Picard-Vessiot extensions, differential Galois groupoids and Hopf algebroids

Laiachi El Kaoutit

Universidad de Granada. Spain.
EMAIL: kaoutit@ugr.es
URL: https://www.ugr.es/~kaoutit/
(A talk based on arXiv: https://arxiv.org/pdf/1607.07633.pdf)

ABSTRACT ASPECTS OF HIGHER REPRESENTATION THEORY.

Brussels, January 23rd, 2019.

The category of differential modules.

The category of differential modules.

Let $A = \mathbb{C}[X]$ be the polynomial algebra in one variable X over the field of complex numbers, which we consider as the coordinate ring of the affine complex line $\mathbb{A}^1_{\mathbb{C}}$.

The category of differential modules.

Let $A = \mathbb{C}[X]$ be the polynomial algebra in one variable X over the field of complex numbers, which we consider as the coordinate ring of the affine complex line $\mathbb{A}^1_{\mathbb{C}}$.

A *differential module* over the differential algebra $(A, \partial := \partial/\partial X)$ is a finitely generated right A-module equipped with a $\mathbb{C}$-linear map $\partial : M \to M$ such that

$$\partial(xa) = \partial x.a + \partial a.x, \text{ for every } a \in A \text{ and } x \in M.$$

The linear map ∂ is called *the differential of M*.

The category of differential modules.

Let $A = \mathbb{C}[X]$ be the polynomial algebra in one variable X over the field of complex numbers, which we consider as the coordinate ring of the affine complex line $\mathbb{A}^1_{\mathbb{C}}$.

A *differential module* over the differential algebra $(A, \partial := \partial/\partial X)$ is a finitely generated right A-module equipped with a $\mathbb{C}$-linear map $\partial : M \to M$ such that

$$\partial(xa) \ = \ \partial x.a + \partial a.x, \text{ for every } \ a \in A \text{ and } x \in M.$$

The linear map ∂ is called *the differential of M*.

NOTE: Every differential module is in fact free of finite rank as an A-module.

The category of differential modules.

Let $A = \mathbb{C}[X]$ be the polynomial algebra in one variable X over the field of complex numbers, which we consider as the coordinate ring of the affine complex line $\mathbb{A}^1_{\mathbb{C}}$.

A *differential module* over the differential algebra $(A, \partial := \partial/\partial X)$ is a finitely generated right A-module equipped with a $\mathbb{C}$-linear map $\partial : M \to M$ such that

$$\partial(xa) = \partial x.a + \partial a.x, \text{ for every } a \in A \text{ and } x \in M.$$

The linear map ∂ is called *the differential of M*.

NOTE: Every differential module is in fact free of finite rank as an A-module.

To each differential module one can associate a *linear differential matrix equation*:

The category of differential modules.

Let $A = \mathbb{C}[X]$ be the polynomial algebra in one variable X over the field of complex numbers, which we consider as the coordinate ring of the affine complex line $\mathbb{A}^1_{\mathbb{C}}$.

A *differential module* over the differential algebra $(A, \partial := \partial/\partial X)$ is a finitely generated right A-module equipped with a $\mathbb{C}$-linear map $\partial : M \to M$ such that

$$\partial(xa) = \partial x.a + \partial a.x, \text{ for every } a \in A \text{ and } x \in M.$$

The linear map ∂ is called *the differential of M*.

NOTE: Every differential module is in fact free of finite rank as an A-module.

To each differential module one can associate a *linear differential matrix equation*: Denote by $\{e_1, \ldots, e_m\}$ any basis of M over A, the differential ∂ is then given by a matrix $mat(M) = (a_{ij}) \in M_m(A)$ such that

$$\partial e_i = -\sum_{j=1}^{m} e_j a_{ji}.$$

The category of differential modules.

The category of differential modules.

So if we identify an element $y \in M$ with its coordinate column in A^m, we have that

$$\partial y = \begin{pmatrix} \partial y_1 \\ \vdots \\ \partial y_m \end{pmatrix} - mat(M) \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}.$$

The category of differential modules.

So if we identify an element $y \in M$ with its coordinate column in A^m, we have that

$$\partial y = \begin{pmatrix} \partial y_1 \\ \vdots \\ \partial y_m \end{pmatrix} - mat(M) \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}.$$

Thus $\ker(\partial)$ is the solution space of the following *linear differential matrix equation*:

$$\begin{pmatrix} y_1' \\ \vdots \\ y_m' \end{pmatrix} = mat(M) \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}.$$

The category of differential modules.

So if we identify an element $y \in M$ with its coordinate column in A^m, we have that

$$\partial y = \begin{pmatrix} \partial y_1 \\ \vdots \\ \partial y_m \end{pmatrix} - mat(M) \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}.$$

Thus $\ker(\partial)$ is the solution space of the following *linear differential matrix equation*:

$$\begin{pmatrix} y_1' \\ \vdots \\ y_m' \end{pmatrix} = mat(M) \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}.$$

A *morphism of differential modules* $f : (M, \partial) \to (N, \partial)$ is an A-linear map $f : M \to N$ which commutes with differentials: $\partial \circ f = f \circ \partial$.

The category of differential modules.

The category of differential modules.

The category $\mathbf{Diff}_A$ of all differential modules, enjoys the following properties:

The category of differential modules.

The category $\mathbf{Diff}_A$ of all differential modules, enjoys the following properties:

- It is a $\mathbb{C}$-linear locally finite abelian category.

The category of differential modules.

The category $\mathbf{Diff}_A$ of all differential modules, enjoys the following properties:

- It is a $\mathbb{C}$-linear locally finite abelian category.
- It is a rigid symmetric monoidal category. The tensor product of two objects $(M, \partial), (N, \partial)$ in $\mathbf{Diff}_A$ is again a differential module with differential:

$$\partial : M \otimes_A N \longrightarrow M \otimes_A N, \quad \Big(\partial(x \otimes_A y) \;=\; \partial(x) \otimes_A y + x \otimes_A \partial(y)\Big)$$

The category of differential modules.

The category $\mathbf{Diff}_A$ of all differential modules, enjoys the following properties:

- It is a $\mathbb{C}$-linear locally finite abelian category.
- It is a rigid symmetric monoidal category. The tensor product of two objects $(M, \partial), (N, \partial)$ in $\mathbf{Diff}_A$ is again a differential module with differential:

$$\partial : M \otimes_A N \longrightarrow M \otimes_A N, \quad \Big(\partial(x \otimes_A y) = \partial(x) \otimes_A y + x \otimes_A \partial(y)\Big)$$

- The forgetful functor $\omega : \mathbf{Diff}_A \to \mathrm{proj}(A)$ is strict monoidal $\mathbb{C}$-linear faithful exact functor. Moreover, we have an isomorphism of $\mathbb{C}$-algebras: $\mathrm{End}_{\mathbf{Diff}_A}\Big((A, \partial)\Big) \cong \mathbb{C}$.

