Finite dimensional representations of abstract groupoids, representative functions and commutative Hopf algebroids

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Brauer groups, Hopf algebras and monoidal categories.
In honour of Stef Caenepeel on the occasion of his 60 birthday.

Turin, May 2016.

The context, motivations and overviews.

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Representative functions on group.

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Representative functions on group.

Let G be an abstract group, and ρ a finite dimensional representation in $\mathbb{k}$-vector spaces.
Consider the subalgebra $\mathscr{V}(\rho)$ of the algebra $\mathrm{Maps}(G, \mathbb{k}) := \mathbb{k}^G$ generated by the functions of the form:

$$a_{ij}^{-} : G \to \mathbb{k}$$

where for $g \in G$, $\rho(g) = (a_{ij}^{g})_{i,j}$ is the invertible matrix defining $\rho(g)$.

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The *algebra of representative functions on G*, is then defined to be

$$\mathscr{R}_{\Bbbk}(G) \; = \; \sum_{\rho \in \mathbf{rep}_{\Bbbk}(G)} \mathscr{V}(\rho) \; \subseteq \Bbbk^G.$$

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This is a *commutative Hopf* $\Bbbk$*-algebra* which ”codifies” almost all informations about the group G (excluding extreme cases, of course).

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where the second one is the *character group* $\chi_{\mathbb{k}}(H)$ of a Hopf algebra H, defined as the group of algebra maps from H to $\mathbb{k}$ (the fibre group at the base field, if we think of H as an affine $\mathbb{k}$-group).

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There is a natural isomorphism, i.e., contravariant adjunction

$$CHAlg_{\mathbb{k}}\Big(-; \mathscr{R}_{\mathbb{k}}(+)\Big) \cong Grp\Big(+; \chi_{\mathbb{k}}(-)\Big)$$

which is known as *a duality between groups and Hopf algebras*.

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In fact we have a diagram

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\begin{tikzcd}
\mathit{Grp} \arrow[rr, bend left, "\mathscr{R}_{\Bbbk}"] \arrow[rr, phantom, "\text{duality}"] & & \mathit{CHAlg}_{\Bbbk} \arrow[ll, bend left, "\chi_{\Bbbk}"] \\
\mathit{CTGrp} \arrow[u, hook] \arrow[rr, bend left, "\textit{continous}"] \arrow[rr, phantom, "\text{anti-equivalence}"] & & \widetilde{\mathit{CHAlg}}_{\mathbb{R}} \arrow[u, hook, "\Bbbk=\mathbb{R}"] \arrow[ll, bend left] \\
\mathit{CLGrp} \arrow[u, hook] \arrow[rr, bend left, "\textit{smooth}"] \arrow[rr, phantom, "\text{anti-equivalence}"] & & \widetilde{\mathit{CHalg}}_{\mathbb{R}} \arrow[u, hook] \arrow[ll, bend left]
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where $\widetilde{CHAlg}_{\Bbbk}$ the subcategory of commutative real Hopf algebras with *gauge* (i.e., a Hopf integral coming from the Haar measure) and with dense character group in the linear dual.

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In this talk, we will see how to construct the functor $\mathscr{R}_{\Bbbk}$ and show the main steps in building up the duality between the category of transitive groupoids and the category of *(geometrically) transitive commutative Hopf algebroids*.

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For a given (small) groupoid

$$\mathscr{G} : \ G_1 \underset{t}{\overset{s}{\underset{\longleftarrow\iota\,}{\rightrightarrows}}} G_0 \ ,$$

we consider the category of all $\mathscr{G}$*-representations* as the symmetric monoidal $\Bbbk$-linear abelian category of functors $\left[\mathscr{G}, \mathrm{Vect}_{\Bbbk}\right]$ with identity object $\mathcal{I} : G_0 \to \mathrm{Vect}_{\Bbbk}, x \to \Bbbk, g \to 1_{\Bbbk}$.

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For any $\mathscr{G}$-representation $\mathcal{V}$ the image of an object $x \in G_0$ is denoted by $\mathcal{V}_x$, and referred to as *the fibre of* $\mathcal{V}$ *at* x.

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The disjoint union of all the fibres of a $\mathscr{G}$-representation $\mathcal{V}$ is denoted by $\overline{\mathcal{V}} = \bigcup_{x \in G_0} \mathcal{V}_x$ and *the canonical projection* by $\pi_{\mathcal{V}} : \overline{\mathcal{V}} \to G_0$. This called *the associated vector* $\mathscr{G}$*-bundle of the representation* $\mathcal{V}$.

