 \, \overline{u}_Z u_Z - \xi_W M_W^2 \, \overline{u}_+ u_+ - \xi_W M_W^2 \, \overline{u}_- u_-$$

$$[\text{SUU}] \left\{ \begin{aligned} &- e\xi_Z M_Z \, \overline{u}_Z \left[\frac{1}{2s_W c_W} H u_Z - \frac{1}{2s_W}(\phi^+ u_- + \phi^- u_+) \right] \\ &- e\xi_W M_W \, \overline{u}_+ \left[\frac{1}{2s_W}(H + \mathrm{i}\chi)u_+ - \phi^+ \left(u_\gamma - \frac{c_W^2 - s_W^2}{2s_W c_W} u_Z \right) \right] \\ &- e\xi_W M_W \, \overline{u}_- \left[\frac{1}{2s_W}(H - \mathrm{i}\chi)u_- - \phi^- \left(u_\gamma - \frac{c_W^2 - s_W^2}{2s_W c_W} u_Z \right) \right] \end{aligned} \right.$$

Electroweak symmetry breaking | fermion masses

- We need masses for quarks and leptons without breaking gauge symmetry

⇒ Introduce Yukawa interactions:

$$\mathcal{L}_{\rm Y} = -\lambda_d \begin{pmatrix} \overline{u}_L & \overline{d}_L \end{pmatrix} \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} d_R - \lambda_u \begin{pmatrix} \overline{u}_L & \overline{d}_L \end{pmatrix} \begin{pmatrix} \phi^{0*} \\ -\phi^- \end{pmatrix} u_R$$
$$- \lambda_\ell \begin{pmatrix} \overline{\nu}_L & \overline{\ell}_L \end{pmatrix} \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \ell_R - \lambda_\nu \begin{pmatrix} \overline{\nu}_L & \overline{\ell}_L \end{pmatrix} \begin{pmatrix} \phi^{0*} \\ -\phi^- \end{pmatrix} \nu_R \quad + \text{h.c.}$$

where $\Phi^c \equiv i\sigma_2 \Phi^* = \begin{pmatrix} \phi^{0*} \\ -\phi^- \end{pmatrix}$ transforms under SU(2) like $\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$

⇒ After EW SSB, fermions acquire masses:

$$\mathcal{L}_{\rm Y} \supset -\frac{1}{\sqrt{2}}(v + H) \left\{ \lambda_d \, \overline{d}d + \lambda_u \, \overline{u}u + \lambda_\ell \, \overline{\ell}\ell + \lambda_\nu \, \overline{\nu}\nu \right\} \quad \Rightarrow \quad m_f = \lambda_f \frac{v}{\sqrt{2}}$$

Additional generations | Yukawa matrices

- There are 3 generations of quarks and leptons in Nature. They are identical copies with the same properties under $SU(2)_L \otimes U(1)_Y$ differing only in their masses

  ⇒ Take a general case of n generations and let u_i^I, d_i^I, ν_i^I, ℓ_i^I be the members of family i ($i = 1, \ldots, n$). Superindex I (interaction basis) was omitted so far

  ⇒ General gauge invariant Yukawa Lagrangian:

$$\boxed{\mathcal{L}_Y} = -\sum_{ij} \left\{ \begin{pmatrix} \bar{u}_{iL}^I & \bar{d}_{iL}^I \end{pmatrix} \left[\begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \lambda_{ij}^{(d)} d_{jR}^I + \begin{pmatrix} \phi^{0*} \\ -\phi^- \end{pmatrix} \lambda_{ij}^{(u)} u_{jR}^I \right] + \begin{pmatrix} \bar{\nu}_{iL}^I & \bar{\ell}_{iL}^I \end{pmatrix} \left[\begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \lambda_{ij}^{(\ell)} \ell_{jR}^I + \begin{pmatrix} \phi^{0*} \\ -\phi^- \end{pmatrix} \lambda_{ij}^{(\nu)} \nu_{jR}^I \right] \right\} + \text{h.c.}$$

  where $\lambda_{ij}^{(d)}$, $\lambda_{ij}^{(u)}$, $\lambda_{ij}^{(\ell)}$, $\lambda_{ij}^{(\nu)}$ are arbitrary Yukawa matrices

⋆ Neutrinos are special: ν_R's do not exist in the SM (then how about oscillations!?). Light ν masses are possible Beyond SM if ν_L and N_R are Majorana fermions or in SMEFT (dim-5 Weinberg operator), involving Lepton Number Violation

Additional generations | mass matrices

- After EW SSB, in n_G-dimensional matrix form:

$$\mathcal{L}_Y \supset -\left(1 + \frac{H}{v}\right) \left\{ \bar{\mathbf{d}}_L^I \, \mathbf{M}_d \, \mathbf{d}_R^I + \bar{\mathbf{u}}_L^I \, \mathbf{M}_u \, \mathbf{u}_R^I + \bar{\mathbf{l}}_L^I \, \mathbf{M}_\ell \, \mathbf{l}_R^I + \bar{\nu}_L^I \, \mathbf{M}_\nu \, \nu_R^I + \text{h.c.} \right\}$$

  with mass matrices

$$(\mathbf{M}_d)_{ij} \equiv \lambda_{ij}^{(d)} \frac{v}{\sqrt{2}} \quad (\mathbf{M}_u)_{ij} \equiv \lambda_{ij}^{(u)} \frac{v}{\sqrt{2}} \quad (\mathbf{M}_\ell)_{ij} \equiv \lambda_{ij}^{(\ell)} \frac{v}{\sqrt{2}} \quad (\mathbf{M}_\nu)_{ij} \equiv \lambda_{ij}^{(\nu)} \frac{v}{\sqrt{2}}$$

  ⇒ Diagonalization determines mass eigenstates d_j, u_j, ℓ_j, ν_j in terms of interaction states d_j^I, u_j^I, ℓ_j^I, ν_j^I, respectively

  ⇒ Each $\mathbf{M}_f$ can be written as

$$\mathbf{M}_f = \mathbf{H}_f \mathcal{U}_f = \mathbf{S}_f^\dagger \mathcal{M}_f \mathbf{S}_f \mathcal{U}_f \quad \Longleftrightarrow \quad \mathbf{M}_f \mathbf{M}_f^\dagger = \mathbf{H}_f^2 = \mathbf{S}_f^\dagger \mathcal{M}_f^2 \mathbf{S}_f$$

  with $\mathbf{H}_f \equiv \sqrt{\mathbf{M}_f \mathbf{M}_f^\dagger}$ a Hermitian positive definite matrix and $\mathcal{U}_f$ unitary