The category of differential modules.

The category $\mathbf{Diff}_A$ of all differential modules, enjoys the following properties:

- It is a $\mathbb{C}$-linear locally finite abelian category.
- It is a rigid symmetric monoidal category. The tensor product of two objects $(M, \partial), (N, \partial)$ in $\mathbf{Diff}_A$ is again a differential module with differential:

$$\partial : M \otimes_A N \longrightarrow M \otimes_A N, \quad \Big(\partial(x \otimes_A y) \; = \; \partial(x) \otimes_A y + x \otimes_A \partial(y) \Big)$$

- The forgetful functor $\omega : \mathbf{Diff}_A \to \mathrm{proj}(A)$ is strict monoidal $\mathbb{C}$-linear faithful exact functor. Moreover, we have an isomorphism of $\mathbb{C}$-algebras: $\mathrm{End}_{\mathbf{Diff}_A}\Big((A, \partial) \Big) \; \cong \; \mathbb{C}$.

We denote by $U := \mathbb{C}[X][Y, \frac{\partial}{\partial X}]$ the noncommutative *ring of differential operators* of A, i.e., the first Weyl algebra.

The category of differential modules.

The category $\mathbf{Diff}_A$ of all differential modules, enjoys the following properties:

- It is a $\mathbb{C}$-linear locally finite abelian category.
- It is a rigid symmetric monoidal category. The tensor product of two objects $(M, \partial), (N, \partial)$ in $\mathbf{Diff}_A$ is again a differential module with differential:

$$\partial : M \otimes_A N \longrightarrow M \otimes_A N, \quad \Big(\partial(x \otimes_A y) = \partial(x) \otimes_A y + x \otimes_A \partial(y) \Big)$$

- The forgetful functor $\omega : \mathbf{Diff}_A \to \mathrm{proj}(A)$ is strict monoidal $\mathbb{C}$-linear faithful exact functor. Moreover, we have an isomorphism of $\mathbb{C}$-algebras: $\mathrm{End}_{\mathbf{Diff}_A}\Big((A, \partial) \Big) \cong \mathbb{C}$.

We denote by $U := \mathbb{C}[X][Y, \frac{\partial}{\partial X}]$ the noncommutative *ring of differential operators* of A, i.e., the first Weyl algebra.
As a (right) A-module, U is free with basis $\{Y^n\}_{n \in \mathbb{N}}$, and with left A-action given by the rule

$$aY = Ya + \frac{\partial a}{\partial X}, \quad \text{for every } a \in A.$$

The finite dual of the of first Weyl algebra

The finite dual of the of first Weyl algebra

The algebra U is, up to isomorphisms, the universal enveloping algebroid of the Lie-Rinehart algebra $(A, \mathbf{Der}_{\mathbb{C}}(A))$ (the A-module of global sections of the transitive Lie algebroid given by the tangent bundle $\mathcal{T}\mathbb{A}^1_{\mathbb{C}}$).

The finite dual of the of first Weyl algebra

The algebra U is, up to isomorphisms, the universal enveloping algebroid of the Lie-Rinehart algebra $(A, \mathbf{Der}_{\mathbb{C}}(A))$ (the A-module of global sections of the transitive Lie algebroid given by the tangent bundle $\mathcal{T}\mathbb{A}^1_{\mathbb{C}}$).

The pair (A, U) is a co-commutative (right) Hopf algebroid and its structure maps are given by:

$$\Delta(Y) = 1 \otimes_A Y + Y \otimes_A 1, \quad \varepsilon(Y) = 0, \text{ and } Y_- \otimes_A Y_+ = 1 \otimes_A Y - Y \otimes_A 1.$$

The finite dual of the of first Weyl algebra

The algebra U is, up to isomorphisms, the universal enveloping algebroid of the Lie-Rinehart algebra $(A, \mathbf{Der}_{\mathbb{C}}(A))$ (the A-module of global sections of the transitive Lie algebroid given by the tangent bundle $\mathcal{T}\mathbb{A}^1_{\mathbb{C}}$).

The pair (A, U) is a co-commutative (right) Hopf algebroid and its structure maps are given by:

$$\Delta(Y) = 1 \otimes_A Y + Y \otimes_A 1, \quad \varepsilon(Y) = 0, \text{ and } Y_- \otimes_A Y_+ = 1 \otimes_A Y - Y \otimes_A 1.$$

Furthermore, there is an isomorphism of symmetric monoidal categories:

$$\mathbf{Diff}_A \underset{\otimes}{\cong} \mathrm{mod}_U$$

where mod_U is the full subcategory of the category of right U-modules whose underlying A-modules are f.g.p. ones.

The finite dual of the of first Weyl algebra

The algebra U is, up to isomorphisms, the universal enveloping algebroid of the Lie-Rinehart algebra $(A, \mathbf{Der}_{\mathbb{C}}(A))$ (the A-module of global sections of the transitive Lie algebroid given by the tangent bundle $\mathcal{T}\mathbb{A}^1_{\mathbb{C}}$).

The pair (A, U) is a co-commutative (right) Hopf algebroid and its structure maps are given by:

$$\Delta(Y) = 1 \otimes_A Y + Y \otimes_A 1, \quad \varepsilon(Y) = 0, \text{ and } Y_- \otimes_A Y_+ = 1 \otimes_A Y - Y \otimes_A 1.$$

Furthermore, there is an isomorphism of symmetric monoidal categories:

$$\mathbf{Diff}_A \underset{\otimes}{\cong} \mathrm{mod}_U$$

where mod_U is the full subcategory of the category of right U-modules whose underlying A-modules are f.g.p. ones.

Applying Tannaka-Krein reconstruction process to the pair $(\mathbf{Diff}_A, \omega)$, leads to a commutative Hopf algebroid (A, U°): *the finite dual of U*.

The finite dual as a subalgebra of the convolution algebra

The finite dual as a subalgebra of the convolution algebra

By the construction of U°, there is an equivalence of rigid symmetric monoidal categories

$$\mathrm{mod}_U \underset{\otimes}{\simeq} \mathrm{comod}_{U^\circ}$$

where comod_{U° is the full subcategory of the category of U°-comodules whose underlying A-modules are f.g.p. ones.

The finite dual as a subalgebra of the convolution algebra

By the construction of U°, there is an equivalence of rigid symmetric monoidal categories

$$\mathrm{mod}_U \underset{\otimes}{\simeq} \mathrm{comod}_{U^\circ}$$

where comod_{U° is the full subcategory of the category of U°-comodules whose underlying A-modules are f.g.p. ones.

