Basic theory of abstract groupoids and their linear representations. Applications.

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- for any two objects X, Y in C_0, there is a set of arrows, or *morphisms* denoted by $C(X, Y)$, where any of the sets $C(X, X)$ contains a special element denote 1_X called *the identity morphism* of X. The set of all morphism is denoted by C_1.

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If the 'class' of object C_0 is actually a set, then C is said to be a *small category*.

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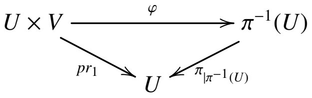
- Given a poset $(\mathcal{P}, \leq)$ one can consider the category whose collection of objects is the set $\mathcal{P}$ it self and $\mathcal{P}(p, q)$ contains one element if $p \leq q$ and empty otherwise. Take for instance a topological space X and consider it poset $Open(X)$ of all open subsets with inclusion as partial order.

More examples of categories.

More examples of categories. Let X be a topological space and $\mathbb{k}$ a topological base field (e.g., $\mathbb{k} = \mathbb{R}$ or $\mathbb{C}$). A $\mathbb{k}$-vector bundle $\mathcal{E} = (E, \pi)$ over X consists of:

1. a family $\{E_x\}_{x \in X}$ of finite-dimensional $\mathbb{k}$-vector spaces,
2. The disjoint union $E = \biguplus_{x \in X} E_x$ admits a topology, which induces the natural topology on each (fibre) E_x, such that the canonical projection $\pi : E \to X$ is continuous.
3. for every point $x \in X$, there exist a neighbourhood U of x, finite-dimensional $\mathbb{k}$-vector space V and a homeomorphism $\varphi : U \times V \to \pi^{-1}(U)$ such that the diagram



commutes, and such that for every point the obvious map $\varphi_y : V \to E_y$ is $\mathbb{k}$-linear. such that $\mathcal{E}_U$ is a isomorphic to a trivial bundle.

Morphism of vector bundles are intuitively defined and the category $VB_{\mathbb{k}}(X)$ so is obtained, is referred to as the category of $\mathbb{k}$-vector bundle over X.

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$$\mathcal{F}_1(g \circ f) = \mathcal{F}_1(g) \circ \mathcal{F}_1(f) \text{ (covariant)}, \quad \mathcal{F}(1_X) = 1_{\mathcal{F}_0(X)}.$$

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- Let $\Bbbk|\Bbbk_0$ be a field extension and V a $\Bbbk$-vector space. We can then consider the functor $- \otimes_{\Bbbk_0} V : Vect_{\Bbbk_0} \to Vect_{\Bbbk}$.

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$$\zeta_{-} : \mathcal{D}(F(Y), -) \longmapsto \mathcal{D}(F(X), -),$$
$$\Big(\zeta_P : \mathcal{D}(F(Y), P) \longmapsto \mathcal{D}(F(X), P),\ [g \longmapsto g \circ \mathcal{F}(f)]\Big)$$

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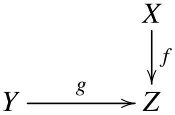
For a given small category $\mathcal{C}$ the associated simplicial set $\cdots C_2 \; \substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow} \; C_1 \; \substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow} \; C_0$ is called *the nerve of* $\mathcal{C}$.

Pull-backs

Let C be a category and consider a diagram



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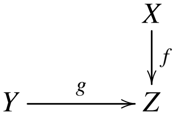

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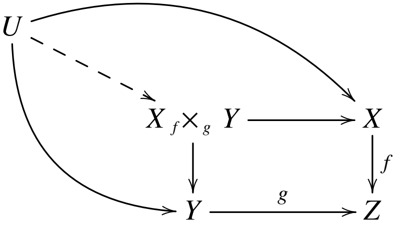

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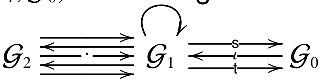

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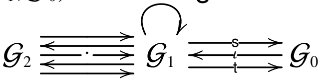

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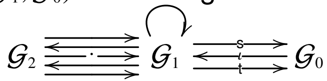

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$$\mathcal{G}_2 \;\substack{\longrightarrow\\ \longleftarrow \\ \overset{\cdot}{\longrightarrow} \\ \longleftarrow \\ \longrightarrow}\; \mathcal{G}_1 \circlearrowleft \;\substack{\overset{s}{\longrightarrow} \\ \overset{\iota}{\longleftarrow} \\ \overset{t}{\longrightarrow}}\; \mathcal{G}_0$$

where $\mathcal{G}_2 := \mathcal{G}_1 \, {}_s\!\times_t \, \mathcal{G}_1 \overset{\cdot}{\longrightarrow} \mathcal{G}_1$ is the multiplication (opposite to the composition), and the rest of the maps are the obvious ones.

