

Finite dimensional representations of abstract groupoids, representative functions and commutative Hopf algebroids

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Brauer groups, Hopf algebras and monoidal categories.
In honour of Stef Caenepeel on the occasion of his 60 birthday.

Turin, May 2016.

The context, motivations and overviews.

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Representative functions on group.

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Let G be an abstract group, and ρ a finite dimensional representation in \mathbb{k} -vector spaces.

Consider the subalgebra $\mathcal{V}(\rho)$ of the algebra $\text{Maps}(G, \mathbb{k}) := \mathbb{k}^G$ generated by the functions of the form:

$$a_{ij}^- : G \rightarrow \mathbb{k}$$

where for $g \in G$, $\rho(g) = (a_{ij}^g)_{i,j}$ is the invertible matrix defining $\rho(g)$.

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The *algebra of representative functions on G* , is then defined to be

$$\mathcal{R}_{\mathbb{k}}(G) = \sum_{\rho \in \text{rep}_{\mathbb{k}}(G)} \mathcal{V}(\rho) \subseteq \mathbb{k}^G.$$

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This is a *commutative Hopf \mathbb{k} -algebra* which "codifies" almost all informations about the group G (excluding extreme cases, of course).

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There is a natural isomorphism, i.e., contravariant adjunction

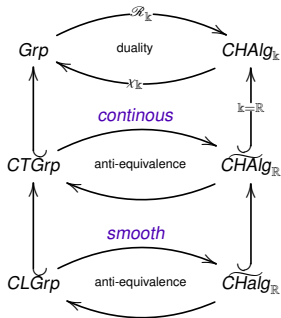
$$CHAlg_{\mathbb{k}}(-; \mathcal{R}_{\mathbb{k}}(+)) \cong Grp(+; \chi_{\mathbb{k}}(-))$$

which is known as *a duality between groups and Hopf algebras*.

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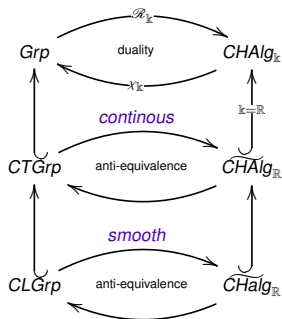
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where $\widetilde{CHAlg}_{\mathbb{k}}$ the subcategory of commutative real Hopf algebras with *gauge* (i.e., a Hopf integral coming from the Haar measure) and with dense character group in the linear dual.

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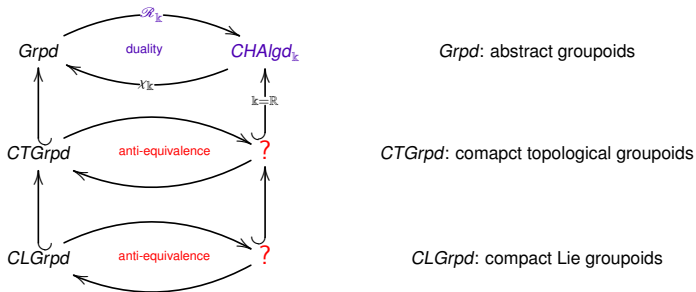
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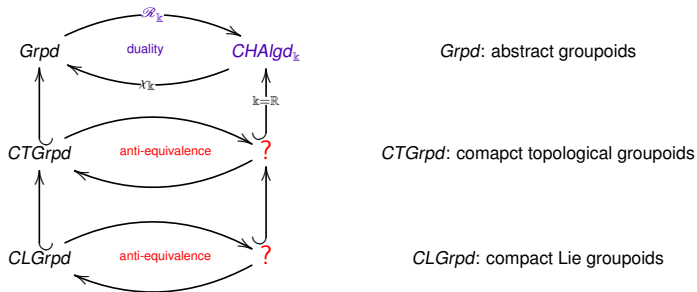
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In this talk, we will see how to construct the functor \mathcal{R}_k and show the main steps in building up the duality between the category of transitive groupoids and the category of (geometrically) transitive commutative Hopf algebroids.