Moreover, there is an injective $(A \otimes_{\mathbb{C}} A)$-algebra map

$$U^\circ \overset{\zeta}{\hookrightarrow} U^*,$$

whose codomain is the right A-linear dual of U.

The finite dual as a subalgebra of the convolution algebra

By the construction of U°, there is an equivalence of rigid symmetric monoidal categories

$$\mathrm{mod}_U \underset{\otimes}{\simeq} \mathrm{comod}_{U^\circ}$$

where comod_{U° is the full subcategory of the category of U°-comodules whose underlying A-modules are f.g.p. ones.

Moreover, there is an injective $(A \otimes_{\mathbb{C}} A)$-algebra map

$$U^\circ \overset{\zeta}{\hookrightarrow} U^*,$$

whose codomain is the right A-linear dual of U.

The algebra U^* is the convolution algebra of the Hopf algebroid U, which have a structure of complete commutative Hopf algebroid. In this way the completion $\hat{\zeta} : \hat{U}^\circ \to U^*$ of ζ becomes a morphism of complete Hopf algebroids.

The finite dual as a subalgebra of the convolution algebra

By the construction of U°, there is an equivalence of rigid symmetric monoidal categories

$$\mathrm{mod}_U \underset{\otimes}{\simeq} \mathrm{comod}_{U^\circ}$$

where comod_{U° is the full subcategory of the category of U°-comodules whose underlying A-modules are f.g.p. ones.

Moreover, there is an injective $(A \otimes_{\mathbb{C}} A)$-algebra map

$$U^\circ \xhookrightarrow{\ \zeta\ } U^*,$$

whose codomain is the right A-linear dual of U.

The algebra U^* is the convolution algebra of the Hopf algebroid U, which have a structure of complete commutative Hopf algebroid. In this way the completion $\hat{\zeta} : \hat{U}^\circ \to U^*$ of ζ becomes a morphism of complete Hopf algebroids.

Following Kapranov's result, the associated formal scheme of U^*, can be seen as the "formal integration" of the Lie algebroid $(\mathcal{T}\mathbb{A}^1_{\mathbb{C}}, \mathbb{A}^1_{\mathbb{C}})$.

The differential Galois groupoid of a differential module

The differential Galois groupoid of a differential module

Fix a differential module $M \in \mathbf{Diff}_A$ with a dual basis $\{e_i, e_i^*\}_{1 \leq i \leq m}$.
There exist elements $\{f_{ij}\}_{1 \leq i,j \leq m}$, $\det_M \in U^{\circ}$, whose image are the following A-linear maps:

The differential Galois groupoid of a differential module

Fix a differential module $M \in \mathbf{Diff}_A$ with a dual basis $\{e_i, e_i^*\}_{1 \leq i \leq m}$. There exist elements $\{f_{ij}\}_{1 \leq i,j \leq m}$, $\det_M \in U^\circ$, whose image are the following A-linear maps:

$$\zeta(f_{ij})(u) = e_j^*(e_i\, u),$$

$$\zeta(\det_M)(u) = \sum_{\sigma \in S_m} (-1)^{sg(\sigma)} e_m^*\big(e_{\sigma(m)}\, u_{(m)}\big) \cdots e_1^*\big(e_{\sigma(1)}\, u_{(1)}\big),$$

for every $u \in U$. Additionally, the element $\det_M$ is invertible.

The differential Galois groupoid of a differential module

Fix a differential module $M \in \mathbf{Diff}_A$ with a dual basis $\{e_i, e_i^*\}_{1 \leq i \leq m}$. There exist elements $\{f_{ij}\}_{1 \leq i,j \leq m}$, $\det_M \in U^\circ$, whose image are the following A-linear maps:

$$\zeta(f_{ij})(u) = e_j^*(e_i\, u),$$

$$\zeta(\det_M)(u) = \sum_{\sigma \in S_m} (-1)^{sg(\sigma)} e_m^*\big(e_{\sigma(m)}\, u_{(m)}\big) \cdots e_1^*\big(e_{\sigma(1)}\, u_{(1)}\big),$$

for every $u \in U$. Additionally, the element $\det_M$ is invertible.

We denote by $U^\circ_{(M)}$ the $(A \otimes_{\mathbb{C}} A)$-subalgebra of U° generated by the elements $\{f_{ij}\}_{1 \leq i,j \leq m}$ and $\det_M^{-1}$.

The differential Galois groupoid of a differential module

Fix a differential module $M \in \textbf{Diff}_A$ with a dual basis $\{e_i, e_i^*\}_{1 \leq i \leq m}$. There exist elements $\{f_{ij}\}_{1 \leq i,j \leq m}$, $\det_M \in U^\circ$, whose image are the following A-linear maps:

$$\zeta(f_{ij})(u) = e_j^*(e_i\, u),$$

$$\zeta(\det_M)(u) = \sum_{\sigma \in S_m} (-1)^{sg(\sigma)} e_m^*\big(e_{\sigma(m)}\, u_{(m)}\big) \cdots e_1^*\big(e_{\sigma(1)}\, u_{(1)}\big),$$

for every $u \in U$. Additionally, the element $\det_M$ is invertible.

We denote by $U^\circ_{(M)}$ the $(A \otimes_{\mathbb{C}} A)$-subalgebra of U° generated by the elements $\{f_{ij}\}_{1 \leq i,j \leq m}$ and $\det_M^{-1}$.

It turns out that $U^\circ_{(M)}$ is a sub Hopf algebroid of U°.

The differential Galois groupoid of a differential module

The differential Galois groupoid of a differential module

Consider now the full subcategory $\langle M \rangle_{\otimes}$ of $\mathbf{Diff}_A$ of finite sub-quotients differential modules of M.

The differential Galois groupoid of a differential module

Consider now the full subcategory $\langle M \rangle_{\otimes}$ of $\mathbf{Diff}_A$ of finite sub-quotients differential modules of M.

An object (X, ∂) of $\mathbf{Diff}_A$ belongs to $\langle M \rangle_{\otimes}$ if it is a quotient of the form $X = X_2/X_1$, where

$$X_1 \subseteq X_2 \subseteq \bigoplus_{l,\, k} T^{(k,\, l)}(M), \quad \left(T^{(k,\, l)}(M) := M^{\otimes k} \otimes_A (M^*)^{\otimes l} \right)$$

(finite direct sum). Since $\mathbf{Diff}_A$ is an abelian category, a differential module (X, ∂) belongs to $\langle M \rangle_{\otimes}$ if and only if it is a sub-object of an object finitely generated by those $T^{(k,\, l)}(M)$'s.

The differential Galois groupoid of a differential module

Consider now the full subcategory $\langle M \rangle_{\otimes}$ of $\mathbf{Diff}_A$ of finite sub-quotients differential modules of M.