Whenever pull-backs are allowed *groupoid objects* are allowed as well:

Categories	Groupoid objects
Top	Topological groupoids
Diff-manifolds	Lie groupoids
Algebraic Varieties	Algebraic groupoids
Groups	Crossed modules
(pre) Sheaves	(pre) Stacks

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- Any equivalence relation $\mathcal{R} \subseteq X \times X$ defines what is known as the *equivalence relation groupoid* whose structure is analogue to the previous one.
- The *action groupoid* is a groupoid of the form $(X \times G, X)$ where X a right G-set. The source is the action while the target is the first projection. The multiplication and the inverse are given by: $(x, g)\,(y, h) \;=\; (x, gh), \quad (x, g)^{-1} \;=\; (gx, g^{-1})$.

The Poincaré groupoid.

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Let X be a topological space, and denoted by $[\gamma]$ the homotopy equivalence class of a path $\gamma : [0,1] \to X$ whose source is denoted by $s([\gamma]) := \gamma(0)$ and its target by $t([\gamma]) := \gamma(1)$. Consider, for every $x \in X$, the class $\iota_x := [i_x]$ of the constant i_x path on x.

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The partial multiplication and the inverse of homotopy-classes are given by: $[\gamma]\,[\delta] \;=\; [\gamma \cdot \delta]$ and $[\gamma]^{-1} = [-\gamma]$, where

$$\gamma \cdot \delta := \begin{cases} \gamma(2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ \delta(2t-1) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases} \qquad (-\gamma)(t) = \gamma(1-t)$$

The groupoid so is constructed is denoted by $\pi(X)$.

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$$\phi_1 \circ \iota = \iota \circ \phi_0, \quad \phi_0 \circ \mathsf{s} = \mathsf{s} \circ \phi_1, \quad \phi_0 \circ \mathsf{t} = \mathsf{t} \circ \phi_1, \quad \phi_1(fg) = \phi_1(f)\phi_1(g),$$

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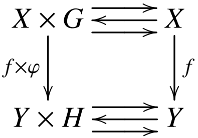
Some examples.

- For any groupoid $\mathcal{G}$ then inclusion $\mathcal{G}^{(i)} \hookrightarrow \mathcal{G}$ is a morphism of groupoids.
- Let $\varphi : G \to H$ be a morphism of groups, consider X and Y respectively a right G and H sets with equivariant map $f : X \to Y$. Then we have a morphism of groupoids



Left and right stars.

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$$t^{-1}(\{x\}) := \{g \in \mathcal{G}_1 |\ t(g) = x\}, \quad s^{-1}(\{x\}) := \left\{g \in \mathcal{G}_1 |\ s(g) = x\right\}$$

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The *isotropy groupoid* is the bundle of groups $\mathcal{G}^{(i)} = \bigcup_{x \in \mathcal{G}_0} \mathcal{G}^x \to \mathcal{G}_0$.

Some classical definition.

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The ancient notion of symmetry.

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(Vitruvius, 1st Century BC):

"*Symmetry is proportioned correspondence of the elements of the work itself, a response, in any given part, of the separate parts to the appearance of the entire figure as a whole. Just as in the human body there is a harmonious quality of shapeliness expressed in terms of the cubit, foot, palm, digit, and other small units, so it is in completing the work of architecture*".

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The modern notion of symmetry.

(Hermann Weyl, 1952):

"*Given a spatial configuration F, those automorphisms of space which leave F unchanged form a group Γ, and this group describes exactly the symmetry possessed by F*".