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For a given (small) groupoid

$$\mathcal{G} : G_1 \begin{array}{c} \xrightarrow{s} \\ \xleftarrow{t} \\ \xrightarrow{t} \end{array} G_0 ,$$

we consider the category of all \mathcal{G} -representations as the symmetric monoidal \mathbb{k} -linear abelian category of functors $[\mathcal{G}, \text{Vect}_{\mathbb{k}}]$ with identity object $\mathcal{I} : G_0 \rightarrow \text{Vect}_{\mathbb{k}}, x \rightarrow \mathbb{k}, g \rightarrow 1_{\mathbb{k}}$.

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For any \mathcal{G} -representation \mathcal{V} the image of an object $x \in G_0$ is denoted by \mathcal{V}_x , and referred to as *the fibre of \mathcal{V} at x* .

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The disjoint union of all the fibres of a \mathcal{G} -representation \mathcal{V} is denoted by $\overline{\mathcal{V}} = \bigcup_{x \in G_0} \mathcal{V}_x$ and *the canonical projection* by $\pi_{\mathcal{V}} : \overline{\mathcal{V}} \rightarrow G_0$. This called *the associated vector \mathcal{G} -bundle of the representation \mathcal{V}* .

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- (1) Given any set X , one can associated the so called *the groupoid of pairs* \mathcal{G}^X , its set of arrows is defined by $G_1 = X \times X$ and the set of objects by $G_0 = X$; the source and the target are $s = pr_2$ and $t = pr_1$, the second and the first projections, and the map of identity arrows is ι the diagonal map.

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- (2) Let $v : X \rightarrow Y$ be a map. Then we can consider the groupoid $X \times_v X \begin{matrix} \xrightarrow{pr_2} \\ \xleftarrow{pr_1} \end{matrix} X$, where the set of arrows is the fibre product .

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- (3) Assume that $\mathcal{R} \subseteq X \times X$ is an equivalence relation on the set X . One can construct a groupoid $\mathcal{R} \begin{matrix} \xrightarrow{pr_2} \\ \xleftarrow{pr_1} \end{matrix} X$, with structure maps as before. This groupoid is known as *the groupoid of equivalence relation*.

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- (4) Any group G can be seen as a groupoid by taking $G_1 = G$ and $G_0 = \{*\}$. Now if X is a right G -set with action $\rho : X \times G \rightarrow X$, then one can define the so called *the action groupoid*: $G_1 = X \times G$ and $G_0 = X$, the source and the target are $s = \rho$ and $t = pr_1$, the identity map sends $x \mapsto (e, x) = \iota_x$, where e is the identity element of G .

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Let \mathcal{V} be a \mathcal{G} -representation in $[\mathcal{G}, \text{vect}_k]$, we define its *dimension function* as the map

$$d_{\mathcal{V}} : G_0 \longrightarrow \mathbb{N}, \quad \left(x \longmapsto \dim_k(\mathcal{V}_x) \right),$$

which clearly extends to a map $d_{\mathcal{V}} : \pi_0(\mathcal{G}) \rightarrow \mathbb{N}$.

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A \mathcal{G} -representation \mathcal{V} in $[\mathcal{G}, \text{vect}_{\mathbb{k}}]$ is called a *finite dimensional representation*, provided that the dimension function $d_{\mathcal{V}}$ has a finite image, that is, $d_{\mathcal{V}}(G_0)$ is a finite subset of the set of positive integers \mathbb{N} .

