New characterizations of geometrically transitive Hopf algebroids

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These definitions are in fact geometric counterparts of the *transitivity* of an abstract groupoid. Namely, recall that an abstract groupoid

$$G_1 \underset{\mathsf{t}}{\overset{\mathsf{s}}{\rightrightarrows}} G_0 \quad (\iota : G_0 \to G_1)$$

is said to be *transitive* if the map $(\mathsf{s}, \mathsf{t}) : G_1 \to G_0 \times G_0$ is surjective.

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More characterizations, involving finite dimensional $\Bbbk$-representations of $\mathscr{G}$, are also possible to be proven.

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We will adopt the functorial point of view, that is a $\Bbbk$-groupoid will be though as a *presheaf of groupoids* on the category of affine $\Bbbk$-schemes $\mathcal{A}lg_{\Bbbk}^{op}$, implicitly endowed within the *fp* topology.

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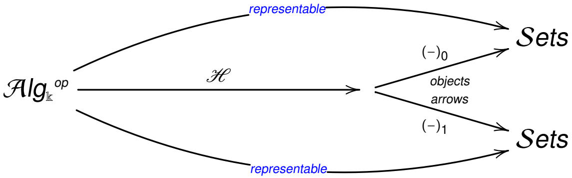

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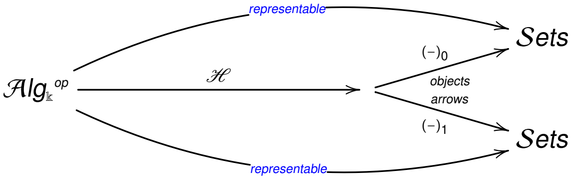

This is determined by a pair of commutative $\Bbbk$-algebras $(A, \mathcal{H})$ such that for any object C in $\mathscr{Alg}_{\Bbbk}$, we have in a functorial way, a groupoid structure

$$\mathscr{H}(C) : \qquad \mathscr{Alg}_{\Bbbk}(\mathcal{H}, C) \underset{t^*}{\overset{s^*}{\underset{\longleftarrow\ \varepsilon^*\ \longrightarrow}{\rightrightarrows}}} \mathscr{Alg}_{\Bbbk}(A, C).$$

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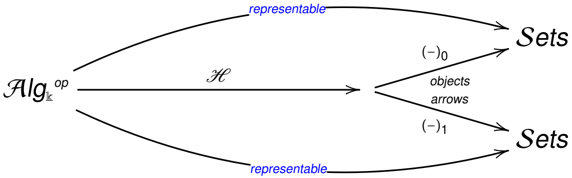

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Equivalently, there are morphisms of $\mathbb{k}$-algebras:

$$\underset{\text{source, target and identity arrow}}{A \underset{t}{\overset{s}{\underset{\varepsilon}{\rightrightarrows\!\!\leftarrow}}} \mathcal{H}} \qquad \underset{\text{composition}}{{}_s\mathcal{H}_t \xrightarrow{\Delta} {}_s\mathcal{H}_t \otimes_A {}_s\mathcal{H}_t,} \qquad \underset{\text{inverse arrow}}{{}_s\mathcal{H}_t \xrightarrow{\mathscr{S}} {}_t\mathcal{H}_s.}$$

satisfying the corresponding compatibility axioms of (the fibre) groupoid: co-associativity, co-unitary and idempotency properties.

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- *The presheaf of groupoids whose fibres are the induced groupoids. This is the presheaf of the form $\mathscr{A} \times \mathscr{B} \times \mathscr{A} \rightleftarrows \mathscr{A}$, where $\mathscr{B}$ and $\mathscr{A}$ are as before (but without action). For instance, if $\mathscr{B} = \mathscr{G}_m$ is the multiplicative $\Bbbk$-group, then the associated Hopf algebroid is of the form $\big(A, (A \otimes_{\Bbbk} A)[X, X^{-1}]\big)$.*

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As we will see later on, two objects of any presheaf associated to a GT Hopf algebroid are locally isomorphic.