Example of Weyl's symmetry: The snowflake

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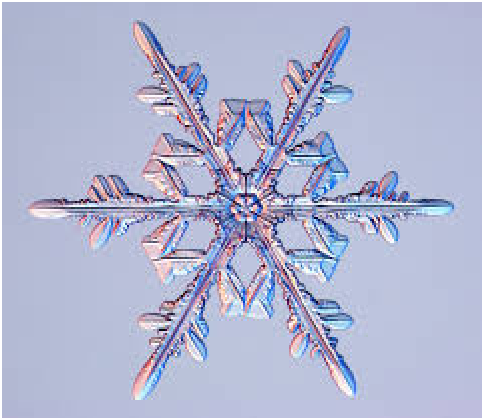

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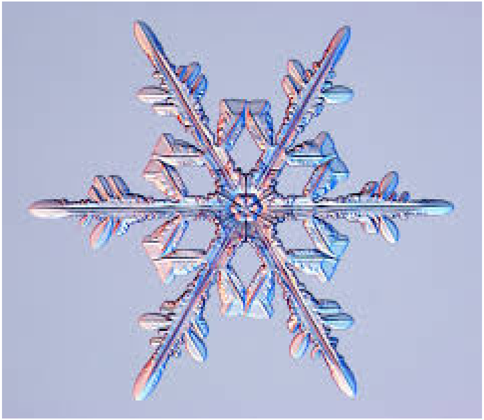

By considering all possible transformation interchanging the equivalent part, the symmetry of this spatial configuration is governed by the dihedral group D_6.

The Hydrogen Transition.

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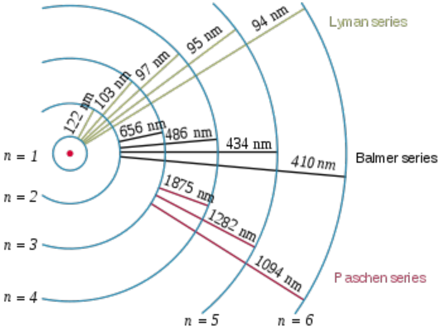

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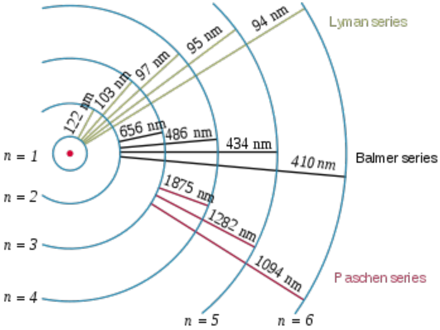

Spectral lines of the Hydrogen Atom

Groupoid and the birth of non-commutative geometry.

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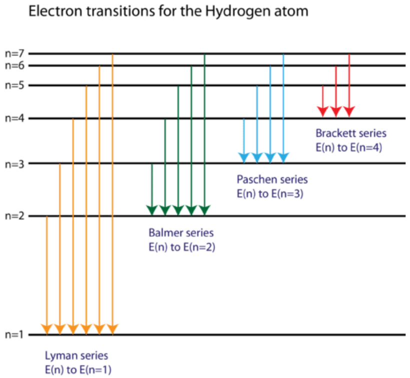

The different levels of energies $E(n)_{1\leq n\leq 7}$, form a groupoids of pairs, this seems was first observed by Alain Connes and was perhaps one of his motivation to formulate his *non commutative geometry*.

Molecular vibrations and vector bundle.

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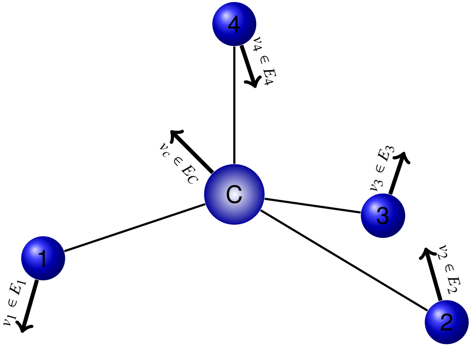

Figura: Molecular model of Carbon Tetrachloride.

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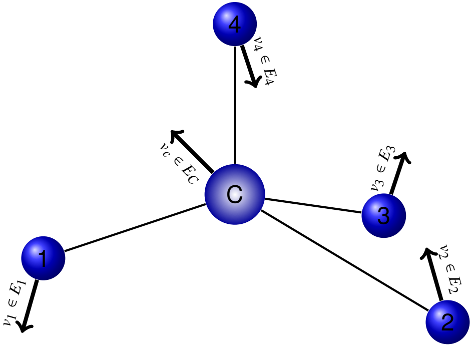

Figura: Molecular model of Carbon Tetrachloride.