An object (X, ∂) of $\mathbf{Diff}_A$ belongs to $\langle M \rangle_{\otimes}$ if it is a quotient of the form $X = X_2/X_1$, where

$$X_1 \subseteq X_2 \subseteq \bigoplus_{l,\, k} T^{(k,\, l)}(M), \quad \Big(T^{(k,\, l)}(M) := M^{\otimes k} \otimes_A (M^*)^{\otimes l} \Big)$$

(finite direct sum). Since $\mathbf{Diff}_A$ is an abelian category, a differential module (X, ∂) belongs to $\langle M \rangle_{\otimes}$ if and only if it is a sub-object of an object finitely generated by those $T^{(k,\, l)}(M)$'s.

Denote by

$$\omega_{|\langle M \rangle_{\otimes}} : \langle M \rangle_{\otimes} \longrightarrow \mathrm{proj}(A)$$

the restriction of the fibre functor ω, and by $(A, \mathscr{R}(\langle M \rangle_{\otimes}))$ its associated commutative Hopf algebroid.

The differential Galois groupoid of a differential module

Consider now the full subcategory $\langle M \rangle_{\otimes}$ of $\mathbf{Diff}_A$ of finite sub-quotients differential modules of M.

An object (X, ∂) of $\mathbf{Diff}_A$ belongs to $\langle M \rangle_{\otimes}$ if it is a quotient of the form $X = X_2/X_1$, where

$$X_1 \subseteq X_2 \subseteq \bigoplus_{l,\,k} T^{(k,\,l)}(M), \quad \Big(T^{(k,\,l)}(M) := M^{\otimes k} \otimes_A (M^*)^{\otimes l} \Big)$$

(finite direct sum). Since $\mathbf{Diff}_A$ is an abelian category, a differential module (X, ∂) belongs to $\langle M \rangle_{\otimes}$ if and only if it is a sub-object of an object finitely generated by those $T^{(k,\,l)}(M)$'s.

Denote by

$$\omega_{|\langle M \rangle_{\otimes}} : \langle M \rangle_{\otimes} \longrightarrow \mathrm{proj}(A)$$

the restriction of the fibre functor ω, and by $(A, \mathscr{R}(\langle M \rangle_{\otimes}))$ its associated commutative Hopf algebroid.

Then, the embedding $\langle M \rangle_{\otimes} \hookrightarrow \mathbf{Diff}_A$, leads to the canonical morphism of Hopf algberoids

$$(A, \mathscr{R}(\langle M \rangle_{\otimes})) \longrightarrow (A, U^{\circ}).$$

The differential Galois groupoid of a differential module

The differential Galois groupoid of a differential module

There is an isomorphism of finitely generated Hopf algebroids:

$$(A, \mathscr{R}(\langle M \rangle_{\otimes})) \cong (A, U^{\circ}_{(M)})$$

The differential Galois groupoid of a differential module

There is an isomorphism of finitely generated Hopf algebroids:

$$(A, \mathscr{R}(\langle M \rangle_{\otimes})) \cong (A, U^{\circ}_{(M)})$$

Thus, we have equivalences of rigid symmetric monoidal $\mathbb{C}$-linear abelian categories:

$$\langle M \rangle_{\otimes} \underset{\otimes}{\simeq} \mathrm{comod}_{\mathscr{R}(\langle M \rangle_{\otimes})} \underset{\otimes}{\simeq} \mathrm{comod}_{U^{\circ}_{(M)}}.$$

The differential Galois groupoid of a differential module

There is an isomorphism of finitely generated Hopf algebroids:

$$(A, \mathscr{R}(\langle M \rangle_{\otimes})) \cong (A, U^{\circ}_{(M)})$$

Thus, we have equivalences of rigid symmetric monoidal $\mathbb{C}$-linear abelian categories:

$$\langle M \rangle_{\otimes} \underset{\otimes}{\simeq} \mathrm{comod}_{\mathscr{R}(\langle M \rangle_{\otimes})} \underset{\otimes}{\simeq} \mathrm{comod}_{U^{\circ}_{(M)}}.$$

Let $\mathscr{H}_M : \mathrm{Alg}_{\mathbb{C}} \to \mathrm{Grpds}$ be the presheaf of groupoids associated to the Hopf algebroid $(A, U^{\circ}_{(M)})$ and consider $\mathscr{H}_M(\mathbb{C})$ its fibre at Spec($\mathbb{C}$), that is, the *character groupoid* of $(A, U^{\circ}_{(M)})$.

The differential Galois groupoid of a differential module

There is an isomorphism of finitely generated Hopf algebroids:

$$(A, \mathscr{R}(\langle M \rangle_{\otimes})) \cong (A, U^{\circ}_{(M)})$$

Thus, we have equivalences of rigid symmetric monoidal $\mathbb{C}$-linear abelian categories:

$$\langle M \rangle_{\otimes} \underset{\otimes}{\simeq} \mathrm{comod}_{\mathscr{R}(\langle M \rangle_{\otimes})} \underset{\otimes}{\simeq} \mathrm{comod}_{U^{\circ}_{(M)}}.$$

Let $\mathscr{H}_M : \mathrm{Alg}_{\mathbb{C}} \to \mathrm{Grpds}$ be the presheaf of groupoids associated to the Hopf algebroid $(A, U^{\circ}_{(M)})$ and consider $\mathscr{H}_M(\mathbb{C})$ its fibre at Spec$(\mathbb{C})$, that is, the *character groupoid* of $(A, U^{\circ}_{(M)})$.

The algebraic groupoid $\mathscr{H}_M(\mathbb{C})$ is referred to as *the differential Galois groupoid of the linear differential matrix equation* attached to (M, ∂).

The differential Galois groupoid of a differential module

There is an isomorphism of finitely generated Hopf algebroids:

$$(A, \mathscr{R}(\langle M \rangle_{\otimes})) \cong (A, U^{\circ}_{(M)})$$

Thus, we have equivalences of rigid symmetric monoidal $\mathbb{C}$-linear abelian categories:

$$\langle M \rangle_{\otimes} \underset{\otimes}{\simeq} \mathrm{comod}_{\mathscr{R}(\langle M \rangle_{\otimes})} \underset{\otimes}{\simeq} \mathrm{comod}_{U^{\circ}_{(M)}}.$$

Let $\mathscr{H}_M : \mathrm{Alg}_{\mathbb{C}} \to \mathrm{Grpds}$ be the presheaf of groupoids associated to the Hopf algebroid $(A, U^{\circ}_{(M)})$ and consider $\mathscr{H}_M(\mathbb{C})$ its fibre at Spec$(\mathbb{C})$, that is, the *character groupoid* of $(A, U^{\circ}_{(M)})$.

