Transitive groupoids, principal bisets and weak equivalences.

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Even at the abstract level, transitive groupoids still enjoying very rich structure, as we will see here.

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That is, a pair of two sets $\mathscr{G} := (G_1, G_0)$ with diagram

$$G_1 \underset{\mathfrak{t}}{\overset{\mathfrak{s}}{\rightrightarrows}} G_0 \;\; (\iota : G_0 \to G_1),$$

where $\mathfrak{s}$ and $\mathfrak{t}$ are resp. the source and the target of a given arrow, and ι assigns to each object its identity arrow. All together with an associative and unital multiplication $G_2 := G_1 \, {}_{\mathfrak{s}}\!\times_{\mathfrak{t}} \, G_1 \to G_1$ as well as a map $G_1 \to G_1$ which associated to each arrow its inverse.

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Given a groupoid $\mathscr{G}$, consider an object $x \in G_0$, *the isotropy group of $\mathscr{G}$ at x*, is the group:

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The orbit set of a groupoid $\mathscr{G}$ is the quotient set of G_0 by the following equivalence relation:

$$x \sim y, \text{ if and only if, there exists } g \in G_1 \text{ with } \mathfrak{s}(g) = x, \mathfrak{t}(g) = y.$$

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That is, the set $\pi_0(\mathscr{G})$ of all *connected components* of $\mathscr{G}$.

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- One can associated to a given set X the so called *the groupoid of pairs*, its set of arrows is defined by $G_1 = X \times X$ and the set of objects by $G_0 = X$; the sourse and the target are $\mathfrak{s} = pr_2$ and $\mathfrak{t} = pr_1$, the second and first projections, and the map of identity arrows is ι the diagonal map. The multiplication and the inverse maps are given by
$$(x, x')\,(x', x'') = (x, x''), \quad \text{and} \quad (x, x')^{-1} = (x', x).$$

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$$(x, x')\,(x', x'') \;=\; (x, x''), \quad \text{and} \quad (x, x')^{-1} \;=\; (x', x).$$
- Let $\nu : X \to Y$ be a map. Consider the fibre product $X\,{}_{\nu}\!\times_{\nu}\, X$ as a set of arrows of the groupoid $X\,{}_{\nu}\!\times_{\nu}\, X \overset{pr_2}{\underset{pr_1}{\overset{\longleftarrow\iota}{\rightrightarrows}}} X$, where as before $\mathfrak{s} = pr_2$ and $\mathfrak{t} = pr_1$, and the map of identity arrows is ι the diagonal map. The multiplication and the inverse are clear.

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- Let $\nu : X \to Y$ be a map. Consider the fibre product $X\,{}_\nu\!\times_\nu\, X$ as a set of arrows of the groupoid $X\,{}_\nu\!\times_\nu\, X \underset{pr_1}{\overset{pr_2}{\rightrightarrows}} X$, where as before $\mathfrak{s} = pr_2$ and $\mathfrak{t} = pr_1$, and the map of identity arrows is ι the diagonal map. The multiplication and the inverse are clear.
- Assume that $\mathcal{R} \subseteq X \times X$ is an equivalence relation on the set X. One can construct a groupoid $\mathcal{R} \underset{pr_1}{\overset{pr_2}{\rightrightarrows}} X$, with structure maps as before. This is an important class of groupoids known as *the groupoid of equivalence relation*.

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  The multiplication is given by $(x, g)(x', g') = (x, gg')$, whenever $xg = x'$:

$$\begin{tikzcd}
x(gg') = x'g' \arrow[r, bend left, "{(x',g')}"] \arrow[rr, bend right, dashed, "{(x,gg')}"'] & x' = xg \arrow[r, bend left, "{(x,g)}"] & x
\end{tikzcd}$$

  The inverse is defined by $(x, g)^{-1} = (xg, g^{-1})$.

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$$X\, {}_{\varsigma}\!\times_{\mathfrak{t}}\, G_1\, {}_{\mathfrak{s}}\!\times_{\varsigma}\, X = \Big\{(x, g, x') \in X \times G_1 \times X \mid \ \varsigma(x) = \mathfrak{t}(g), \varsigma(x') = \mathfrak{s}(g)\Big\}.$$

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The pair $\mathscr{G}^{\varsigma} = (G^{\varsigma}{}_1, G^{\varsigma}{}_0)$ is a groupoid, with structure maps:

$$\mathfrak{s} = pr_3, \quad \mathfrak{t} = pr_1, \quad \iota_x = (\varsigma(x), \iota_{\varsigma(x)}, \varsigma(x)), \text{ for every } x \in X.$$

The multiplication is defined by

$$(x, g, y)\,(x', g', y') \;=\; (x, gg', y'),$$

whenever $y = x'$, and the inverse is given by $(x, g, y)^{-1} = (y, g^{-1}, x)$.