In a small displacement from equilibrium, each of the atoms moves in its own three-dimensional vector space: E_1, E_2, E_3, E_4 and E_C.

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Now, let us see how the group S_4 acts on the set of displacements. Consider, for example, the action of the element $(123) \in S_4$. On the molecule itself, at equilibrium, (123) leaves C fixed, rotates the chlorine atoms $1, 2$ and 3 and leaves 4 fixed:

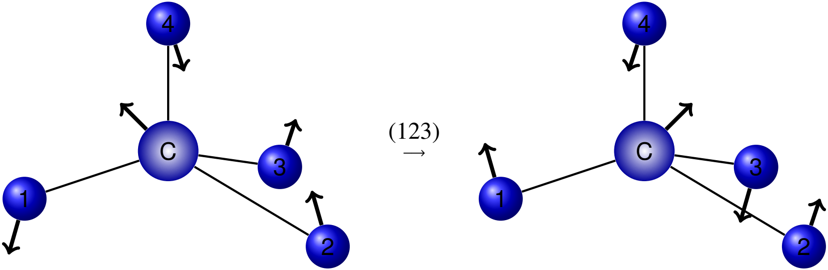

Figura: The action of the element $(123) \in S_4$ on the displacements of Carbon Tetrachloride.

Molecular vibrations and vector bundle.

Molecular vibrations and vector bundle. Set $M = \{1, 2, 2, 3, 4, C\}$ to be the set of atoms, in the previous example. Then $(\mathcal{E}, \pi)$, where $E = \biguplus_{x \in M} E_x$ and $\pi : E \to M$ is the obvious maps, is a S_4-equivariant vector bundle, or *homogeneous vector bundle*, whose associated module of global sections:

$$\Gamma(\mathcal{E}) := \left\{ \sigma : M \to E | \, \pi \circ \sigma = \text{identity} \right\}$$

is the space of displacements of the molecule as a whole, and the action of S_4 on $\Gamma(\mathcal{E})$ might be considered as the action of the symmetry group on the space of displacements.

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In general let us assume that a group G acts on set M and consider it associated *action groupoid* $\mathcal{G} := (G \times M, M)$. Then any G-equivariant vector bundle over M leads to a linear representation on $\mathcal{G}$. The converse also holds true, thus, any finite-dimensional (having the same dimension at each fibre) linear representation of $\mathcal{G}$, gives rise to a G-equivariant vector bundle.

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There is in fact an equivalence of (symmetric monoidal) categories between the category of G-equivariant bundles over M and that of linear representations of $\mathcal{G}$.

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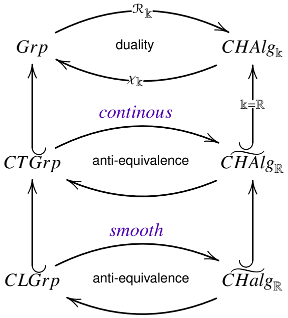
The classical picture of groups is represented by the following diagram



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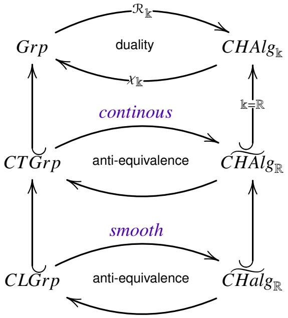

where $\widetilde{CHAlg}_{\Bbbk}$ the subcategory of commutative real Hopf algebras with *gauge* (i.e., a Hopf integral coming from the Haar measure) and with dense character group in the linear dual.

The ultimate goal of our research on linear representations of groupoids is to give a new approach to Tannaka-Krein duality for compact topological groupoids, and try to recognize the theory of compact Lie groupoids as a part of the theory of real affine algebraic groupoids.