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The *induction functor* $\boldsymbol{\phi}_* : \mathrm{Comod}_{\mathcal{H}} \to \mathrm{Comod}_{\mathcal{K}}$ associated to a morphism of Hopf algebroid $\boldsymbol{\phi} : (A, \mathcal{H}) \to (B, \mathcal{K})$ is defined by

$$\mathscr{O}_{\mathcal{H}}(-) \otimes_A {}_{\phi_0}B : \mathrm{Comod}_{\mathcal{H}} \to \mathrm{Comod}_{\mathcal{K}}.$$

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$$\Phi := (\phi_1^*, \phi_0^*) : \mathscr{K} \longrightarrow \mathscr{H}$$

between the associated presheaves of groupoids.

The base change morphism. Given an algebra map $\phi : A \to B$, then the pair of algebras $(B, \mathcal{H}_\phi) := (B, B \otimes_A \mathcal{H} \otimes_A B)$ is a Hopf algebroid known as *the base change Hopf algebroid*, and $(\phi, \phi_1) : (A, \mathcal{H}) \to (B, \mathcal{H}_\phi)$ is a morphism of Hopf algebroids, where

$$\phi_1 : \mathcal{H} \to \mathcal{H}_\phi, \quad u \mapsto 1_B \otimes_A u \otimes_A 1_B.$$

The *induction functor* $\boldsymbol{\phi}_* : \mathrm{Comod}_{\mathcal{H}} \to \mathrm{Comod}_{\mathcal{K}}$ associated to a morphism of Hopf algebroid $\boldsymbol{\phi} : (A, \mathcal{H}) \to (B, \mathcal{K})$ is defined by

$$\mathscr{O}_{\mathcal{H}}(-) \otimes_A {}_{\phi_0}B : \mathrm{Comod}_{\mathcal{H}} \to \mathrm{Comod}_{\mathcal{K}}.$$

This is by definition a monoidal symmetric functor.

Isotropy Hopf alegbras and their conjugations.

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Fix an object $x \in A(\Bbbk)$ and consider the presheaf of groups

$$\mathscr{H}_x : \mathscr{A}lg_{\Bbbk}^{op} \longrightarrow Grp, \quad C \longrightarrow \mathscr{H}(C)^{x^*(1_C)},$$

which assigns to any algebra C the the isotropy group $\mathscr{H}(C)^{x^*(1_C)}$ of $x^*(1_C) = 1_c \circ x \in A(C)$, where $1_c : \Bbbk \to C$ is the identity map of C.

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The presheaf $\mathscr{H}_x$ of isotropy groups, is represented by the Hopf $\Bbbk$-algebra $(\Bbbk_x, \mathcal{H}_x)$, where $\mathcal{H}_x := \Bbbk_x \otimes_A {}_{s}\mathcal{H}_{t} \otimes_A \Bbbk_x$ is the base change of $x : A \to \Bbbk$.

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Examples

- *For a presheaf of groupoids of pairs $(\mathscr{A} \times \mathscr{A}, \mathscr{A})$ there is only one type of isotropy groups, namely, the presheaf of trivial groups. The associated isotropy type Hopf algebra is then $\Bbbk$.*

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- *For the presheaf of groupoids $(\mathscr{A} \times \mathscr{G}_m \times \mathscr{A}, \mathscr{A})$ there is also only one type of isotropy groups, namely, $\mathscr{G}_m$. Thus $\Bbbk[X, X^{-1}]$ is the isotropy type Hopf algebra of $(A, (A \otimes_{\Bbbk} A)[X, X^{-1}])$.*

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Let x, y be two objects in $\mathscr{H}(\Bbbk)$. We say that *the isotropy Hopf algebras* $(\Bbbk_x, \mathcal{H}_x)$ *and* $(\Bbbk_y, \mathcal{H}_y)$ *are conjugated*, provided there is an isomorphism of Hopf algebras $\mathfrak{g} : (\Bbbk_x, \mathcal{H}_x) \to (\Bbbk_y, \mathcal{H}_y)$ such that the following diagram