The algebraic groupoid $\mathscr{H}_M(\mathbb{C})$ is referred to as *the differential Galois groupoid of the linear differential matrix equation* attached to (M, ∂).

Consider the following algebraic groupoid:

$$\mathscr{G}^m : \quad \mathbb{A}^1_{\mathbb{C}} \times GL_m(\mathbb{C}) \times \mathbb{A}^1_{\mathbb{C}} \underset{pr_1}{\overset{pr_3}{\underset{\longleftarrow \iota}{\rightrightarrows}}} \mathbb{A}^1_{\mathbb{C}},$$

This is the induced groupoid of $GL_m(\mathbb{C})$ along the map $\mathbb{A}^1_{\mathbb{C}} \to \{*\}$.

The differential Galois groupoid of a differential module

The differential Galois groupoid of a differential module

There is a “monomorphism” of affine algebraic groupoids

$$\mathscr{H}_M(\mathbb{C}) \hookrightarrow \mathscr{G}^m \ .$$

In particular, any isotropy group of $\mathscr{H}_M(\mathbb{C})$ is identified with a closed sub-group of the algebraic group $GL_m(\mathbb{C})$.

The differential Galois groupoid of a differential module

There is a “monomorphism” of affine algebraic groupoids

$$\mathscr{H}_M(\mathbb{C}) \hookrightarrow \mathscr{G}^m .$$

In particular, any isotropy group of $\mathscr{H}_M(\mathbb{C})$ is identified with a closed sub-group of the algebraic group $GL_m(\mathbb{C})$.

Example

Let (M, ∂) be a differential module whose underlying module $M = A.\mathfrak{m}$ is a free A-module of rank one, endowed with the differential matrix $mat(M) = a \in A$, that is, $\partial(\mathfrak{m}) = a(X)\mathfrak{m}$. Then the Hopf algebroid $U^{\circ}_{(M)}$ is generated as an $(A \otimes_{\mathbb{C}} A)$-algebra by the invertible element $\det_M$. Thus $U^{\circ}_{(M)}$ is isomorphic to the Hopf algebroid $(A \otimes_{\mathbb{C}} A)[T, T^{-1}] \cong A \otimes_{\mathbb{C}} \mathbb{C}[T, T^{-1}] \otimes_{\mathbb{C}} A$, which is the base extension of the Hopf $\mathbb{C}$-algebra $\mathbb{C}[T, T^{-1}]$ (the coordinate algebra of the multiplicative group).

The differential Galois groupoid of a differential module

There is a “monomorphism” of affine algebraic groupoids

$$\mathscr{H}_M(\mathbb{C}) \hookrightarrow \mathscr{G}^m .$$

In particular, any isotropy group of $\mathscr{H}_M(\mathbb{C})$ is identified with a closed sub-group of the algebraic group $GL_m(\mathbb{C})$.

Example

Let (M, ∂) be a differential module whose underlying module $M = A.\mathfrak{m}$ is a free A-module of rank one, endowed with the differential matrix $mat(M) = a \in A$, that is, $\partial(\mathfrak{m}) = a(X)\mathfrak{m}$. Then the Hopf algebroid $U^{\circ}_{(M)}$ is generated as an $(A \otimes_{\mathbb{C}} A)$-algebra by the invertible element $\det_M$. Thus $U^{\circ}_{(M)}$ is isomorphic to the Hopf algebroid $(A \otimes_{\mathbb{C}} A)[T, T^{-1}] \cong A \otimes_{\mathbb{C}} \mathbb{C}[T, T^{-1}] \otimes_{\mathbb{C}} A$, which is the base extension of the Hopf $\mathbb{C}$-algebra $\mathbb{C}[T, T^{-1}]$ (the coordinate algebra of the multiplicative group).

NOTE: Changing the differential ∂ on that free one rank modules, gives the “same” Hopf algebroid $U^{\circ}_{(M)}$. The fact, is that this differential ∂ of M do not induces a differential algebra structure on $U^{\circ}_{(M)}$, but on a certain quotient of this one.

The Picard-Vessiot extensions (after André)

The Picard-Vessiot extensions (after André)

Let us fix a differential module (M, ∂) with rank m, and consider as above the category $\langle M \rangle_{\otimes}$. This category admits a tensor generator (e.g., the differential module $M \oplus M^{*}$) and have a fibre functor over $\mathrm{Spec}(A) \neq \emptyset$, namely, the forgetful functor $\omega := \omega_{|\langle M \rangle_{\otimes}}$.

The Picard-Vessiot extensions (after André)

Let us fix a differential module (M, ∂) with rank m, and consider as above the category $\langle M \rangle_\otimes$. This category admits a tensor generator (e.g., the differential module $M \oplus M^*$) and have a fibre functor over $\mathrm{Spec}(A) \neq \emptyset$, namely, the forgetful functor $\omega := \omega_{|\langle M \rangle_\otimes}$.

By Deligne's results, we know that there is a fiber functor over the base field $\mathbb{C}$:

$$\omega' : \langle M \rangle_\otimes \longrightarrow \mathrm{vect}_\mathbb{C}.$$

The Picard-Vessiot extensions (after André)

Let us fix a differential module (M, ∂) with rank m, and consider as above the category $\langle M \rangle_{\otimes}$. This category admits a tensor generator (e.g., the differential module $M \oplus M^*$) and have a fibre functor over $\mathrm{Spec}(A) \neq \emptyset$, namely, the forgetful functor $\omega := \omega_{|\langle M \rangle_{\otimes}}$.

By Deligne's results, we know that there is a fiber functor over the base field $\mathbb{C}$:

$$\omega' : \langle M \rangle_{\otimes} \longrightarrow \mathrm{vect}_{\mathbb{C}}.$$

Following André's approach, the existence of this fibre functor is a crucial step in building up the *Picard-Vessiot extension of* (A, ∂) *for the differential module* (M, ∂).

The Picard-Vessiot extensions (after André)

Let us fix a differential module (M, ∂) with rank m, and consider as above the category $\langle M \rangle_\otimes$. This category admits a tensor generator (e.g., the differential module $M \oplus M^*$) and have a fibre functor over $\mathrm{Spec}(A) \neq \emptyset$, namely, the forgetful functor $\omega := \omega_{|\langle M \rangle_\otimes}$.

By Deligne's results, we know that there is a fiber functor over the base field $\mathbb{C}$:

$$\omega' : \langle M \rangle_\otimes \longrightarrow \mathrm{vect}_{\mathbb{C}}.$$

Following André's approach, the existence of this fibre functor is a crucial step in building up the *Picard-Vessiot extension of* (A, ∂) *for the differential module* (M, ∂).