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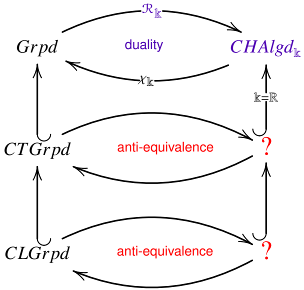

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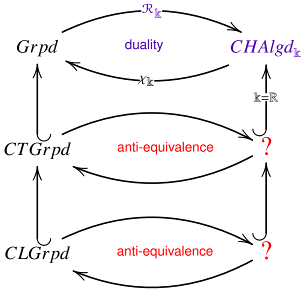

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In this talk, we will see how to construct the functor $\mathcal{R}_{\Bbbk}$ and show the main steps in building up the duality between the category of transitive groupoids and the category of *(geometrically) transitive commutative Hopf algebroids*.

Let $\Bbbk$ denotes a ground base field, $\mathrm{Vect}_{\Bbbk}$ its category of vector spaces, and $\mathrm{vect}_{\Bbbk}$ the full subcategory of finite dimensional ones.

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For a given groupoid

$$\mathcal{G} : \mathcal{G}_1 \underset{t}{\overset{s}{\rightrightarrows}} \mathcal{G}_0 \; (\iota : \mathcal{G}_0 \to \mathcal{G}_1),$$

we consider the category of all $\mathcal{G}$-*representations* as the symmetric monoidal $\Bbbk$-linear abelian category of functors $[\mathcal{G}, \mathrm{Vect}_{\Bbbk}]$ with identity object $\mathcal{I} : \mathcal{G}_0 \to \mathrm{Vect}_{\Bbbk}, x \to \Bbbk, g \to 1_{\Bbbk}$.

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The disjoint union of all the fibres of a $\mathcal{G}$-representation $\mathcal{V}$ is denoted by $\overline{\mathcal{V}} = \biguplus_{x \in G_0} \mathcal{V}_x$ and *the canonical projection* by $\pi_{\mathcal{V}} : \overline{\mathcal{V}} \to \mathcal{G}_0$. This called *the associated vector* $\mathcal{G}$*-bundle of the representation* $\mathcal{V}$.

Let $\mathcal{V}$ be a $\mathcal{G}$-representation in $[\mathcal{G}, \mathrm{vect}_{\Bbbk}]$, we define its *dimension function* as the map

$$d_{\mathcal{V}} : \mathcal{G}_0 \longrightarrow \mathbb{N}, \quad \Big(x \longmapsto dim_{\Bbbk}(\mathcal{V}_x)\Big),$$

which clearly extends to a map $d_{\mathcal{V}} : \pi_0(\mathcal{G}) \to \mathbb{N}$.

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Let $\mathcal{V}$ and $\mathcal{W}$ be two representations in $\mathbf{rep}_{\Bbbk}(\mathcal{G})$. Then

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Therefore, the category $\mathbf{rep}_{\Bbbk}(\mathcal{G})$ is a symmetric rigid monoidal $\Bbbk$-linear abelian category. But NOT *locally finite*, in general.

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Consider the set $X = \{1, 2\}$ and denote by $\mathcal{G}^{\{1,2\}}$ the associated groupoid of pairs. Thus $\mathcal{G}_0 = \{1, 2\}$ and $\mathcal{G}_1 = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$.

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The vector spaces of homomorphisms are given by

$$\mathbf{rep}_{\mathbb{k}}(\mathcal{G}^{\{1,2\}})\Big((n, N),\ (m, M)\Big) \ = \ M_{m,n}(\mathbb{k}),$$

the $\mathbb{k}$-vector space of $m \times n$ matrices with matrix multiplication.

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The other operations in $\mathbf{rep}_{\Bbbk}(\mathcal{G}^{\{1,2\}})$ are

$$(n, N) \oplus (m, M) \ = \ \left(n + m, \begin{pmatrix} N & 0 \\ 0 & M \end{pmatrix}\right), \quad \mathcal{D}(n, N) \ = \ (n, N^t)$$

$$(n, N) \otimes (m, M) = \Big(nm,\ (N\, b_{ij})_{1 \leq i,j \leq m}\Big), \text{ where } M = (b_{ij}), \text{ and } \mathcal{I} = (1, 1).$$

$$\mathrm{Tr}(n, N) \ = \ n.$$

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Furthermore, $\mathbf{rep}_{\Bbbk}(\mathcal{G})$ admits a non trivial fibre functor to the category of finite dimensional vector spaces. Namely, fix an object $x \in \mathcal{G}_0$, and consider the functor

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Summarizing $(\mathbf{rep}_{\Bbbk}(\mathcal{G}), \boldsymbol{\omega}_x)$ is a (neutral) Tannakian category in the sense of Saavedra-Rivano, Deligne and Milne.