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(1) Given any set X, one can associated the so called *the groupoid of pairs* $\mathscr{G}^X$, its set of arrows is defined by $G_1 = X \times X$ and the set of objects by $G_0 = X$; the sourse and the target are $s = pr_2$ and $t = pr_1$, the second and the first projections, and the map of identity arrows is ι the diagonal map.

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(2) Let $\nu : X \to Y$ be a map. Then we can consider the groupoid
$$\begin{tikzcd} X \, {}_{\nu}\!\times_{\nu} X \arrow[r, shift left=2, "pr_2"] \arrow[r, shift right=2, "pr_1"'] & X \arrow[l, "\iota" description] \end{tikzcd}$$
, where the set of arrows is the fibre product .

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(3) Assume that $\mathcal{R} \subseteq X \times X$ is an equivalence relation on the set X. One can construct a groupoid $\mathcal{R} \overset{pr_2}{\underset{pr_1}{\overset{\xleftarrow{\iota}}{\rightrightarrows}}} X$, with structure maps as before. This groupoid is known as *the groupoid of equivalence relation*.

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(4) Any group G can be seen as a groupoid by taking $G_1 = G$ and $G_0 = \{*\}$. Now if X is a right G-set with action $\rho : X \times G \to X$, then one can define the so called *the action groupoid*: $G_1 = X \times G$ and $G_0 = X$, the source and the target are $s = \rho$ and $t = pr_1$, the identity map sends $x \mapsto (e, x) = \iota_x$, where e is the identity element of G.

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Let $\mathcal{V}$ be a $\mathscr{G}$-representation in $\left[\mathscr{G},\, \mathrm{vect}_{\Bbbk}\right]$, we define its *dimension function* as the map

$$d_{\mathcal{V}} : G_0 \longrightarrow \mathbb{N}, \quad \Big(x \longmapsto dim_{\Bbbk}\big(\mathcal{V}_x\big)\Big),$$

which clearly extends to a map $d_{\mathcal{V}} : \pi_0(\mathscr{G}) \to \mathbb{N}$.

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We denote the resulting category by $\mathbf{rep}_{\Bbbk}(\mathscr{G})$. Clearly,

$$\mathbf{rep}_{\Bbbk}(\mathscr{G}) = \left[\mathscr{G}, \mathrm{vect}_{\Bbbk}\right], \text{ when } \pi_0(\mathscr{G}) \text{ is a finite set.}$$

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Let $\mathcal{V}$ and $\mathcal{W}$ be two representations in $\mathbf{rep}_{\Bbbk}(\mathscr{G})$. Then

$$d_{\mathcal{V} \oplus \mathcal{W}} = d_{\mathcal{V}} + d_{\mathcal{W}}, \quad d_{\mathcal{D}\mathcal{V}} = d_{\mathcal{V}}, \quad \text{and} \;\; d_{\mathcal{V} \otimes \mathcal{W}} = d_{\mathcal{V}}\, d_{\mathcal{W}}.$$

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An objet in $\mathbf{rep}_{\mathbb{k}}(\mathscr{G}^{\{1,2\}})$ is then a pair (n, N), where n is a positive integer, and $N \in GL_n(\mathbb{k})$.

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The vector spaces of homomorphisms are given by

$$\mathbf{rep}_{\mathbb{k}}(\mathscr{G}^{\{1,2\}})\Big((n, N),\ (m, M)\Big) = M_{m,n}(\mathbb{k}),$$

the $\mathbb{k}$-vector space of $m \times n$ matrices with matrix multiplication.

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The other operations in $\mathbf{rep}_{\mathbb{k}}(\mathscr{G}^{\{1,2\}})$ are

$$(n, N) \oplus (m, M) \;=\; \left(n + m, \begin{pmatrix} N & 0 \\ 0 & M \end{pmatrix}\right), \quad \mathscr{D}(n, N) \;=\; (n, N^t)$$

$$(n, N) \otimes (m, M) = \Big(nm,\; (N\, b_{ij})_{1 \leq i,j \leq m}\Big), \text{ where } M = (b_{ij}), \text{ and } \mathcal{I} = (1, 1).$$

$$\mathrm{Tr}(n, N) \;=\; n.$$

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Furthermore, $\mathbf{rep}_{\Bbbk}(\mathscr{G})$ admits a non trivial fibre functor to the category of finite dimensional vector spaces. Namely, fix an object $x \in G_0$, and consider the functor

$$\boldsymbol{\omega}_x : \mathbf{rep}_{\Bbbk}(\mathscr{G}) \longrightarrow \mathrm{vect}_{\Bbbk}, \quad \Big(\mathcal{V} \longrightarrow \mathcal{V}_x\Big).$$

Then $\boldsymbol{\omega}_x$ is a non trivial fibre functor, and $\boldsymbol{\omega}_x \cong \boldsymbol{\omega}_y$, for any $x, y \in G_0$.