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We denote the resulting category by $\mathbf{rep}_{\mathbb{k}}(\mathcal{G})$. Clearly,

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$$d_{\mathcal{V} \oplus \mathcal{W}} = d_{\mathcal{V}} + d_{\mathcal{W}}, \quad d_{\mathcal{D}\mathcal{V}} = d_{\mathcal{V}}, \quad \text{and} \quad d_{\mathcal{V} \otimes \mathcal{W}} = d_{\mathcal{V}} d_{\mathcal{W}}.$$

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The vector spaces of homomorphisms are given by

$$\mathbf{rep}_{\mathbb{k}}(\mathcal{G}^{\{1,2\}})((n, N), (m, M)) = M_{m,n}(\mathbb{k}),$$

the \mathbb{k} -vector space of $m \times n$ matrices with matrix multiplication.

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The other operations in $\mathbf{rep}_{\mathbb{k}}(\mathcal{G}^{\{1,2\}})$ are

$$(n, N) \oplus (m, M) = \left(n + m, \begin{pmatrix} N & 0 \\ 0 & M \end{pmatrix} \right), \quad \mathcal{D}(n, N) = (n, N^t)$$

$$(n, N) \otimes (m, M) = \left(nm, (N b_{ij})_{1 \leq i, j \leq m} \right), \text{ where } M = (b_{ij}), \text{ and } \mathcal{I} = (1, 1).$$

$$\mathrm{Tr}(n, N) = n.$$

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Furthermore, $\mathbf{rep}_{\mathbb{k}}(\mathcal{G})$ admits a non trivial fibre functor to the category of finite dimensional vector spaces. Namely, fix an object $x \in G_0$, and consider the functor

$$\omega_x : \mathbf{rep}_{\mathbb{k}}(\mathcal{G}) \longrightarrow \mathbf{vect}_{\mathbb{k}}, \quad (\mathcal{V} \longrightarrow \mathcal{V}_x).$$

Then ω_x is a non trivial fibre functor, and $\omega_x \cong \omega_y$, for any $x, y \in G_0$.

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Summarizing $(\mathbf{rep}_{\mathbb{k}}(\mathcal{G}), \omega_x)$ is a (neutral) Tannakian category in the sense of Saavedra-Rivano, Deligne and Milne.

Finite dimensional representations of groupoids.

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Let \mathcal{G} be a groupoid and denote by $A_0(\mathcal{G}) := \mathbb{k}^{G_0}$ its *base algebra* and by $A_1(\mathcal{G}) := \mathbb{k}^{G_1}$ its *total algebra*. By reflecting the groupoid structure of \mathcal{G} , we have a diagram of algebras:

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Let \mathcal{V} be a finite dimensional \mathcal{G} -representation and denote by $d_{\mathcal{V}}(G_0) := \{n_1, n_2, \dots, n_N\}$ ordered as $n_1 < n_2 < \dots < n_N$ (where obviously the maximal and minimal indices depend upon \mathcal{V}).

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The set of objects G_0 is then a disjoint union $G_0 = \bigcup_{i=1}^N G_{\mathcal{V}}^i$, where each of the $G_{\mathcal{V}}^i$'s is the inverse image $G_{\mathcal{V}}^i := d_{\mathcal{V}}^{-1}(\{n_i\})$, for any $i = 1, \dots, N$.

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We can then define the functor which acts on objects by:

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By identifying a \mathcal{G} -representation in $\mathbf{rep}_{\mathbb{k}}(\mathcal{G})$ with its associated vector \mathcal{G} -bundle, we can consider the \mathbb{k} -vector space of "global sections":

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Both functors ω and Γ are symmetric monoidal faithful functors. Moreover, there is a tensorial natural isomorphism $\omega \cong \Gamma$.