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Let $\mathcal{G}$ be a groupoid and denote by $A_0(\mathcal{G}) := \Bbbk^{\mathcal{G}_0}$ its *base algebra* and by $A_1(\mathcal{G}) := \Bbbk^{\mathcal{G}_1}$ its *total algebra*. By reflecting the groupoid structure of $\mathcal{G}$, we have a diagram of algebras:

$$A_0(\mathcal{G}) \underset{t^*}{\overset{s^*}{\rightrightarrows}} A_1(\mathcal{G}), \quad \iota^*: A_1(\mathcal{G}) \to A_0(\mathcal{G}).$$

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The set of objects $\mathcal{G}_0$ is then a disjoint union $\mathcal{G}_0 = \bigcup_{i=1}^{N} G_{\mathcal{V}}^i$, where each of the $G_{\mathcal{V}}^i$'s is the inverse image $G_{\mathcal{V}}^i := d_{\mathcal{V}}^{-1}\left(\{n_i\}\right)$, for any $i = 1, \cdots, N$.

This leads to a decomposition of the base algebra $A_0(\mathcal{G})$:

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We can then define the functor which acts on objects by:

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By identifying a $\mathcal{G}$-representation in $\mathbf{rep}_{\Bbbk}(\mathcal{G})$ with its associated vector $\mathcal{G}$-bundle, we can consider the $\Bbbk$-vector space of "global sections":

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Both functors $\boldsymbol{\omega}$ and Γ are symmetric monoidal faithful functors. Moreover, there is a tensorial natural isomorphism $\boldsymbol{\omega} \cong \Gamma$.

Tannakian reconstruction process and the universal Hopf algebroid.

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PR-2 The functor: $(A \otimes A)\text{-CAlg}_{\mathbb{k}} \longrightarrow \text{Sets}$,

$$\Big(A \overset{s}{\underset{t}{\rightrightarrows}} C \Big) \longrightarrow \mathrm{Iso}^{\otimes}(t^*\omega,\ s^*\omega)$$

is representable.

The universal solution for both PR-1-2 is given by the following A-bimodule

$$\mathcal{L}_{\Bbbk}(\mathcal{T}, \omega) := \frac{\bigoplus_{P \in \mathcal{T}} \omega(P)^* \otimes_{T_P} \omega(P)}{\mathcal{J}},$$

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The representative functions functor.

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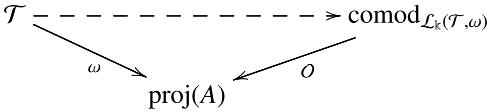
It turns out that $(A, \mathcal{L}_{\Bbbk}(\mathcal{T},\omega))$ is a *commutative Hopf algebroid*, such that there is a commutative diagram:



where $\mathrm{comod}_{\mathcal{L}_{\Bbbk}(\mathcal{T},\omega)}$ is the full subcategory of comodules with finitely generated and projective underlying A-modules.

Let $\mathcal{G}$ be a groupoid and consider the pair $(\mathbf{rep}_{\Bbbk}(\mathcal{G}), \boldsymbol{\omega})$. Applying the previous general constructions, we obtain a commutative Hopf algebroid $\big(A_0(\mathcal{G}), \mathcal{L}_{\Bbbk}(\mathbf{rep}_{\Bbbk}(\mathcal{G}), \boldsymbol{\omega})\big)$, which we denote by $\big(A_0(\mathcal{G}), \mathcal{R}_{\Bbbk}(\mathcal{G})\big)$.

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The terminology "functions" is justified by the following $(A_0(\mathcal{G}) \otimes_{\Bbbk} A_0(\mathcal{G}))$-algebra map:

$$\xi : \mathcal{R}_{\Bbbk}(\mathcal{G}) \longrightarrow A_1(\mathcal{G}),$$

where $A_1(\mathcal{G})$ is an $A_0(\mathcal{G}) \otimes_{\Bbbk} A_0(\mathcal{G})$-algebra, as before by reflecting the groupoid structure of $\mathcal{G}$.