Finite dimensional representations of groupoids.

The transitive case. Recall that a groupoid $\mathscr{G}$ is said to be *transitive* if for any two objects $x, y \in G_0$, there is an arrow $g \in G_1$ such that $s(g) = x$ and $t(g) = y$, or equivalently, $\pi_0(\mathscr{G})$ is a singleton, i.e., a set with only one element.

Let $\mathscr{G}$ be a transitive groupoid. Then, the category $\mathbf{rep}_{\Bbbk}(\mathscr{G})$ is a symmetric rigid monoidal locally finite $\Bbbk$-linear abelian category.

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Summarizing $(\mathbf{rep}_{\Bbbk}(\mathscr{G}), \boldsymbol{\omega}_x)$ is a (neutral) Tannakian category in the sense of Saavedra-Rivano, Deligne and Milne.

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Let $\mathscr{G}$ be a groupoid and denote by $A_0(\mathscr{G}) := \Bbbk^{G_0}$ its *base algebra* and by $A_1(\mathscr{G}) := \Bbbk^{G_1}$ its *total algebra*. By reflecting the groupoid structure of $\mathscr{G}$, we have a diagram of algebras:

$$\begin{tikzcd} A_0(\mathscr{G}) \arrow[r, shift left=2, "s^*"] \arrow[r, shift right=2, "t^*"'] & A_1(\mathscr{G}) \arrow[l, "\iota^*" description] \end{tikzcd}.$$

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Let $\mathcal{V}$ be a finite dimensional $\mathscr{G}$-representation and denote by $d_{\mathcal{V}}(G_0) := \left\{ n_1, n_2, \cdots, n_N \right\}$ ordered as $n_1 < n_2 < \cdots < n_N$ (where obviously the maximal and minimal indices depend upon $\mathcal{V}$).

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The set of objects G_0 is then a disjoint union $G_0 = \bigcup_{i=1}^{N} G_{\mathcal{V}}^{i}$, where each of the $G_{\mathcal{V}}^{i}$'s is the inverse image $G_{\mathcal{V}}^{i} := d_{\mathcal{V}}^{-1}\Big(\{n_i\}\Big)$, for any $i = 1, \cdots, N$.

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We can then define the functor which acts on objects by:

$$\boldsymbol{\omega} : \mathbf{rep}_{\Bbbk}(\mathscr{G}) \longrightarrow \mathrm{proj}(A_0(\mathscr{G})), \quad \mathcal{V} \longrightarrow P_{\mathcal{V}} = B_1^{n_1} \times \cdots \times B_N^{n_N}$$

an $A_0(\mathscr{G})$-module which corresponds to the above decomposition.

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By identifying a $\mathscr{G}$-representation in $\mathbf{rep}_{\mathbb{k}}(\mathscr{G})$ with its associated vector $\mathscr{G}$-bundle, we can consider the $\mathbb{k}$-vector space of "global sections":

$$\Gamma(\mathcal{V}) := \Big\{ s : G_0 \to \overline{\mathcal{V}} \,|\; \pi_{\mathcal{V}} \circ s = id_{G_0} \Big\}.$$

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Both functors $\boldsymbol{\omega}$ and Γ are symmetric monoidal faithful functors.
Moreover, there is a tensorial natural isomorphism $\boldsymbol{\omega} \cong \Gamma$.

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Tannakian reconstruction process and the universal Hopf algebroid.

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Associated to these data, there are at least two universal problems, which have a common solution:

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PR-1 The functor: $A\text{-}\mathrm{Corings} \longrightarrow \mathrm{Sets}$,

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PR-2 The functor: $(A \otimes A)\text{-}\mathrm{CAlg}_{\Bbbk} \longrightarrow \mathrm{Sets}$,

$$\Big(A \underset{t}{\overset{s}{\rightrightarrows}} C \Big) \longrightarrow \mathrm{Iso}^{\otimes}\big(t^{*}\omega,\, s^{*}\omega\big)$$

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The universal solution for both PR-1-2 is given by the following A-bimodule

$$\mathcal{L}_{\Bbbk}(\mathcal{T}, \omega) := \frac{\bigoplus_{P \in \mathcal{T}} \omega(P)^* \otimes_{T_P} \omega(P)}{\mathcal{J}},$$