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Associated to these data, there are at least two universal problems, which have a common solution:

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PR-1 The functor: $A\text{-Corings} \rightarrow \text{Sets}$,

$$C \rightarrow \left\{ \text{the set of functorial right } C\text{-comodules structure on } \omega \right\}$$

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PR-2 The functor: $(A \otimes A)\text{-CAlg}_{\mathbb{k}} \rightarrow \text{Sets}$,

$$\left(A \begin{array}{c} \xrightarrow{s} \\ \xrightarrow{t} \end{array} C \right) \rightarrow \text{Iso}^{\otimes}(\mathfrak{t}^* \omega, \mathfrak{s}^* \omega)$$

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The universal solution for both PR-1-2 is given by the following A -bimodule

$$\mathcal{L}_{\mathbb{k}}(\mathcal{T}, \omega) := \frac{\bigoplus_{P \in \mathcal{T}} \omega(P)^* \otimes_{T_P} \omega(P)}{\mathcal{J}},$$

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where T_P is the endomorphism algebra of an object $P \in \mathcal{T}$ and \mathcal{J} is the A -sub-bimodule generated by

$$\mathcal{J} := \left\langle \psi \lambda \otimes_{T_P} p - \psi \otimes_{T_Q} \lambda p \right\rangle_{\left\{ \psi \in \omega(Q)^*, p \in \omega(P), \lambda: P \rightarrow Q \in \mathcal{T} \right\}}$$

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It turns out that $(A, \mathcal{L}_{\mathbb{k}}(\mathcal{T}, \omega))$ is a *commutative Hopf algebroid*, such that there is a commutative diagram:

$$\begin{array}{ccc} \mathcal{T} & \overset{\text{---}}{\dashrightarrow} & \text{comod}_{\mathcal{L}_{\mathbb{k}}(\mathcal{T}, \omega)} \\ & \searrow \omega & \swarrow \circ \\ & \text{proj}(A) & \end{array}$$

where $\text{comod}_{\mathcal{L}_{\mathbb{k}}(\mathcal{T}, \omega)}$ is the full subcategory of comodules with finitely generated and projective underlying A -modules.

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Let \mathcal{G} be a groupoid and consider the pair $(\mathbf{rep}_{\mathbb{k}}(\mathcal{G}), \omega)$. Applying the previous general constructions, we obtain a commutative Hopf algebroid $(A_0(\mathcal{G}), \mathcal{L}_{\mathbb{k}}(\mathbf{rep}_{\mathbb{k}}(\mathcal{G}), \omega))$, which we denote by $(A_0(\mathcal{G}), \mathcal{R}_{\mathbb{k}}(\mathcal{G}))$.

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The terminology "functions" is justified by the following $(A_0(\mathcal{G}) \otimes_{\mathbb{k}} A_0(\mathcal{G}))$ -algebra map:

$$\xi : \mathcal{R}_{\mathbb{k}}(\mathcal{G}) \longrightarrow A_1(\mathcal{G}),$$

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The representative functions establish a contravariant functor:

$$\mathcal{R}_{\mathbb{k}} : \mathbf{Grpd} \longrightarrow \mathbf{CHA}_{\mathbb{k}}$$

from the category of abstract groupoids to the category of commutative Hopf algebroids.

Examples of representative functions algebra.

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- If G is a groupoids with only one object, that is, a group, then $\mathcal{R}_{\mathbb{k}}(G)$ is the usual Hopf algebra of representative functions on the group G . This is isomorphic to the finite dual $\mathbb{k}[G]^o$ of the group algebra $\mathbb{k}[G]$.

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- Let $X = \{1, 2\}$ be a set of two elements and consider as before the groupoid $\mathcal{G}^{\{1,2\}}$ of pairs and denote by $A := \mathbb{k} \times \mathbb{k}$ its base algebra. Then

$$\mathcal{R}_{\mathbb{k}}(\mathcal{G}) = \frac{\bigoplus_{n \in \mathbb{N}} A^n \otimes_{M_n(\mathbb{k})} A^n}{\langle v \otimes_{M_n(\mathbb{k})} \lambda w - \lambda^t v \otimes_{M_m(\mathbb{k})} w \rangle_{v \in A^n, w \in A^m, \lambda \in M_{m \times n}(\mathbb{k})}}.$$