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The representative functions establish a contravariant functor:

$$\mathcal{R}_{\Bbbk} : Grpd \longrightarrow CHAlg_{\Bbbk}$$

from the category of abstract groupoids to the category of commutative Hopf algebroids.

Examples.

(•) If G is a groupoids with only one object, that is, a group, then $\mathcal{R}_{\Bbbk}(G)$ is the usual Hopf algebra of representative funcions on the group G. This is isomorphic to the finite dual $\Bbbk[G]^o$ of the group algebra $\Bbbk[G]$.

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- (•) Let $X = \{1, 2\}$ be a set of two elements and consider as before the groupoid $\mathcal{G}^{\{1,2\}}$ of pairs and denote by $A := \Bbbk \times \Bbbk$ its base algebra. Then

$$\mathcal{R}_{\Bbbk}(\mathcal{G}) = \frac{\bigoplus_{n \in \mathbb{N}} A^n \otimes_{M_n(\Bbbk)} A^n}{\langle v \otimes_{M_n(\Bbbk)} \lambda w - \lambda^t v \otimes_{M_m(\Bbbk)} w \rangle_{v \in A^n,\, w \in A^m,\, \lambda \in M_{m \times n}(\Bbbk)}}.$$

Examples.

- (•) If G is a groupoids with only one object, that is, a group, then $\mathcal{R}_{\Bbbk}(G)$ is the usual Hopf algebra of representative funcions on the group G. This is isomorphic to the finite dual $\Bbbk[G]^o$ of the group algebra $\Bbbk[G]$.
- (•) Let $X = \{1, 2\}$ be a set of two elements and consider as before the groupoid $\mathcal{G}^{\{1,2\}}$ of pairs and denote by $A := \Bbbk \times \Bbbk$ its base algebra. Then

$$\mathcal{R}_{\Bbbk}(\mathcal{G}) = \frac{\bigoplus_{n \in \mathbb{N}} A^n \otimes_{M_n(\Bbbk)} A^n}{\langle v \otimes_{M_n(\Bbbk)} \lambda w - \lambda^t v \otimes_{M_m(\Bbbk)} w \rangle_{v \in A^n,\, w \in A^m,\, \lambda \in M_{m \times n}(\Bbbk)}}.$$

- (•) Let $\mathcal{G}\colon\ G \times X \underset{pr_1}{\overset{\varrho}{\underset{\iota}{\rightrightarrows}}} X$ be an action groupoid. Then there is a morphism of Hopf algebroids:

$$(\Bbbk^X, \Bbbk^X \otimes \mathcal{R}_{\Bbbk}(G) \otimes \Bbbk^X) \longrightarrow (\Bbbk^X, \mathcal{R}_{\Bbbk}(\mathcal{G})).$$

  Furthermore, if the action is transitive, then any isotropy Hopf algebra $(\Bbbk_x, \mathcal{R}_{\Bbbk}(\mathcal{G})^x)$, for $x \in X$, is isomorphic to $(\Bbbk, \mathcal{R}_{\Bbbk}(G))$.
  In general, $\mathcal{R}_{\Bbbk}(\mathcal{G})$ is not a split Hopf algebroid (i.e., not isomorphic to $\Bbbk^X \otimes \mathcal{R}_{\Bbbk}(G)$)

The properties of $\mathcal{R}_{\Bbbk}(\mathcal{G})$ when $\mathcal{G}$ is transitive.

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- $(A_0(\mathcal{G}), \mathcal{R}_{\Bbbk}(\mathcal{G}))$ is a *transitive* Hopf algebroid, in the sense that each of the fibers of its associated presheaf of groupoids is actually a transitive groupoids
  (i.e., each of the groupoids $(\mathcal{R}_{\Bbbk}(\mathcal{G})(C), A_0(\mathcal{G})(C))$ is transitive, for any commutative algebra C). The notation is $R(C) := \mathrm{Alg}_{\Bbbk}(R, C)$.

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- The fibre functor $\boldsymbol{\omega} : \mathbf{rep}_{\Bbbk}(\mathcal{G}) \to \mathrm{proj}(A_0(\mathcal{G}))$ induces a monoidal equivalence between the category $\mathbf{rep}_{\Bbbk}(\mathcal{G})$ and the category $\mathrm{comod}_{\mathcal{R}_{\Bbbk}(\mathcal{G})}$ of comodules with finitely generated underlying module.