For any point $x \in \mathbb{A}^1_{\mathbb{C}}$, consider the associated isotropy Hopf $\mathbb{C}$-algebra $U^{\circ}_{(M),\, x}$ of the Hopf algebroid $U^{\circ}_{(M)}$.

The Picard-Vessiot extensions (after André)

Let us fix a differential module (M, ∂) with rank m, and consider as above the category $\langle M\rangle_{\otimes}$. This category admits a tensor generator (e.g., the differential module $M \oplus M^*$) and have a fibre functor over $\mathrm{Spec}(A) \neq \emptyset$, namely, the forgetful functor $\omega := \omega_{|\langle M\rangle_{\otimes}}$.

By Deligne's results, we know that there is a fiber functor over the base field $\mathbb{C}$:

$$\omega' : \langle M\rangle_{\otimes} \longrightarrow \mathrm{vect}_{\mathbb{C}}.$$

Following André's approach, the existence of this fibre functor is a crucial step in building up the *Picard-Vessiot extension of* (A, ∂) *for the differential module* (M, ∂).

For any point $x \in \mathbb{A}^1_{\mathbb{C}}$, consider the associated isotropy Hopf $\mathbb{C}$-algebra $U^{\circ}_{(M),\,x}$ of the Hopf algebroid $U^{\circ}_{(M)}$.
This is by definition the base extension Hopf algebra

$$(\mathbb{C}, U^{\circ}_{(M),\,x} := \mathbb{C}_x \otimes_A U^{\circ}_{(M)} \otimes_A \mathbb{C}_x),$$

where $\mathbb{C}_x$ is $\mathbb{C}$ viewed as an A-algebra via the $\mathbb{C}$-algebra map $x : A \to \mathbb{C}$.

The Picard-Vessiot extensions

The Picard-Vessiot extensions

Observe that $(A, U^{\circ}_{(M)})$ is *geometrically transitive* flat Hopf algebroid over $\mathbb{C}$ and that $\mathscr{H}_M(\mathbb{C}) \neq \emptyset$, which is a transitive groupoid.

The Picard-Vessiot extensions

Observe that $(A, U^{\circ}_{(M)})$ is *geometrically transitive* flat Hopf algebroid over $\mathbb{C}$ and that $\mathscr{H}_{M}(\mathbb{C}) \neq \emptyset$, which is a transitive groupoid.

This implies that the canonical Hopf algebroid extension

$$\mathbf{x} : (A, U^{\circ}_{(M)}) \to (\mathbb{C}, U^{\circ}_{(M),\, x})$$

is a weak equivalence, which means that the induced functor

$$\mathbf{x}_{*} : \mathrm{Comod}_{U^{\circ}_{(M)}} \to \mathrm{Comod}_{U^{\circ}_{(M),\, x}}$$

establishes an equivalence of symmetric monoidal categories.

The Picard-Vessiot extensions

Observe that $(A, U^{\circ}_{(M)})$ is *geometrically transitive* flat Hopf algebroid over $\mathbb{C}$ and that $\mathscr{H}_M(\mathbb{C}) \neq \emptyset$, which is a transitive groupoid.

This implies that the canonical Hopf algebroid extension

$$\mathbf{x} : (A, U^{\circ}_{(M)}) \to (\mathbb{C}, U^{\circ}_{(M),\, x})$$

is a weak equivalence, which means that the induced functor

$$\mathbf{x}_* : \mathrm{Comod}_{U^{\circ}_{(M)}} \to \mathrm{Comod}_{U^{\circ}_{(M),\, x}}$$

establishes an equivalence of symmetric monoidal categories.

In this way we obtain a chain of symmetric monoidal $\mathbb{C}$-linear faithful and exact functors:

$$\omega_x : \langle M \rangle_{\otimes} \xrightarrow[\otimes\text{-}\simeq]{} \mathrm{comod}_{U^{\circ}_{(M)}} \xrightarrow[\otimes\text{-}\simeq]{\mathbf{x}_*} \mathrm{comod}_{U^{\circ}_{(M),\, x}} \xrightarrow{\mathscr{O}} \mathrm{vect}_{\mathbb{C}},$$

where $\mathscr{O}$ is the forgetful functor.

The Picard-Vessiot extensions

The Picard-Vessiot extensions

There is a point $x \in \mathbb{A}^1_{\mathbb{C}}$ such that $\omega' = \omega_x$, up to a canonical natural isomorphism. In particular, the extended fibre functor

$$\omega' \otimes_{\mathbb{C}} A : \langle M \rangle_{\otimes} \longrightarrow \mathrm{proj}(A)$$

over A is naturally isomorphic to ω. Therefore, we have isomorphisms of Hopf algebroids:

$$(A, \mathscr{R}(\omega)) \cong (A, \mathscr{R}(\omega' \otimes_{\mathbb{C}} A)) \cong (A, U^{\circ}_{(M)}).$$

The Picard-Vessiot extensions

There is a point $x \in \mathbb{A}^1_{\mathbb{C}}$ such that $\omega' = \omega_x$, up to a canonical natural isomorphism. In particular, the extended fibre functor

$$\omega' \otimes_{\mathbb{C}} A : \langle M \rangle_{\otimes} \longrightarrow \mathrm{proj}(A)$$

over A is naturally isomorphic to ω. Therefore, we have isomorphisms of Hopf algebroids:

$$(A, \mathscr{R}(\omega)) \cong (A, \mathscr{R}(\omega' \otimes_{\mathbb{C}} A)) \cong (A, U^{\circ}_{(M)}).$$

So far we have two fibred functors:

$$\omega := \omega_{|\langle M \rangle_{\otimes}}, \; \omega' \otimes_{\mathbb{C}} A : \langle M \rangle_{\otimes} \longrightarrow \mathrm{proj}(A).$$

The Picard-Vessiot extensions

There is a point $x \in \mathbb{A}^1_{\mathbb{C}}$ such that $\omega' = \omega_x$, up to a canonical natural isomorphism. In particular, the extended fibre functor

$$\omega' \otimes_{\mathbb{C}} A : \langle M \rangle_{\otimes} \longrightarrow \mathrm{proj}(A)$$

over A is naturally isomorphic to ω. Therefore, we have isomorphisms of Hopf algebroids:

$$(A, \mathscr{R}(\omega)) \cong (A, \mathscr{R}(\omega' \otimes_{\mathbb{C}} A)) \cong (A, U^{\circ}_{(M)}).$$

So far we have two fibred functors:

$$\omega := \omega_{|\langle M \rangle_{\otimes}}, \ \omega' \otimes_{\mathbb{C}} A : \langle M \rangle_{\otimes} \longrightarrow \mathrm{proj}(A).$$

This leads to a principal bi-bundle $(\mathscr{R}(\omega, \omega' \otimes_{\mathbb{C}} A), \boldsymbol{\alpha}, \boldsymbol{\beta})$ over $U^{\circ}_{(M)}$, where $\boldsymbol{\alpha}, \boldsymbol{\beta} : A \to \mathscr{R}(\omega, \omega' \otimes_{\mathbb{C}} A)$ are its structural algebra maps. This is the $(A \otimes_{\mathbb{C}} A)$-algebra representing the bi-torsor $\underline{\mathrm{Isom}}^{\otimes}_{A \otimes A}(\omega,\ \omega' \otimes_{\mathbb{C}} A)$.