  – Every $\mathbf{H}_f$ can be diagonalized by a unitary matrix $\mathbf{S}_f$

  – The resulting $\mathcal{M}_f$ is diagonal and positive definite

Additional generations | fermion masses and mixings

- In terms of diagonal mass matrices (mass eigenstate basis):

$$\mathcal{M}_d = \mathrm{diag}(m_d, m_s, m_b, \dots)\,, \quad \mathcal{M}_u = \mathrm{diag}(m_u, m_c, m_t, \dots)$$
$$\mathcal{M}_\ell = \mathrm{diag}(m_e, m_\mu, m_\tau, \dots)\,, \quad \mathcal{M}_\nu = \mathrm{diag}(m_{\nu_e}, m_{\nu_\mu}, m_{\nu_\tau}, \dots)$$

$$\mathcal{L}_Y \supset -\left(1+\frac{H}{v}\right)\left\{\bar{\mathbf{d}}\,\mathcal{M}_d\,\mathbf{d} + \bar{\mathbf{u}}\,\mathcal{M}_u\,\mathbf{u} + \bar{\mathbf{l}}\,\mathcal{M}_\ell\,\mathbf{l} + \bar{\nu}\,\mathcal{M}_\nu\,\nu\right\}$$

where fermion couplings to Higgs are proportional to masses and

$$\mathbf{d}_L \equiv \mathbf{S}_d\,\mathbf{d}_L^I \qquad \mathbf{u}_L \equiv \mathbf{S}_u\,\mathbf{u}_L^I \qquad \mathbf{l}_L \equiv \mathbf{S}_\ell\,\mathbf{l}_L^I \qquad \nu_L \equiv \mathbf{S}_\nu\,\nu_L^I$$
$$\mathbf{d}_R \equiv \mathbf{S}_d\mathcal{U}_d\,\mathbf{d}_R^I \qquad \mathbf{u}_R \equiv \mathbf{S}_u\mathcal{U}_u\,\mathbf{u}_R^I \qquad \mathbf{l}_R \equiv \mathbf{S}_\ell\mathcal{U}_\ell\,\mathbf{l}_R^I \qquad \nu_R \equiv \mathbf{S}_\nu\mathcal{U}_\nu\,\nu_R^I$$

$$\left.\begin{array}{l}\text{Neutral Currents preserve chirality}\\ \bar{\mathbf{f}}_L^I\,\mathbf{f}_L^I = \bar{\mathbf{f}}_L\,\mathbf{f}_L \text{ and } \bar{\mathbf{f}}_R^I\,\mathbf{f}_R^I = \bar{\mathbf{f}}_R\,\mathbf{f}_R\end{array}\right\} \Rightarrow \mathcal{L}_{NC} \text{ does not change flavor}$$

$\Rightarrow$ GIM mechanism [Glashow, Iliopoulos, Maiani '70]

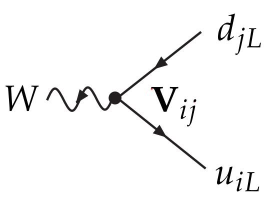
Additional generations | quark sector

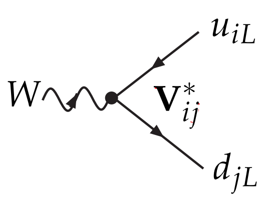
- However, in Charged Currents (also chirality preserving and only LH):

$$\bar{\mathbf{u}}_L^I\,\mathbf{d}_L^I = \bar{\mathbf{u}}_L\,\mathbf{S}_u\,\mathbf{S}_d^\dagger\,\mathbf{d}_L = \bar{\mathbf{u}}_L\mathbf{V}\mathbf{d}_L$$

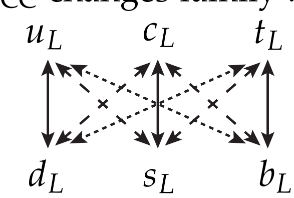
with $\mathbf{V} \equiv \mathbf{S}_u\,\mathbf{S}_d^\dagger$ the (unitary) CKM mixing matrix [Cabibbo '63; Kobayashi, Maskawa '73]

$$\Rightarrow \quad \mathcal{L}_{CC} = \frac{g}{2\sqrt{2}}\sum_{ij}\bar{u}_i\gamma^\mu(1-\gamma_5)\,\mathbf{V}_{ij}\,d_j\,W_\mu^\dagger + \text{h.c.}$$





$\mathcal{L}_{CC}$ changes family !!



$\Rightarrow$ If u_i or d_j had degenerate masses one could choose $\mathbf{S}_u = \mathbf{S}_d$ (field redefinition) and quark families would not mix. But they are *not degenerate*, so they mix!

$\Rightarrow$ $\mathbf{S}_u$ and $\mathbf{S}_d$ are not observable. Just masses and CKM mixings are observable

Additional generations | quark sector

- How many physical parameters in this sector?
  - Quark masses and CKM mixings determined by mass (or Yukawa) matrices
  - A general $n_G \times n_G$ unitary matrix, like the CKM, is given by

$$n_G^2 \text{ real parameters} = n_G(n_G-1)/2 \text{ moduli} + n_G(n_G+1)/2 \text{ phases}$$

Some phases are unphysical since they can be absorbed by field redefinitions:

$$u_i \to e^{i\phi_i}\, u_i\,, \quad d_j \to e^{i\theta_j}\, d_j \quad \Rightarrow \quad \mathbf{V}_{ij} \to \mathbf{V}_{ij}\, e^{i(\theta_j-\phi_i)}$$

Therefore $2n_G - 1$ unphysical phases and the physical parameters are:

$$\boxed{(n_G-1)^2 = n_G(n_G-1)/2 \text{ moduli } + (n_G-1)(n_G-2)/2 \text{ phases}}$$

Additional generations | quark sector

$\Rightarrow$ Case of $n_G = 2$ generations: 1 parameter, the Cabibbo angle θ_C:

$$\mathbf{V} = \begin{pmatrix} \cos\theta_C & \sin\theta_C \\ -\sin\theta_C & \cos\theta_C \end{pmatrix}$$

$\Rightarrow$ Case of $n_G = 3$ generations: 3 angles + 1 phase. In the standard parameterization:

$$\mathbf{V} = \begin{pmatrix} \mathbf{V}_{ud} & \mathbf{V}_{us} & \mathbf{V}_{ub} \\ \mathbf{V}_{cd} & \mathbf{V}_{cs} & \mathbf{V}_{cb} \\ \mathbf{V}_{td} & \mathbf{V}_{ts} & \mathbf{V}_{tb} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta_{13}} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{13}} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{13}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{13}} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{13}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta_{13}} & c_{23}c_{13} \end{pmatrix} \Rightarrow$$

δ_{13} only source of CP violation in the SM !

with $c_{ij} \equiv \cos\theta_{ij} \geq 0, \quad s_{ij} \equiv \sin\theta_{ij} \geq 0 \quad (i < j = 1,2,3)$ and $0 \leq \delta_{13} \leq 2\pi$

Complete EWSM Lagrangian | fields and interactions

$$\boxed{\mathcal{L} = \mathcal{L}_F + \mathcal{L}_{\rm YM} + \mathcal{L}_\Phi + \mathcal{L}_Y + \mathcal{L}_{\rm GF} + \mathcal{L}_{\rm FP}}$$

$$\begin{aligned} \mathcal{L}_F &\supset \mathcal{L}_{\rm CC} + \mathcal{L}_{\rm NC} \\ \mathcal{L}_{\rm YM} &\supset \mathcal{L}_{VVV} + \mathcal{L}_{VVVV} \\ \mathcal{L}_\Phi &\supset \text{gauge boson masses} \\ \mathcal{L}_Y &\supset \text{fermion masses and mixings} \end{aligned}$$

- Fields: [F] fermions; [S] scalars (Higgs and unphysical Goldstones); [V] vector bosons; [U] unphysical ghosts

- Interactions: [FFV] [FFS] [SSV] [SVV] [SSVV]
  [VVV] [VVVV] [SSS] [SSSS]
  [SUU] [UUVV]

Complete EWSM Lagrangian | Feynman rules

- Feynman rules for generic couplings normalized to e (all momenta incoming):