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- Let $\mathcal{G}: G \times X \begin{matrix} \xrightarrow{c} \\ \xleftarrow{p|_1} \end{matrix} X$ be an action groupoid. Then there is a morphism of Hopf algebroids:

$$(\mathbb{k}^X, \mathbb{k}^X \otimes \mathcal{R}_{\mathbb{k}}(G) \otimes \mathbb{k}^X) \longrightarrow (\mathbb{k}^X, \mathcal{R}_{\mathbb{k}}(\mathcal{G})).$$

Furthermore, if the action is transitive, then any isotropy Hopf algebra $(\mathbb{k}_x, \mathcal{R}_{\mathbb{k}}(\mathcal{G})^x)$, for $x \in X$, is isomorphic to $(\mathbb{k}, \mathcal{R}_{\mathbb{k}}(G))$.

In general, $\mathcal{R}_{\mathbb{k}}(\mathcal{G})$ is not a split Hopf algebroid (i.e., not isomorphic to $\mathbb{k}^X \otimes \mathcal{R}_{\mathbb{k}}(G)$)

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The properties of $\mathcal{R}_k(\mathcal{G})$ when \mathcal{G} is transitive.

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- ▶ $(A_0(\mathcal{G}), \mathcal{R}_{\mathbb{k}}(\mathcal{G}))$ is a *transitive* Hopf algebroid, in the sense that each of the fibers of its associated presheaf of groupoids is actually a transitive groupoids (i.e., each of the groupoids $(\mathcal{R}_{\mathbb{k}}(\mathcal{G})(C), A_0(\mathcal{G})(C))$ is transitive, for any commutative algebra C). The notation is $R(C) := \text{Alg}_{\mathbb{k}}(R, C)$.

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- ▶ The fibre functor $\omega : \mathbf{rep}_k(\mathcal{G}) \rightarrow \text{proj}(A_0(\mathcal{G}))$ induces a monoidal equivalence between the category $\mathbf{rep}_k(\mathcal{G})$ and the category $\text{comod}_{\mathcal{R}_k(\mathcal{G})}$ of comodules with finitely generated underlying module.

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- ▶ Any comodule M in $\text{comod}_{\mathcal{R}_k(\mathcal{G})}$ is a locally free $A_0(\mathcal{G})$ -module with constant rank, in the sense that, if for some $x \in G_0$, we have $\dim_k(M_x) = n$, then so is the dimension of any other fibre.

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The character functor and the counit. Let (A, \mathcal{H}) be a commutative Hopf algebroid such that $A \neq 0$ and $\text{Alg}_{\mathbb{k}}(A, \mathbb{k}) \neq \emptyset$. The groupoid

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Furthermore, for any groupoid \mathcal{G} , we have a natural transformation:

$$\mathcal{G} \longrightarrow \chi_{\mathbb{k}} \circ \mathcal{R}_{\mathbb{k}}(\mathcal{G}),$$

which is not in general a monomorphism. When it is, the groupoid $\chi_{\mathbb{k}}(\mathcal{R}_{\mathbb{k}}(\mathcal{G}))$ is sometimes called *the algebraic cover* of \mathcal{G} .

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is an isomorphism of Hopf algebroids;

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- ▶ there is a natural morphism of Hopf transitive Hopf algebroids:

$$(A, \mathcal{H}) \longrightarrow (A_0(\chi_{\mathbb{k}}(\mathcal{H})), \mathcal{B}_{\mathbb{k}}(\chi_{\mathbb{k}}(\mathcal{H})))$$

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Notations: Denotes by $TGrpd$ the full subcategory of transitive groupoids, and by $TCHAlg_{\mathbb{k}}$ the full subcategory of transitive commutative Hopf algebroids.

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As we have seen before, the representative functions functor define a contravariant functor

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On the other hand, the character functor obviously define a contravariant functor

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There is a natural isomorphism

$$TGrpd(-, \chi_{\mathbb{k}}(+)) \cong TCHAlgd_{\mathbb{k}}(+, \mathcal{R}_{\mathbb{k}}(-)).$$

HAPPY BIRTHDAY STEF