The Picard-Vessiot extensions

There is a point $x \in \mathbb{A}^1_{\mathbb{C}}$ such that $\omega' = \omega_x$, up to a canonical natural isomorphism. In particular, the extended fibre functor

$$\omega' \otimes_{\mathbb{C}} A : \langle M \rangle_{\otimes} \longrightarrow \mathrm{proj}(A)$$

over A is naturally isomorphic to ω. Therefore, we have isomorphisms of Hopf algebroids:

$$(A, \mathscr{R}(\omega)) \cong (A, \mathscr{R}(\omega' \otimes_{\mathbb{C}} A)) \cong (A, U^{\circ}_{(M)}).$$

So far we have two fibred functors:

$$\omega := \omega_{|\langle M \rangle_{\otimes}}, \ \omega' \otimes_{\mathbb{C}} A : \langle M \rangle_{\otimes} \longrightarrow \mathrm{proj}(A).$$

This leads to a principal bi-bundle $(\mathscr{R}(\omega, \omega' \otimes_{\mathbb{C}} A), \boldsymbol{\alpha}, \boldsymbol{\beta})$ over $U^{\circ}_{(M)}$, where $\boldsymbol{\alpha}, \boldsymbol{\beta} : A \to \mathscr{R}(\omega, \omega' \otimes_{\mathbb{C}} A)$ are its structural algebra maps. This is the $(A \otimes_{\mathbb{C}} A)$-algebra representing the bi-torsor $\underline{\mathrm{Isom}}^{\otimes}_{A \otimes A}(\omega, \ \omega' \otimes_{\mathbb{C}} A)$.

This principal bi-bundle is (up to isomorphisms) the unit principal bi-bundle $\mathscr{U}(U^{\circ}_{(M)})$ given by the Hopf algebroid $U^{\circ}_{(M)}$ its self.

The Picard-Vessiot extensions

The Picard-Vessiot extensions

Let us consider the quotient algebra

$$\mathcal{P} := U^{\circ}_{(M)} / \langle \mathsf{s} - \mathsf{t} \rangle$$

by the Hopf ideal $\langle \mathsf{s} - \mathsf{t} \rangle$. This is the (total isotropy) Hopf A-algebra of $U^{\circ}_{(M)}$ with unit $\boldsymbol{\iota} : A \to \mathcal{P}$ the algebra map induced from $\boldsymbol{\alpha}, \boldsymbol{\beta}$.

The Picard-Vessiot extensions

Let us consider the quotient algebra

$$\mathcal{P} := U^{\circ}_{(M)}/\langle \mathsf{s} - \mathsf{t} \rangle$$

by the Hopf ideal $\langle \mathsf{s} - \mathsf{t} \rangle$. This is the (total isotropy) Hopf A-algebra of $U^{\circ}_{(M)}$ with unit $\boldsymbol{\iota} : A \to \mathcal{P}$ the algebra map induced from $\boldsymbol{\alpha}, \boldsymbol{\beta}$.

The algebra $\mathcal{P}$ admits a differential $\boldsymbol{\partial} : \mathcal{P} \to \mathcal{P}$, extracted from the differential of M such that the map $\boldsymbol{\iota} : (A, \partial) \longrightarrow (\mathcal{P}, \boldsymbol{\partial})$ becomes a morphism of differential algebras.

The Picard-Vessiot extensions

Let us consider the quotient algebra

$$\mathcal{P} := U^{\circ}_{(M)}/\langle \mathsf{s} - \mathsf{t} \rangle$$

by the Hopf ideal $\langle \mathsf{s} - \mathsf{t} \rangle$. This is the (total isotropy) Hopf A-algebra of $U^{\circ}_{(M)}$ with unit $\boldsymbol{\iota} : A \to \mathcal{P}$ the algebra map induced from $\boldsymbol{\alpha}, \boldsymbol{\beta}$.

The algebra $\mathcal{P}$ admits a differential $\boldsymbol{\partial} : \mathcal{P} \to \mathcal{P}$, extracted from the differential of M such that the map $\boldsymbol{\iota} : (A, \partial) \longrightarrow (\mathcal{P}, \boldsymbol{\partial})$ becomes a morphism of differential algebras.

Denote by $\mathrm{Aut}_{(A,\,\partial)}\big((\mathcal{P}, \boldsymbol{\partial})\big)$ the *group of differential A-algebra automorphisms*, that is, algebra automorphisms $\boldsymbol{\sigma} : \mathcal{P} \to \mathcal{P}$ such that $\boldsymbol{\sigma} \circ \boldsymbol{\iota} = \boldsymbol{\iota}$ and $\boldsymbol{\partial} \circ \boldsymbol{\sigma} = \boldsymbol{\sigma} \circ \boldsymbol{\partial}$.

The Picard-Vessiot extensions

Let us consider the quotient algebra

$$\mathcal{P} := U^{\circ}_{(M)}/\langle \mathsf{s} - \mathsf{t} \rangle$$

by the Hopf ideal $\langle \mathsf{s} - \mathsf{t} \rangle$. This is the (total isotropy) Hopf A-algebra of $U^{\circ}_{(M)}$ with unit $\boldsymbol{\iota} : A \to \mathcal{P}$ the algebra map induced from $\boldsymbol{\alpha}, \boldsymbol{\beta}$.

The algebra $\mathcal{P}$ admits a differential $\boldsymbol{\partial} : \mathcal{P} \to \mathcal{P}$, extracted from the differential of M such that the map $\boldsymbol{\iota} : (A, \partial) \longrightarrow (\mathcal{P}, \boldsymbol{\partial})$ becomes a morphism of differential algebras.

Denote by $\mathrm{Aut}_{(A,\, \partial)}\big((\mathcal{P}, \boldsymbol{\partial})\big)$ the *group of differential A-algebra automorphisms*, that is, algebra automorphisms $\boldsymbol{\sigma} : \mathcal{P} \to \mathcal{P}$ such that $\boldsymbol{\sigma} \circ \boldsymbol{\iota} = \boldsymbol{\iota}$ and $\boldsymbol{\partial} \circ \boldsymbol{\sigma} = \boldsymbol{\sigma} \circ \boldsymbol{\partial}$.