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- Any comodule M in $\mathrm{comod}_{\mathcal{R}_{\Bbbk}(\mathcal{G})}$ is a locally free $A_0(\mathcal{G})$-module with constant rank, in the sense that, if for some $x \in \mathcal{G}_0$, we have $dim_{\Bbbk}(M_x) = n$, then so is the dimension of any other fibre.

The character functor and the counit. Let $(A, \mathcal{H})$ be a commutative Hopf algebroid such that $A \neq 0$ and $Alg_{\Bbbk}(A, \Bbbk) \neq \emptyset$. The groupoid

$$\mathscr{H}(\Bbbk): \; Alg_{\Bbbk}(\mathcal{H}, \Bbbk) \overset{s^*}{\underset{t^*}{\underset{\longleftarrow \iota^* \longrightarrow}{\rightrightarrows}}} Alg_{\Bbbk}(A, \Bbbk),$$

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Furthermore, for any groupoid $\mathscr{G}$, we have a natural transformation:

$$\mathscr{G} \longrightarrow \chi_{\Bbbk} \circ \mathscr{R}_{\Bbbk}(\mathscr{G}),$$

which is not in general a monomorphism. When it is, the groupoid $\chi_{\Bbbk}(\mathscr{R}_{\Bbbk}(\mathscr{G}))$ is sometimes called *the algebraic cover of* $\mathscr{G}$.

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$$\mathcal{L}_{\mathbb{k}}(comod_{\mathcal{H}}, \mathcal{O}) \longrightarrow \mathcal{H}, \quad \text{where } \mathcal{O} : comod_{\mathcal{H}} \to \mathrm{proj}(A),$$
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- there is a natural morphism of Hopf transitive Hopf algebroids:

$$(A, \mathcal{H}) \longrightarrow (A_0(\chi_{\mathbb{k}}(\mathcal{H})), \mathcal{R}_{\mathbb{k}}(\chi_{\mathbb{k}}(\mathcal{H})))$$

Notations: Denotes by $TGrpd$ the full subcategory of transitive groupoids, and by $TCHAlgd_{\Bbbk}$ the full subcategory of transitive commutative Hopf algebroids.

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As we have seen before, the represenative functions functor define a contravariant functor

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There is a natural isomorphism

$$TGrpd(- , \chi_{\Bbbk}(+)) \;\cong\; TCHAlgd_{\Bbbk}(+ , \mathcal{R}_{\Bbbk}(-)).$$

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Alan Weinstein:

“*I hope to have convinced the reader that groupoids are worth knowing about and worth looking out for.*”

“*Spero di aver convinto il lettore che i gruppoidi sono qualcosa che valga la pena conoscere e investigare.*”

“*Espero haber convencido al lector de que merece la pena conocer los groupoids y quedarse a la expectativa.*”

“*J’espère avoir convaincu le lecteur que les groupoïdes valent la peine d’être connus et méritent d’être recherchés.*”

References

- Katherine, Brading and Elena, Castellani, eds. (2003), *Symmetries in Physics: Philosophical Reflections.* Cambridge: Cambridge University Press.
- R.Brown, *From groups to groupoids: A brief survey.* Bull. London Math. Soc. **19** (2), (1987), 113–134.
- Alain, Connes, *Noncommutative Geometry*. (1994), San Diego: Academic Press.
- L. El Kaoutit, *On geometrically transitive Hopf algebroids*, arXiv:1508.05761v2 [math:AC] (2015).
- L. El Kaoutit and N. Kowalzig, *Morita theory for Hopf algebroids, principal bibundles, and weak equivalences*. Documenta Math., **22**, 551–609 (2017).
- Alexandre Guay and Brian Hepburn, *Symmetry and Its Formalisms: Mathematical Aspects*, Philosophy of Science, **76** (April 2009), 160?178.
- Alan, Weinstein, *Groupoids: Unifying Internal and External Symmetry*, Notices of the American Mathematical Society 43: 744–752.
- Hermann, Weyl, *Symmetry*. (1952) Princeton, N.J: Princeton University Press.