$(\mathrm{i}\mathcal{L})$			
	$[\mathrm{FFV}_\mu]$	$\mathrm{i}e\gamma^\mu(g_V - g_A\gamma_5) = \mathrm{i}e\gamma^\mu(g_L P_L + g_R P_R)$	
	[FFS]	$\mathrm{i}e(g_S - g_P\gamma_5) = \mathrm{i}e(c_L P_L + c_R P_R)$	
	$[\mathrm{SV}_\mu \mathrm{V}_\nu]$	$\mathrm{i}eKg_{\mu\nu}$	
	$[\mathrm{S}(p_1)\mathrm{S}(p_2)\mathrm{V}_\mu]$	$\mathrm{i}eG(p_1-p_2)_\mu$	
	$[\mathrm{V}_\mu(k_1)\mathrm{V}_\nu(k_2)\mathrm{V}_\rho(k_3)]$	$\mathrm{i}eJ\left[g_{\mu\nu}(k_2-k_1)_\rho + g_{\nu\rho}(k_3-k_2)_\mu + g_{\mu\rho}(k_1-k_3)_\nu\right]$	
	$[\mathrm{V}_\mu(k_1)\mathrm{V}_\nu(k_2)\mathrm{V}_\rho(k_3)\mathrm{V}_\sigma(k_4)]$	$\mathrm{i}e^2C\left[2g_{\mu\nu}g_{\rho\sigma} - g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right]$	
	$[\mathrm{SSV}_\mu\mathrm{V}_\nu]$	$\mathrm{i}e^2C_2 g_{\mu\nu}$	also [UUVV]
	[SSS]	$\mathrm{i}eC_3$	also [SUU]
	[SSSS]	$\mathrm{i}e^2C_4$	

Note: $g_{L,R} = g_V \pm g_A$ Attention to symmetry factors!

$c_{L,R} = g_S \pm g_P$ *e.g.* $2\times HZZ$

http://www.ugr.es/local/jillana/SM/FeynmanRulesSM.pdf

EW phenomenology | Input parameters

- Parameters: (${}^*\nu$SM)

$17 + 9^* =$	1	1	1	1	$9 + 3^*$	4	6^*
formal:	g	g'	v	λ	λ_f	$\mathbf{V}_{\text{CKM}}$	$\mathbf{U}_{\text{PMNS}}$
practical:	α	M_W	M_Z	M_H	m_f		

where $e = g s_W = g' c_W$ and

$$\underbrace{\alpha = \frac{e^2}{4\pi} \qquad M_W = \frac{1}{2} g v \qquad M_Z = \frac{M_W}{c_W}}_{g,\ g',\ v} \qquad M_H = \sqrt{2\lambda}\, v \qquad m_f = \frac{v}{\sqrt{2}} \lambda_f$$

$\Rightarrow$ Many (more) experiments

$\Rightarrow$ After Higgs discovery, for the first time *all* parameters measured!

EW phenomenology | Input parameters

- Experimental values [Particle Data Group '20]
  - Fine structure constant:

    $\alpha^{-1} = 137.035\,999\,150\,(33)$ Harvard cyclotron (g_e) [1712.06060]

    $\alpha^{-1} = 137.035\,999\,046\,(27)$ atom interferometry (Cesium) [1812.04130]
    $\alpha^{-1} = 137.035\,999\,206\,(11)$ atom interferometry (Rubidium) [Nature 588, 61(2020)]

  - The SM predicts $M_W < M_Z$ in agreement with measurements:

    $M_Z = (91.1876 \pm 0.0021)$ GeV LEP1/SLD

    $M_W = (80.379 \pm 0.012)$ GeV LEP2/Tevatron/LHC

  - Top quark mass:

    $m_t = (172.76 \pm 0.30)$ GeV Tevatron/LHC

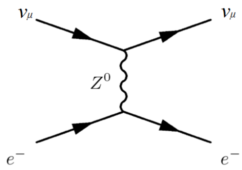
  - Higgs boson mass:

    $M_H = (125.25 \pm 0.17)$ GeV LHC

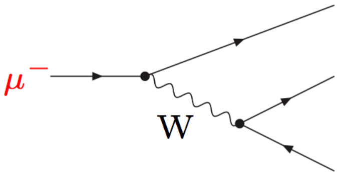
  - ...

EW phenomenology | Observables and experiments

Weak NC discovery (1973)



- Low energy observables ($Q^2 \ll M_Z^2$)
  - ν-nucleon (NuTeV) and νe (CERN) scattering asymmetries CC/NC and $\nu/\bar\nu$ $\Rightarrow \boxed{s_W^2}$
  - Parity and Atomic Parity violation (SLAC, CERN, Jefferson Lab, Mainz) LR asymmetries $e_{R,L}N \to eX$ and Z effects on atomic transitions $\Rightarrow \boxed{s_W^2}$
  - muon decay: $\mu \to e\,\overline{\nu}_e \nu_\mu$ (PSI) lifetime



$$\frac{1}{\tau_\mu} = \Gamma_\mu = \frac{G_F^2 m_\mu^5}{192\pi^3}\, f(m_e^2/m_\mu^2)$$

$$f(x) \equiv 1 - 8x + 8x^3 - x^4 - 12x^2 \ln x = 0.99981295 \qquad \Rightarrow \boxed{G_F}$$

$$\mathrm{i}\mathcal{M} = \left(\frac{\mathrm{i}e}{\sqrt{2}s_W}\right)^2 \bar{e}\gamma^\rho \nu_L \frac{-\mathrm{i}g_{\rho\delta}}{q^2 - M_W^2} \overline{\nu_L}\gamma^\delta \mu \equiv \overbrace{-\mathrm{i}\frac{4G_F}{\sqrt{2}} (\bar{e}\gamma^\rho \nu_L)(\overline{\nu_L}\gamma_\rho \mu)}^{\text{Fermi theory } (-q^2 \ll M_W^2)} \; ; \; \frac{G_F}{\sqrt{2}} = \frac{\pi\alpha}{2 s_W^2 M_W^2}$$

EW phenomenology | Observables and experiments

- Low energy observables

$\Rightarrow$ Fermi constant provides the Higgs VEV (electroweak scale):

$$v = \left(\sqrt{2}G_F\right)^{-1/2} \approx 246\,\text{GeV}$$

and constrains the product $M_W^2 s_W^2$, which implies

$$M_Z^2 > M_W^2 = \frac{\pi\alpha}{\sqrt{2}G_F s_W^2} > \frac{\pi\alpha}{\sqrt{2}G_F} \approx (37.4\text{ GeV})^2$$

$\Rightarrow$ Consistency checks: e.g. from muon lifetime:

$$G_F = 1.166\,378\,7(6) \times 10^{-5}\ \text{GeV}^{-2}$$

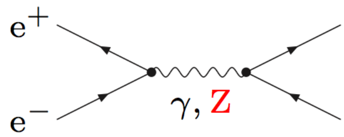
If one compares with (tree level result)

$$\frac{G_F}{\sqrt{2}} = \frac{\pi\alpha}{2s_W^2 M_W^2} = \frac{\pi\alpha}{2(1 - M_W^2/M_Z^2)M_W^2} \approx 1.125 \times 10^{-5}$$

a discrepancy that disappears when *quantum corrections* are included

EW phenomenology | Observables and experiments

- $e^+e^- \to \bar{f}f$ (PEP, PETRA, TRISTAN, ..., LEP1, SLD)



$$\frac{d\sigma}{d\Omega} = N_c^f \frac{\alpha^2}{4s}\beta_f \Big\{ \Big[1+\cos^2\theta + (1-\beta_f^2)\sin^2\theta\Big]\, G_1(s) + 2(\beta_f^2-1)\, G_2(s) + 2\beta_f\cos\theta\, G_3(s)\Big\}$$

$$G_1(s) = Q_e^2Q_f^2 + 2Q_eQ_fv_ev_f\mathrm{Re}\chi_Z(s) + (v_e^2+a_e^2)(v_f^2+a_f^2)|\chi_Z(s)|^2$$