In this way, we obtain a functor valued in groups:

$$\underline{\mathrm{Aut}}_{(A,\, \partial)}\big((\mathcal{P}, \boldsymbol{\partial})\big) : \mathrm{Alg}_{\mathbb{C}} \longrightarrow \mathrm{Grps}, \quad \Big(C \longrightarrow \mathrm{Aut}_{(A \otimes C,\, \partial \otimes C)}\big((\mathcal{P} \otimes C, \boldsymbol{\partial} \otimes C)\big)\Big),$$

whose fibre at $\mathrm{Spec}(\mathbb{C})$ is $\mathrm{Aut}_{(A,\, \partial)}\big((\mathcal{P}, \boldsymbol{\partial})\big)$.

The Picard-Vessiot extensions

The Picard-Vessiot extensions

The differential algebra $(\mathcal{P}, \partial)$ enjoys the following more properties:

The Picard-Vessiot extensions

The differential algebra $(\mathcal{P}, \boldsymbol{\partial})$ enjoys the following more properties:

- $(\mathcal{P}, \boldsymbol{\partial})$ is a simple differential algebra;

The Picard-Vessiot extensions

The differential algebra $(\mathcal{P}, \boldsymbol{\partial})$ enjoys the following more properties:

- $(\mathcal{P}, \boldsymbol{\partial})$ is a simple differential algebra;
- As an A-algebra $\mathcal{P}$ is generated by the entries of the matrix $F = (f_{ij} + \langle \mathrm{s} - \mathrm{t} \rangle)_{1 \leq \mathrm{i},\, \mathrm{j} \leq \mathrm{m}}$ and $(\det_M{}^{-1} + \langle \mathrm{s} - \mathrm{t} \rangle)$ with differential

$$\boldsymbol{\partial} F = mat(M)\, F.$$

  That is, F is a *fundamental matrix of solutions* of the linear differential matrix equation attached to (M, ∂).

The Picard-Vessiot extensions

The differential algebra $(\mathcal{P}, \boldsymbol{\partial})$ enjoys the following more properties:

- $(\mathcal{P}, \boldsymbol{\partial})$ is a simple differential algebra;
- As an A-algebra $\mathcal{P}$ is generated by the entries of the matrix $F = (f_{ij} + \langle \mathrm{s} - \mathrm{t} \rangle)_{1 \leq \mathrm{i},\, \mathrm{j} \leq \mathrm{m}}$ and $(\det_M{}^{-1} + \langle \mathrm{s} - \mathrm{t} \rangle)$ with differential

$$\boldsymbol{\partial} F = mat(M)\, F.$$

  That is, F is a *fundamental matrix of solutions* of the linear differential matrix equation attached to (M, ∂).

Furthermore, we have that

The Picard-Vessiot extensions

The differential algebra $(\mathcal{P}, \boldsymbol{\partial})$ enjoys the following more properties:

- $(\mathcal{P}, \boldsymbol{\partial})$ is a simple differential algebra;
- As an A-algebra $\mathcal{P}$ is generated by the entries of the matrix $F = (f_{ij} + \langle \mathrm{s} - \mathrm{t} \rangle)_{1 \leq \mathrm{i},\, \mathrm{j} \leq \mathrm{m}}$ and $(\det_M{}^{-1} + \langle \mathrm{s} - \mathrm{t} \rangle)$ with differential

$$\boldsymbol{\partial} F = mat(M)\, F.$$

  That is, F is a *fundamental matrix of solutions* of the linear differential matrix equation attached to (M, ∂).

Furthermore, we have that

- there is an isomorphism

$$\underline{\mathrm{Aut}}_{(A,\, \partial)}\big((\mathcal{P}, \boldsymbol{\partial})\big) \cong \underline{\mathrm{Aut}}^{\otimes}(\omega')$$

  of affine group schemes;

The Picard-Vessiot extensions

The differential algebra $(\mathcal{P}, \boldsymbol{\partial})$ enjoys the following more properties:

- $(\mathcal{P}, \boldsymbol{\partial})$ is a simple differential algebra;
- As an A-algebra $\mathcal{P}$ is generated by the entries of the matrix $F = (f_{ij} + \langle \mathrm{s} - \mathrm{t} \rangle)_{1 \leq \mathrm{i}, \mathrm{j} \leq \mathrm{m}}$ and $(\det_M{}^{-1} + \langle \mathrm{s} - \mathrm{t} \rangle)$ with differential

$$\boldsymbol{\partial} F = mat(M)\, F.$$

  That is, F is a *fundamental matrix of solutions* of the linear differential matrix equation attached to (M, ∂).

Furthermore, we have that

- there is an isomorphism

$$\underline{\mathrm{Aut}}_{(A,\,\partial)}\big((\mathcal{P}, \boldsymbol{\partial})\big) \cong \underline{\mathrm{Aut}}^{\otimes}(\omega')$$

  of affine group schemes;
- the algebraic group $\mathrm{Aut}_{(A,\,\partial)}\big((\mathcal{P}, \boldsymbol{\partial})\big)$ is isomorphic to each of the isotropy groups of the algebraic differential Galois groupoid $\mathscr{H}_M(\mathbb{C})$ attached to (M, ∂).

The Picard-Vessiot extensions

The differential algebra $(\mathcal{P}, \boldsymbol{\partial})$ enjoys the following more properties:

- $(\mathcal{P}, \boldsymbol{\partial})$ is a simple differential algebra;
- As an A-algebra $\mathcal{P}$ is generated by the entries of the matrix $F = (f_{ij} + \langle \mathrm{s} - \mathrm{t} \rangle)_{1 \leq \mathrm{i}, \mathrm{j} \leq \mathrm{m}}$ and $(\det_M{}^{-1} + \langle \mathrm{s} - \mathrm{t} \rangle)$ with differential

$$\boldsymbol{\partial} F = mat(M)\, F.$$

  That is, F is a *fundamental matrix of solutions* of the linear differential matrix equation attached to (M, ∂).

Furthermore, we have that

- there is an isomorphism

$$\underline{\mathrm{Aut}}_{(A,\, \partial)}\big((\mathcal{P}, \boldsymbol{\partial})\big) \cong \underline{\mathrm{Aut}}^{\otimes}(\omega')$$

  of affine group schemes;
- the algebraic group $\mathrm{Aut}_{(A,\, \partial)}\big((\mathcal{P}, \boldsymbol{\partial})\big)$ is isomorphic to each of the isotropy groups of the algebraic differential Galois groupoid $\mathscr{H}_M(\mathbb{C})$ attached to (M, ∂).

The pair $(\mathcal{P}, \boldsymbol{\partial})$ is the *Picard-Vessiot extension of* (A, ∂) *for* (M, ∂).

Thank you!