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The groupoid $\mathscr{G}^{\varsigma}$ is known as *the induced groupoid of* $\mathscr{G}$ *by the map* ς, or *the pull-back groupoid of* $\mathscr{G}$ *along* ς with *the canonical morphism* $\phi^{\varsigma} : \mathscr{G}^{\varsigma} \to \mathscr{G}$ of groupoids, given by the second projection.

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In this case, to any group G and a set X, one can associated the groupoid $(X \times G \times X, X)$, as the induced groupoid of $(G, \{*\})$ by the map $* : X \to \{*\}$.

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A *morphism of groupoids* is a functor between the underlying categories.
Any morphism $\phi : \mathscr{H} \to \mathscr{G}$ of groupoids factors through the canonical morphism $\mathscr{G}^{\phi_0} \to \mathscr{G}$, that is we have the following (strict) commutative diagram

$$\begin{tikzcd} \mathscr{H} \arrow[rr, "\phi"] \arrow[dr, dashed, "\phi'"'] & & \mathscr{G} \\ & \mathscr{G}^{\phi_0} \arrow[ur] & \end{tikzcd}$$

of groupoids, where $\phi'_0 = id_{\mathscr{H}_0}$ and

$$\phi'_1 : H_1 \longrightarrow G^{\phi_0}{}_1, \quad \Big(h \longmapsto \big(\mathfrak{t}(h), \phi_1(h), \mathfrak{s}(h)\big)\Big).$$

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The last condition is shown as follows: Given a groupoid $\mathscr{G}$ satisfying the first condition. Fix an object $x \in G_0$ with isotropy group $\mathscr{G}^x$ and choose a family of arrows $\{f_y\}_{y \in G_0}$ such that $f_y \in \mathfrak{t}^{-1}(\{x\})$ and $\mathfrak{s}(f_y) = y$, for $y \neq x$ while $f_x = \iota(x)$, for $y = x$. In this way the morphism

$$\phi^x : \mathscr{G} \xrightarrow{\cong} (G_0 \times \mathscr{G}^x \times G_0, G_0), \quad \Big((g, z) \longmapsto \Big(\big(\mathfrak{s}(g), f_{\mathfrak{t}(g)}\, g\, f^{-1}_{\mathfrak{s}(g)}, \mathfrak{t}(g)\big), z\Big)\Big)$$

establishes an isomorphism of groupoids.

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  Explicitly, let X be a right G-set whose G-action is transitive, that is, the set of orbits is a one element set. Thus, for every pair of elements $x, y \in X$ there exists $g \in G$ such that $y = xg$. This means that the map $X \times G \to X \times X$, $(x, g) \mapsto (x, xg)$ is a surjective, and so the action groupoid $(X \times G, X)$ is transitive.

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Two groupoids $\mathscr{G}$ and $\mathscr{H}$ are said to be *weakly equivalent*, if there is a third groupoid $\mathscr{K}$ and a diagram of weak equivalences :

$$\begin{tikzcd} & \mathscr{K} \arrow[dl] \arrow[dr] & \\ \mathscr{G} & & \mathscr{H}\,. \end{tikzcd}$$

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The following definition is a natural generalization, to the context of groupoids, of the usual notion of group-set.

Definition

Given a groupoid $\mathscr{G}$ and a map $\varsigma : X \to G_0$. We say that (X, ς) is a *right $\mathscr{G}$-set*, if there is a map: *the action*

$$\rho : X \,{}_{\varsigma}\!\times_{\mathsf{t}}\, G_1 \longrightarrow X, \;\; \text{sending } (x, g) \longmapsto xg,$$

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3. $(xg)h = x(gh)$, for every $x \in X$, $g, h \in G_1$ with $\varsigma(x) = \mathfrak{t}(g)$ and $\mathfrak{t}(h) = \mathfrak{s}(g)$.

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A *left action* is analogously defined by interchanging the source with the target. Obviously, any groupoid $\mathscr{G}$ acts over itself on both sides by using the regular action: $G_1\, {}_{\mathfrak{s}}\!\times_{\mathfrak{t}}\, G_1 \to G_1$. That is, $(G_1, \mathfrak{s})$ is a right $\mathscr{G}$-set and $(G_1, \mathfrak{t})$ is a left $\mathscr{G}$-set with this action.

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Let (X, ς) be a right $\mathscr{G}$-set, and consider the pair of sets

$$X \rtimes \mathscr{G} := \Big(X \, {}_{\varsigma}\!\times_{\mathfrak{t}} \, G_1, X \Big)$$

as a groupoid with structure maps

$$\mathfrak{s} = \rho, \quad \mathfrak{t} = pr_1, \quad \iota_x = (x, \iota_{\varsigma(x)}).$$

The multiplication and the inverse maps are defined by

$$(x, g)(x', g') = (x, gg'), \quad \text{and } (x, g)^{-1} = (xg, g^{-1}).$$

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The groupoid $X \rtimes \mathscr{G}$ is known as the *right translation groupoid of X by $\mathscr{G}$*, and there is a canonical morphism $X \rtimes \mathscr{G} \to \mathscr{G}$ of groupoids, given by the second projection.