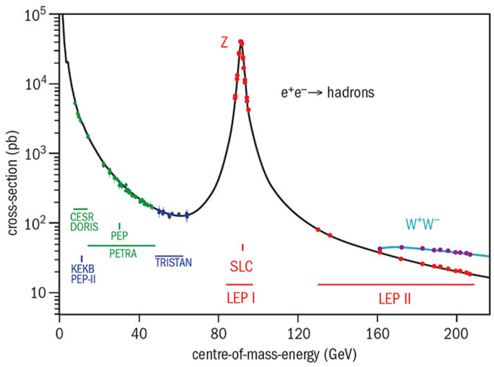
$$G_2(s) = (v_e^2+a_e^2)a_f^2|\chi_Z(s)|^2$$

$$G_3(s) = 2Q_eQ_fa_ea_f\mathrm{Re}\chi_Z(s) + 4v_ev_fa_ea_f|\chi_Z(s)|^2 \quad \Rightarrow \quad \boxed{A_{FB}(s)}$$

$$\chi_Z(s) \equiv \frac{s}{s-M_Z^2+iM_Z\Gamma_Z},\ N_c^f = 1\ (3) \text{ for } f = \text{lepton (quark)}$$

$$\sigma(s) = N_c^f\frac{2\pi\alpha^2}{3s}\beta_f\Big[(3-\beta_f^2)G_1(s) - 3(1-\beta_f^2)G_2(s)\Big]$$

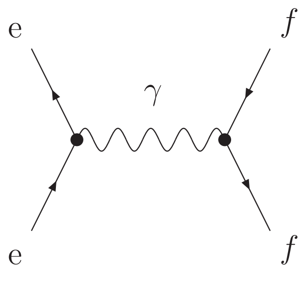
$$\beta_f = \sqrt{1-4m_f^2/s}$$



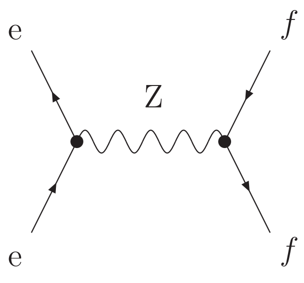
EW phenomenology | $e^+e^- \to \bar{f}f$ ($f \neq e$) in the SM

Unitary gauge:

$$\mathcal{M} = \mathcal{M}_\gamma + \mathcal{M}_Z$$



$$i\mathcal{M}_\gamma = \bar{u}(p_4)\ (-ieQ_f)\gamma^\mu\ v(p_3)\frac{-ig_{\mu\nu}}{s}\ \bar{v}(p_1)\ (-ieQ_e)\gamma^\nu\ u(p_2)$$



$$\begin{aligned} i\mathcal{M}_Z &= \bar{u}(p_4)\ ie\gamma^\mu(v_f-a_f\gamma_5)\ v(p_3)\frac{-i(g_{\mu\nu}-k^\mu k^\nu/M_Z^2)}{s-M_Z^2+iM_Z\Gamma_Z} \\ &\quad \times\bar{v}(p_1)\ ie\gamma^\nu(v_e-a_e\gamma_5)\ u(p_2) \end{aligned}$$

Note: The term with $k^\mu k^\nu$ is irrelevant in the $m_e = 0$ limit

EW phenomenology | $e^+e^- \to \bar{f}f$ ($f \neq e$) in the SM

The differential cross section in the CoM (non polarized case and $m_e = 0$) is:

$$\frac{d\sigma}{d\Omega} = N_c^f \frac{\alpha^2}{4s}\beta_f \left\{ \left[1+\cos^2\theta + (1-\beta_f^2)\sin^2\theta\right] G_1(s) + 2(\beta_f^2-1)G_2(s) + 2\beta_f \cos\theta G_3(s) \right\},$$

$$\begin{aligned} G_1(s) &= Q_e^2Q_f^2 + 2Q_eQ_fv_ev_f\mathrm{Re}\chi_Z(s) + (v_e^2+a_e^2)(v_f^2+a_f^2)|\chi_Z(s)|^2, \\ G_2(s) &= (v_e^2+a_e^2)a_f^2|\chi_Z(s)|^2, \\ G_3(s) &= 2Q_eQ_fa_ea_f\mathrm{Re}\chi_Z(s) + 4v_ev_fa_ea_f|\chi_Z(s)|^2, \end{aligned}$$

where $\chi_Z(s) \equiv \dfrac{s}{s - M_Z^2 + iM_Z\Gamma_Z}$ and $N_c^f = 1$ (3) for $f =$ lepton (quark).

It is easy to identify the terms coming form γ, Z exchange or the interference.
The total cross section and the forward-backward asymmetry in the CoM are

$$\sigma = N_c^f \frac{2\pi\alpha^2}{3s}\beta_f \left[(3-\beta_f^2)G_1(s) - 3(1-\beta_f^2)G_2(s)\right],$$

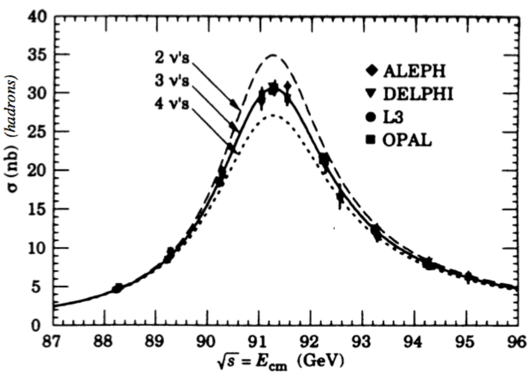
$$A_{FB} = \frac{6\beta_f G_3(s)}{4(3-\beta_f^2)G_1(s) + 12(\beta_f^2-1)G_2(s)} \to \frac{3}{4}\frac{G_3(s)}{G_1(s)} = \frac{3}{4}\frac{2v_ea_e}{v_e^2+a_e^2}\frac{2v_fa_f}{v_f^2+a_f^2} \quad \text{when } \beta_f \to 1$$

EW phenomenology | Observables and experiments

- Z pole observables (LEP1/SLD)

  $M_Z, \Gamma_Z, \sigma_{\text{had}}, A_{FB}, A_{LR}, R_b, R_c, R_\ell \quad \Rightarrow \boxed{M_Z, s_W^2}$

  from $e^+e^- \to \bar{f}f$ at the Z pole ($\gamma - Z$ interference vanishes). Neglecting m_f:



$$\sigma_{\text{had}} = 12\pi \frac{\Gamma(e^+e^-)\Gamma(\text{had})}{M_Z^2\Gamma_Z^2}$$

$$R_b = \frac{\Gamma(b\bar{b})}{\Gamma(\text{had})} \quad R_c = \frac{\Gamma(c\bar{c})}{\Gamma(\text{had})} \quad R_\ell = \frac{\Gamma(\text{had})}{\Gamma(\ell^+\ell^-)}$$

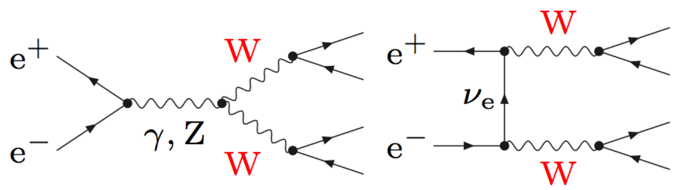
$$\left[\Gamma(Z \to f\bar{f}) \equiv \Gamma(f\bar{f}) = N_c^f \frac{\alpha M_Z}{3}(v_f^2 + a_f^2)\right]$$