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This means that there is an algebra map $g : \mathcal{H} \to \mathbb{k}$ such that

$$g \circ \mathsf{s} = x, \;\; g \circ \mathsf{t} = y, \;\; \text{and} \;\; u_1{}^{-} \otimes_A u_1{}^{0} \otimes_A u_1{}^{+} g(u_2) \; = \; g(u_1) \otimes_A u_2 \otimes_A 1_y \; \in \mathcal{H}_y$$

where, by denoting $\mathsf{z} := \mathfrak{g} \circ \mathfrak{x} : (A, \mathcal{H}) \to (\mathbb{k}_y, \mathcal{H}_y)$, we have

$$\mathsf{z}_0 = x \quad \text{and} \quad \mathsf{z}_1(u) = \mathfrak{g}(1_x \otimes_A u \otimes_A 1_x) := u^{-} \otimes_A u^{0} \otimes_A u^{+}.$$

for every $u \in \mathcal{H}$ (summations understood).

Weak equivalences between Hopf algebroids.

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- A morphism of flat Hopf algebroids $\phi : (A, \mathcal{H}) \to (B, \mathcal{K})$ is said to be a *weak equivalence* if the associated induction functor ϕ_* is an equivalence of categories.

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Two flat Hopf algebroids $(A, \mathcal{H})$ and $(B, \mathcal{K})$ are said to be *weakly equivalent* if there is a third Hopf algebroid $(C, \mathcal{J})$ with diagram

$$\begin{array}{ccccc} & & (C, \mathcal{J}) & & \\ & \nearrow & & \nwarrow & \\ (A, \mathcal{H}) & & & & (B, \mathcal{K}) \end{array}$$

of weak equivalences.

Principal Bi-bundles between flat Hopf algebroids.

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Let $(A, \mathcal{H})$ and $(B, \mathcal{K})$ be two flat Hopf algebroids. A *left principal* $(\mathcal{H}, \mathcal{K})$*-bundle* is a three-tuple (P, α, β) where

$$A \xrightarrow{\alpha} P \xleftarrow{\beta} B$$

is a diagram of $\mathbb{k}$-algebra, and P is an $(\mathcal{H}, \mathcal{K})$-bicomodule algebra, that is, P is left $\mathcal{H}$-comodule algebra via α and a right $\mathcal{K}$-comodule algebra via β, such that

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- β is a faithfully flat extension;
- the canonical map

  $$\mathrm{can}_{\mathcal{H},P} : P \otimes_B P \longrightarrow \mathcal{H} \otimes_A P, \quad \Big(p \otimes_B q \longmapsto p_{(-1)} \otimes_A p_{(0)} q \Big)$$

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Right principal bundles are similarly defined. A *principal bi-bundles* is simoultaniously a left and right principal bundle.

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- Given a morphism of flat Hopf algebroids $\boldsymbol{\phi} : (A, \mathcal{H}) \to (B, \mathcal{K})$, set $P := \mathcal{H} \otimes_{A\ \phi_0} B$ with the $\mathbb{k}$-algebra extensions

$$\alpha : A \to P,\ \alpha(a) = \mathsf{s}(a) \otimes_A 1_B, \quad \beta : B \to P,\ \beta(b) = 1_{\mathcal{H}} \otimes_A b.$$

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Left principal $(\mathcal{H}, \mathcal{K})$-bundles form a category, where each morphism is an isomorphism, i.e. a groupoid. The cotensor product of left principal bundles, is again a left principal bundle. Thus left principal bundles over flat Hopf algebroids form a *bicategory* which in fact is a *bigroupoid*.

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- For any extension $\phi : A \to B$, the associated canonical morphism $\boldsymbol{\phi} : (A,\mathcal{H}) \to (B,\mathcal{H}_\phi)$ of Hopf algebroids is a weak equivalence, where $\mathcal{H}_\phi = B \otimes_A \mathcal{H} \otimes_A B$;

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3. *Any two isotropy Hopf algebra are conjugated.*

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