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Given a right $\mathscr{G}$-set (X, ς), the *orbit set* $X/\mathscr{G}$ of (X, ς) is the orbit set of the translation groupoid $X \rtimes \mathscr{G}$. For instance, if $\mathscr{G} = (X \times G, X)$ is an action groupoid, then obviously the orbit set of this groupoid coincides with the classical set X/G of orbits of X.

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Let $\mathscr{G}$ and $\mathscr{H}$ be two groupoids and $(X, \varsigma, \vartheta)$ a triple consisting of a set X and two maps $\varsigma : X \to G_0, \quad \vartheta : X \to H_0$.

Definition

The triple $(X, \varsigma, \vartheta)$ is said to be an $(\mathscr{H}, \mathscr{G})$-*biset* if there is a left $\mathscr{H}$-action $\lambda : H_1 \, {}_{\mathfrak{s}}\!\times_{\vartheta} \, X \to X$ and right $\mathscr{G}$-action $\rho : X \, {}_{\varsigma}\!\times_{\mathfrak{t}} \, G_1 \to X$ such that

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- For any $x \in X$, $h \in H_1$ and $g \in G_1$ with $\varsigma(x) = \mathfrak{t}(g)$, $\vartheta(x) = \mathfrak{s}(h)$, we have $h(xg) \;=\; (hx)g$.

The two sided translation groupoid associated to a given $(\mathscr{H}, \mathscr{G})$-biset $(X, \varsigma, \vartheta)$ is defined to be the groupoid $\mathscr{H} \ltimes X \rtimes \mathscr{G}$ whose set of objects is X and set of arrows is $H_1\, {}_{\mathfrak{s}}\!\times_{\vartheta} X\, {}_{\varsigma}\!\times_{\mathfrak{s}} G_1$. The structure maps are:

$$\mathfrak{s}(h, x, g) = x, \quad \mathfrak{t}(h, x, g) = hxg^{-1} \quad \text{and} \quad \iota_x = (\iota_{\vartheta(x)}, x, \iota_{\varsigma(x)}).$$

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Associated to a given $(\mathscr{H}, \mathscr{G})$-biset $(X, \varsigma, \vartheta)$, there are to canonical morphism of groupoids:

$$\Sigma : \mathscr{H} \ltimes X \rtimes \mathscr{G} \longrightarrow \mathscr{G}, \qquad \Big((h,x,g),y\Big) \longmapsto \Big(g, \varsigma(y)\Big),$$

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Analogously one defines *right principal* $(\mathscr{H},\mathscr{G})$*-biset. A principal* $(\mathscr{H},\mathscr{G})$*-biset* is both left and right principal biset.

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For instance, $(G_1, \mathfrak{t}, \mathfrak{s})$ is a left and right principal $(\mathscr{G}, \mathscr{G})$-biset, known as the *unit principal biset*, which we denote by $\mathscr{U}(\mathscr{G})$.

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Precisely, assume we are given $(X, \varsigma, \vartheta)$ a left principal $(\mathscr{H}, \mathscr{G})$-biset, and let $\psi : \mathscr{K} \to \mathscr{G}$ be a morphism of groupoids. Consider the set $Y := X\, {}_{\vartheta}\!\times_{\psi_0} K_0$ together with the maps $pr_2 : Y \to K_0$ and $\widetilde{\varsigma} := \varsigma \circ pr_1 : Y \to H_0$.

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which is actually a left principal $(\mathscr{H}, \mathscr{K})$-biset, and known as the *pull-back principal biset of* $(X, \varsigma, \vartheta)$; we denote it by $\psi^*\big((X, \varsigma, \vartheta)\big)$.

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A left principal biset is called a *trivial left principal biset* if it is the pull-back of the unit left principal biset, that is, of the form $\psi^*(\mathscr{U}(\mathscr{G}))$ for some morphism of groupoids $\psi : \mathscr{K} \to \mathscr{G}$.

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- *If $(X, \varsigma, \vartheta)$ is a principal $(\mathscr{H}, \mathscr{G})$-biset, then the canonical morphisms of groupoids*

$$\begin{tikzcd} & \mathscr{H} \ltimes X \rtimes \mathscr{G} \arrow[dl, "\Theta"'] \arrow[dr, "\Sigma"] & \\ \mathscr{H} & & \mathscr{G} \end{tikzcd}$$

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In particular, $\mathscr{G}$ and $\mathscr{H}$ are weakly equivalent.

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- *$\mathscr{G}$ is a transitive groupoid;*
- *For every map $\varsigma : X \to G_0$, the pull-back biset ${\phi^{\varsigma}}^{*}(\mathscr{U}(\mathscr{G}))$ is a principal $(\mathscr{G}, \mathscr{G}^{\varsigma})$-biset.*

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