Forward-Backward and (if polarized e^-) Left-Right asymmetries due to Z:

$$A_{FB} = \frac{\sigma_F - \sigma_B}{\sigma_F + \sigma_B} = \frac{3}{4}A_f\frac{A_e + P_e}{1 + P_eA_e} \qquad A_{LR} = \frac{\sigma_L - \sigma_R}{\sigma_L + \sigma_R} = A_eP_e \quad \text{with } A_f \equiv \frac{2v_fa_f}{v_f^2+a_f^2}$$

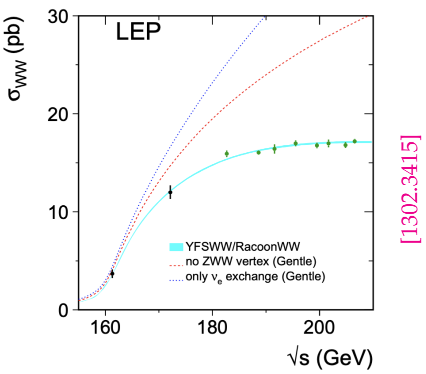
EW phenomenology | Observables and experiments

- W-pair production (LEP2)
  $e^+e^- \to WW \to 4\,f\ (+\gamma)$

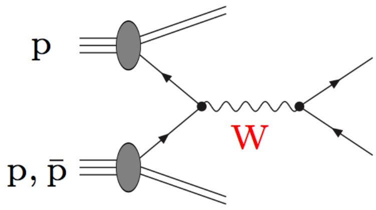


$\Rightarrow \boxed{M_W}$

$\Downarrow$

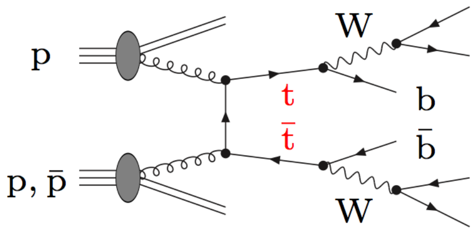


- W production (Tevatron/LHC)
  $pp/p\bar{p} \to W \to \ell\nu_\ell\ (+\gamma)$



$\Rightarrow \boxed{M_W}$

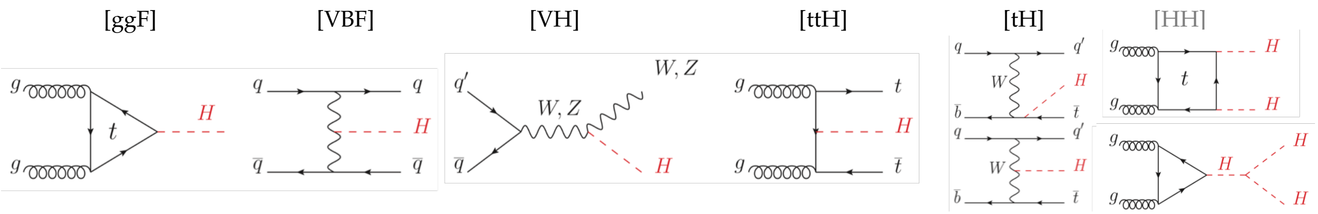
- Top-quark production (Tevatron/LHC)
  $pp/p\bar{p} \to t\bar{t} \to 6\,f$

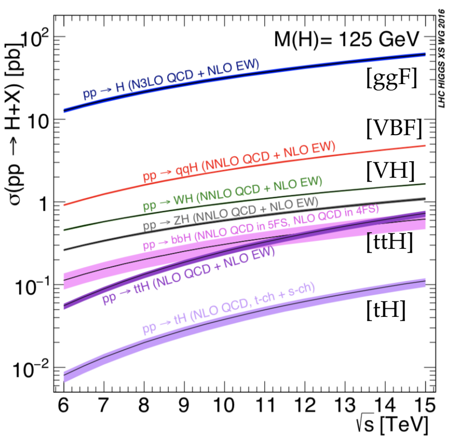


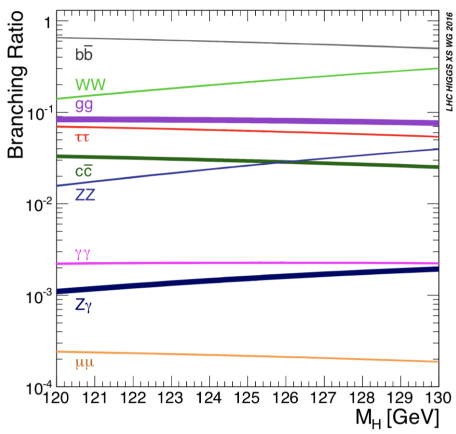
$\Rightarrow \boxed{m_t}$

EW phenomenology | Observables and experiments

- Higgs (LHC)
  Single and Double H production and decay to different channels $\Rightarrow \boxed{M_H}$







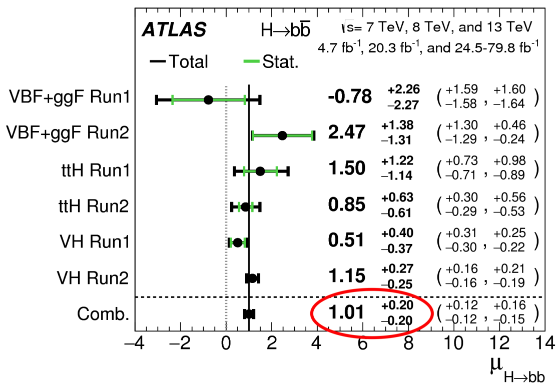
EW phenomenology | Observables and experiments

- Higgs (LHC) [PDG '20]

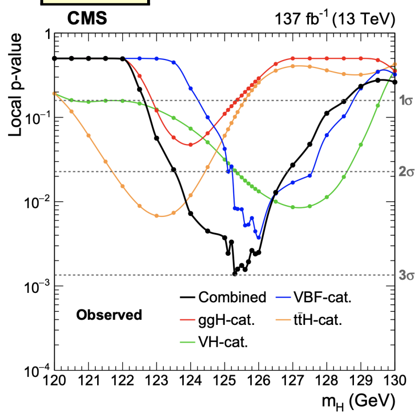
Signal strength $\mu = \frac{(\sigma \cdot \mathrm{BR})_{\mathrm{obs}}}{(\sigma \cdot \mathrm{BR})_{\mathrm{SM}}}$	Run 1	Run 2
ATLAS	1.17 ± 0.27	1.02 ± 0.14
CMS	$1.18^{+0.26}_{-0.23}$	$1.18^{+0.17}_{-0.14}$

Per channel:
$\gamma\gamma$, ZZ, W^+W^-, $\tau^+\tau^-$ $> 5\sigma$

$\boxed{b\bar{b}}$ $> 5\sigma$ [Jul '18!]



$\boxed{\mu^+\mu^-}$ $\sim 3\sigma$ [Jul '20!!]



EW phenomenology | Observables and experiments

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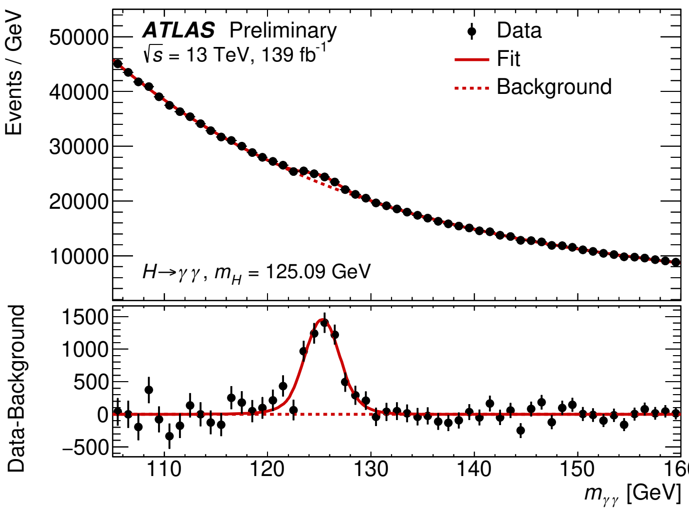
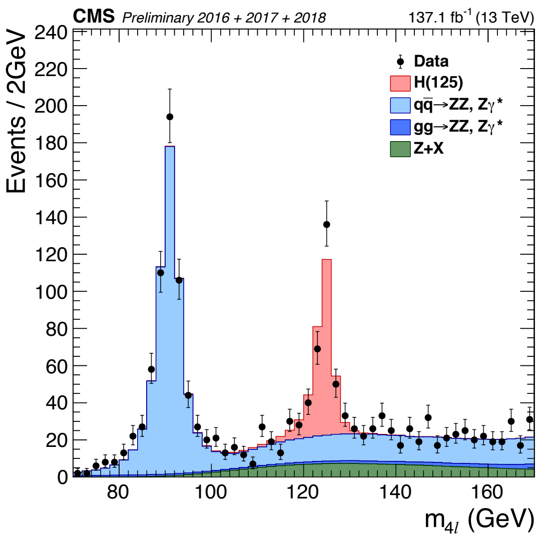
- Higgs mass (LHC) [PDG '20]

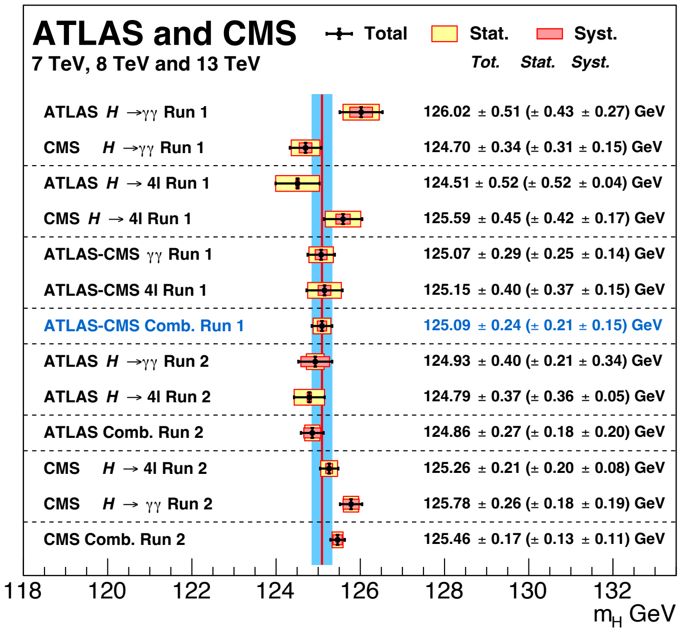
$H \to \gamma\gamma$



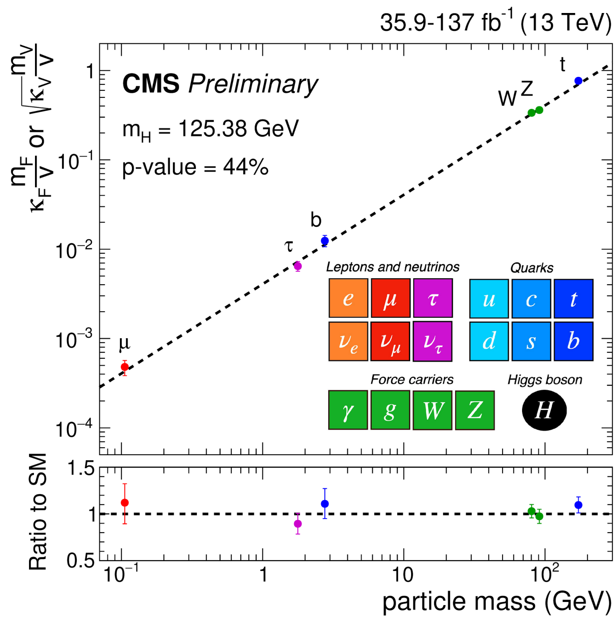
$H \to ZZ^* \to 4\ell$



- Higgs mass (LHC) [PDG '20]



- Higgs couplings (LHC) [2009.04363]



H self-couplings not yet observed

$\Leftarrow \kappa_F$ or $\kappa_V = \dfrac{\text{obs}}{\text{SM}}$

proben over more that 3 orders of mangitude!

Strong interactions

Strong interactions | Properties

- Quantum Chromodynamics (QCD) is *the* theory of strong interactions
- *Quarks* and *gluons* are the fundamental *dof* but they never show up as free states: they are bound in hadrons (confinement):

Baryons ($q_1q_2q_3$ or $\bar{q}_1\bar{q}_2\bar{q}_3$)

name		content	Q [e]	m [GeV]
p	proton	uud	$+1$	0,938
$\bar{\text{p}}$	antiproton	$\bar{\text{u}}\bar{\text{u}}\bar{\text{d}}$	-1	
n	neutron	ddu	0	0,939
$\bar{\text{n}}$	antineutron	$\bar{\text{d}}\bar{\text{d}}\bar{\text{u}}$		
Λ	lambda	uds	0	1,116
$\bar{\Lambda}$	antilambda	$\bar{\text{u}}\bar{\text{d}}\bar{\text{s}}$		
... ∼ 120 ...				

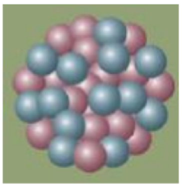
Mesons ($q_1\bar{q}_2$)

name		content	Q [e]	m [GeV]
π^0	neutral pion	u$\bar{\text{u}}$, d$\bar{\text{d}}$	0	0,135
π^+	charged pion	u$\bar{\text{d}}$	$+1$	0,140
π^-		d$\bar{\text{u}}$	-1	
K^+	charged kaon	u$\bar{\text{s}}$	$+1$	0,494
K^-		s$\bar{\text{u}}$	-1	
K^0	neutral kaon	d$\bar{\text{s}}$	0	0,498
$\bar{K}^0$		s$\bar{\text{d}}$		
... ∼ 140 ...				

and **exotics** (glueballs, tetraquarks, pentaquarks, . . .)

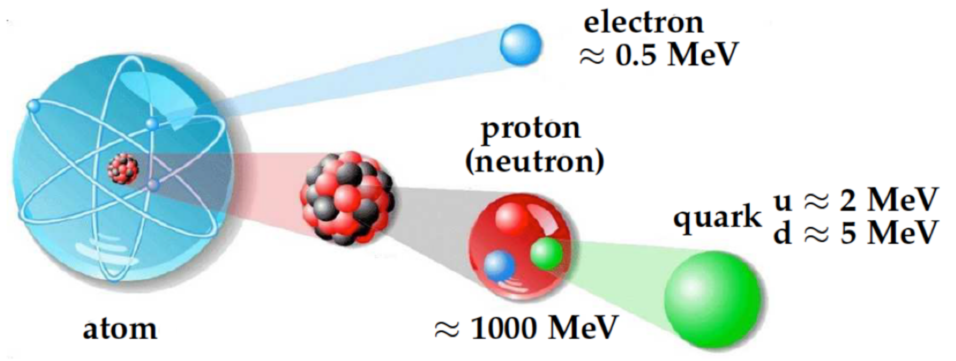
Strong interactions | Properties

- Strong interactions are responsible for:
  - Stability of nuclei (nucleon-nucleon interaction is a residual strong force)



strong attraction is greater than *electric* repulsion

  - $\sim 99\,\%$ of nucleon mass is binding energy, i.e. most of the mass in everything!



QCD | Lagrangian | SU(3) gauge symmetry

$$\mathcal{L}_{\rm QCD} = \underbrace{\overline{\psi}_{fi}\left(\mathrm{i}\not{D}_{ij} - m\delta_{ij}\right)\psi_{fj}}_{\text{quarks}} - \underbrace{\frac{1}{4}F^a_{\mu\nu}F^{a,\mu\nu}}_{\text{gluons}} \qquad \text{(flavor diagonal)}$$

$$F^a_{\mu\nu} = \partial_\mu \mathcal{A}^a_\nu - \partial_\nu \mathcal{A}^a_\mu + g_s f^{abc}\mathcal{A}^b_\mu \mathcal{A}^c_\nu$$

- (Anti–)quarks ψ_f come in $N_c = 3$ colors (anticolors) and there are $n_f = 6$ flavors:

$$\psi_{fi} \qquad \begin{cases} f = \mathrm{u, d, s, c, b, t} & \text{(flavor index)} \\ i = 1, \ldots, N_c = 3 & \text{(color index)} \end{cases} \qquad \text{fundamental irrep } \mathbf{3}\ (\bar{\mathbf{3}})$$

- Gluons $\mathcal{A}^a_\mu$ come in $N_c^2 - 1 = 8$ combinations of color and anticolor:

$$\mathcal{A}^a_\mu \qquad a = 1, \ldots, N_c^2 - 1 = 8 \quad \text{(color index)} \qquad \text{adjoint irrep } \mathbf{8}$$

QCD | Lagrangian | SU(3) gauge symmetry

$$\mathcal{L}_{\text{QCD}} = \underbrace{\overline{\psi}_{fi}\left(\mathrm{i}\not{D}_{ij} - m\delta_{ij}\right)\psi_{fj}}_{\text{quarks}} - \underbrace{\frac{1}{4}F^a_{\mu\nu}F^{a,\mu\nu}}_{\text{gluons}} \qquad \text{(flavor diagonal)}$$

$$F^a_{\mu\nu} = \partial_\mu \mathcal{A}^a_\nu - \partial_\nu \mathcal{A}^a_\mu + g_s f^{abc}\mathcal{A}^b_\mu \mathcal{A}^c_\nu$$

- Quark kinetic terms and quark-gluon interactions come from covariant derivative:

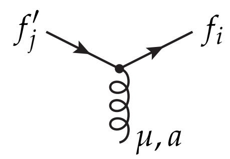
$$(D_\mu)_{ij} = \delta_{ij}\partial_\mu - \mathrm{i}g_s t^a_{ij}\mathcal{A}^a_\mu\,, \quad t^a_{ij} = \frac{1}{2}\lambda^a_{ij} \quad \text{(8 Gell-Mann matrices } 3\times 3\text{)}$$

- Gluon kinetic terms and self-interactions fixed by SU(3) structure constants f^{abc}:

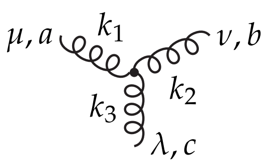
$$\begin{aligned}
\mathcal{L}_{\text{kin}} &= -\frac{1}{4}(\partial_\mu \mathcal{A}^a_\nu - \partial_\nu \mathcal{A}^a_\mu)(\partial^\mu \mathcal{A}^{a,\nu} - \partial^\nu \mathcal{A}^{a,\mu}) \\
\mathcal{L}_{\text{cubic}} &= -\frac{1}{2}g_s f^{abc}\,(\partial_\mu \mathcal{A}^a_\nu - \partial_\nu \mathcal{A}^a_\mu)\mathcal{A}^{b,\mu}\mathcal{A}^{c,\nu} \\
\mathcal{L}_{\text{quartic}} &= -\frac{1}{4}g_s^2 f^{abe} f^{cde}\,\mathcal{A}^a_\mu \mathcal{A}^b_\nu \mathcal{A}^{c,\mu}\mathcal{A}^{d,\nu}
\end{aligned}$$

QCD | Lagrangian | Feynman rules

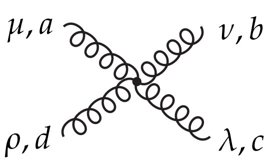
- Quark and gluon external legs and propagators are as usual
- Vertices:



$$= \mathrm{i}g_s t^a_{ij}\gamma^\mu \delta_{ff'}$$



$$= g_s f^{abc}\,[g_{\mu\nu}(k_1 - k_2)_\lambda + g_{\nu\lambda}(k_2 - k_3)_\mu + g_{\lambda\mu}(k_3 - k_1)_\nu]$$



$$= -\mathrm{i}g_s^2 \begin{bmatrix} f^{abe} f^{cde}\,(g_{\mu\lambda}g_{\nu\rho} - g_{\mu\rho}g_{\nu\lambda}) \\ + f^{ace} f^{dbe}\,(g_{\mu\rho}g_{\nu\lambda} - g_{\mu\nu}g_{\lambda\rho}) \\ + f^{ade} f^{bce}\,(g_{\mu\nu}g_{\lambda\rho} - g_{\mu\lambda}g_{\nu\rho}) \end{bmatrix}$$

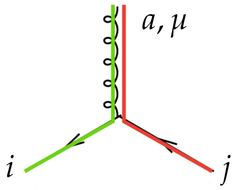
(interactions with Faddeev-Popov ghosts omitted here)

QCD | About color charges

- Quarks carry color charge: $\psi = \psi(x) \otimes \begin{pmatrix} \color{red}{R} \\ \color{green}{G} \\ \color{blue}{B} \end{pmatrix}$

- Antiquarks carry anticolor charge: $\overline{\psi} = \overline{\psi}(x) \otimes \begin{pmatrix} \color{red}{\overline{R}} & \color{green}{\overline{G}} & \color{blue}{\overline{B}} \end{pmatrix}$

- Gluons carry color and anticolor. A gluon emission *repaints* the quark:

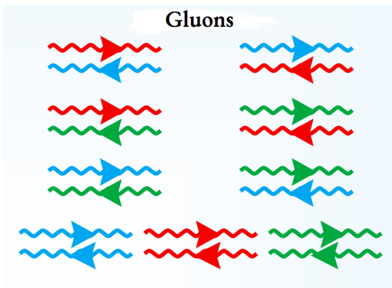
e.g. 

$$\overline{\psi}_i t^1_{ij} \psi_j \sim \tfrac{1}{2} \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \lambda^8 = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & \frac{-2}{\sqrt{3}} \end{pmatrix},$$

QCD | About color charges

- 8 gluons: 

$$\mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{1} \oplus \mathbf{8}$$

(1 in 9 combinations is color-neutral)

$$\begin{cases} R\overline{R} - G\overline{G} \\ R\overline{R} + G\overline{G} - 2B\overline{B} \end{cases}$$

If the color-singlet massless gluon state $R\overline{R} + G\overline{G} + B\overline{B}$ existed, it would give rise to a strong force of infinite range!

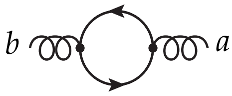
- Likewise, only color-singlet states can exist as free particles:

$$\begin{array}{llll} q\bar{q}' & \mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{1} \oplus 8 & \textbf{: mesons} & \frac{1}{\sqrt{3}} \delta_{ij} \left| q_i \bar{q}'_j \right\rangle \\ qq'q'' & \mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{1} \oplus 8 \oplus 8 \oplus 10 & \textbf{: baryons} & \frac{1}{\sqrt{6}} \epsilon_{ijk} \left| q_i q'_j q''_k \right\rangle \end{array} \quad i, j, k \in \{R, G, B\}$$

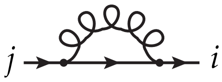
but qq' color singlets do not exist, since $\mathbf{3} \otimes \mathbf{3} = \bar{\mathbf{3}} \oplus \mathbf{6}$

QCD | About color charges

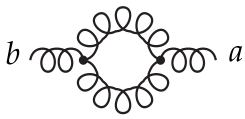
- Color algebra (useful identities): $t^a = \frac{1}{2}\lambda^a, \quad N_C = 3$



$$\propto \mathrm{Tr}(t^a t^b) = T_R \delta^{ab}, \quad T_R = \frac{1}{2}$$



$$\propto t^a_{ik} t^a_{kj} = C_F \delta_{ij}, \quad C_F = \frac{N_c^2 - 1}{2N_c} = \frac{4}{3}$$



$$\propto f^{acd} f^{bcd} = C_A \delta^{ab}, \quad C_A = N_c = 3$$

probability of a gluon to emit $q\bar{q}$ $<$quark to emit a gluon $<$gluon to emit gluons

(gluons interact more strongly than quarks)

QED vs QCD | running coupling

- All coupling constants *run*:

  $\alpha \equiv \dfrac{g^2}{4\pi} = \alpha(Q^2)$, where Q is the momentum scale of the process

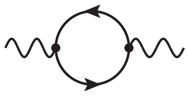
$$Q^2 \frac{\partial \alpha}{\partial Q^2} = \beta(\alpha)\,, \quad \beta(\alpha) \equiv -\alpha^2(\beta_0 + \beta_1 \alpha + \beta_2 \alpha^2 + \ldots)$$

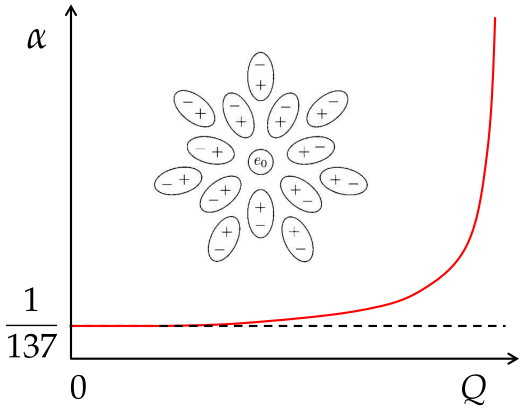
$$\alpha(Q^2) = \frac{\alpha(Q_0^2)}{1 + \beta_0 \alpha(Q_0^2) \ln \dfrac{Q^2}{Q_0^2}} \qquad \text{(Leading Order)}$$

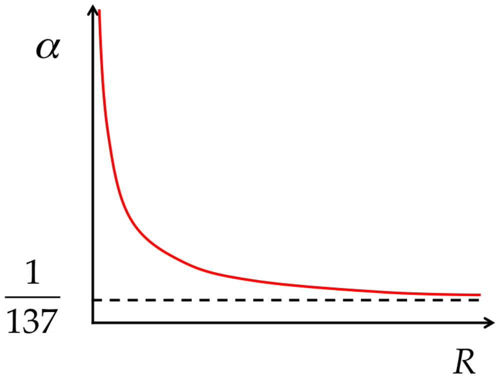
- Physically, this is related to the (anti-)screening of the fundamental charges by quantum fluctuations, depending on the sign of β_0:
  - In QED: $\alpha_{\rm em} = \dfrac{e^2}{4\pi}\,,\ \beta_0(\alpha_{\rm em}) = -\dfrac{1}{3\pi} \quad (< 0)$
  - In QCD: $\alpha_s = \dfrac{g_s^2}{4\pi}\,,\ \beta_0(\alpha_s) = \dfrac{11C_A - 4T_R N_f}{12\pi} = \dfrac{33 - 2N_f}{12\pi} \quad (> 0 \text{ for } N_f \leq 16)$

QED **running coupling**

- In QED, the fluctuating vacuum behaves like a dielectric medium, screening the bare electric charge e_0 at increasing distances $R \sim 1/Q$:



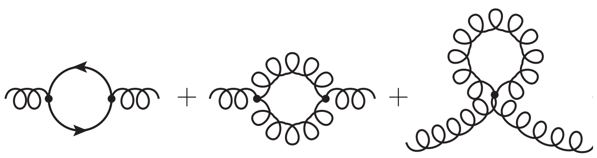




e.g. $\alpha(M_Z^2) \approx 1/128$

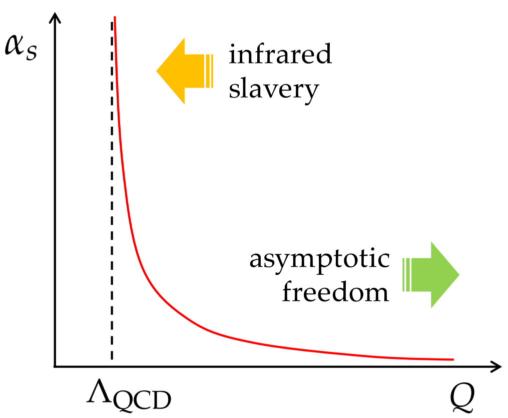
QCD **running coupling**

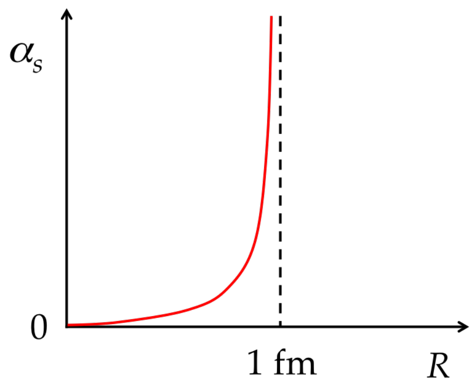
- Contributions to the QCD beta function $\beta(\alpha_s)$ (from QCD vacuum polarization):



(1) screening

(2), (3) anti-screening (non-abelian!)





$\Lambda_{\text{QCD}} \approx 200$ MeV

QCD running coupling

⇒ There is a scale $\Lambda_{\rm QCD}$ where $\alpha_s \to \infty$ (dimensional transmutation) given at LO by

$$\Lambda^2_{\rm QCD} = Q^2 \exp\left\{ -\frac{1}{\beta_0 \alpha_s(Q^2)} \right\} \quad \Leftrightarrow \quad \alpha_s(Q^2) = \frac{1}{\beta_0 \ln \dfrac{Q^2}{\Lambda^2_{\rm QCD}}} \qquad (Q^2 > \Lambda^2_{\rm QCD})$$

$\Lambda_{\rm QCD} \approx 200$ MeV, that is $R \sim 1/Q \approx 1$ fm (the size of a proton!)

- Asymptotic freedom:
  At short distances ($Q \gg \Lambda_{\rm QCD}$) quarks and gluons are almost free, they interact *weakly*: perturbative regime

- Infrared slavery:
  At long distances ($Q \sim \Lambda_{\rm QCD}$) the coupling diverges (Landau pole), quarks and gluons interact very *strongly* (**confinement into hadrons**): non-perturbative regime

⇒ Strong interactions are short-range, despite of gluon being massless

QCD running coupling