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where T_P is the endomorphism algebra of an object $P \in \mathcal{T}$ and $\mathscr{J}$ is the A-sub-bimodule generated by

$$\mathscr{J} := \Big\langle \psi\,\lambda \otimes_{T_P} p - \psi \otimes_{T_Q} \lambda p \Big\rangle_{\left\{\psi\in\omega(Q)^*,\; p\in\omega(P),\; \lambda: P\to Q\in\mathcal{T}\right\}}$$

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It turns out that $\big(A, \mathscr{L}_{\Bbbk}(\mathcal{T}, \omega)\big)$ is a *commutative Hopf algebroid*, such that there is a commutative diagram:

$$\begin{tikzcd}
\mathcal{T} \arrow[rr, dashed] \arrow[dr, "\omega"'] & & \mathrm{comod}_{\mathscr{L}_{\Bbbk}(\mathcal{T},\omega)} \arrow[dl, "\mathcal{O}"] \\
& \mathrm{proj}(A) &
\end{tikzcd}$$

where $\mathrm{comod}_{\mathscr{L}_{\Bbbk}(\mathcal{T},\omega)}$ is the full subcategory of comodules with finitely generated and projective underlying A-modules.

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Let $\mathscr{G}$ be a groupoid and consider the pair $\big(\mathbf{rep}_{\Bbbk}(\mathscr{G}), \boldsymbol{\omega}\big)$. Applying the previous general constructions, we obtain a commutative Hopf algebroid $\Big(A_0(\mathscr{G}), \mathscr{L}_{\Bbbk}\big(\mathbf{rep}_{\Bbbk}(\mathscr{G}), \boldsymbol{\omega}\big)\Big)$, which we denote by $\Big(A_0(\mathscr{G}), \mathscr{R}_{\Bbbk}(\mathscr{G})\Big)$.

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The terminology ”functions” is justified by the following $(A_0(\mathscr{G}) \otimes_{\Bbbk} A_0(\mathscr{G}))$-algebra map:

$$\xi : \mathscr{R}_{\Bbbk}(\mathscr{G}) \longrightarrow A_1(\mathscr{G}),$$

where $A_1(\mathscr{G})$ is an $A_0(\mathscr{G}) \otimes_{\Bbbk} A_0(\mathscr{G})$-algebra, as before by reflecting the groupoid structure of $\mathscr{G}$.

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The representative functions establish a contravariant functor:

$$\mathscr{R}_{\Bbbk} : \mathit{Grpd} \longrightarrow \mathit{CHAlg}_{\Bbbk}$$

from the category of abstract groupoids to the category of commutative Hopf algebroids.

Examples of representative functions algebra.

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(•) If G is a groupoids with only one object, that is, a group, then $\mathscr{R}_{\mathbb{k}}(G)$ is the usual Hopf algebra of representative funcions on the group G. This is isomorphic to the finite dual $\mathbb{k}[G]^o$ of the group algebra $\mathbb{k}[G]$.

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- (•) Let $X = \{1, 2\}$ be a set of two elements and consider as before the groupoid $\mathscr{G}^{\{1,2\}}$ of pairs and denote by $A := \Bbbk \times \Bbbk$ its base algebra. Then

$$\mathscr{R}_{\Bbbk}(\mathscr{G}) = \frac{\bigoplus_{n \in \mathbb{N}} A^n \otimes_{M_n(\Bbbk)} A^n}{\langle v \otimes_{M_n(\Bbbk)} \lambda w - \lambda^t v \otimes_{M_m(\Bbbk)} w \rangle_{v \in A^n,\, w \in A^m,\, \lambda \in M_{m \times n}(\Bbbk)}}.$$

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- (•) Let $\mathscr{G}\colon\ G \times X \underset{pr_1}{\overset{\varrho}{\underset{\iota}{\leftleftarrows\rightrightarrows}}} X$ be an action groupoid. Then there is a morphism of Hopf algebroids:

$$(\Bbbk^X, \Bbbk^X \otimes \mathscr{R}_{\Bbbk}(G) \otimes \Bbbk^X) \longrightarrow (\Bbbk^X, \mathscr{R}_{\Bbbk}(\mathscr{G})).$$

  Furthermore, if the action is transitive, then any isotropy Hopf algebra $(\Bbbk_x, \mathscr{R}_{\Bbbk}(\mathscr{G})^x)$, for $x \in X$, is isomorphic to $(\Bbbk, \mathscr{R}_{\Bbbk}(G))$.
  In general, $\mathscr{R}_{\Bbbk}(\mathscr{G})$ is not a split Hopf algebroid (i.e., not isomorphic to $\Bbbk^X \otimes \mathscr{R}_{\Bbbk}(G)$)

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The properties of $\mathscr{R}_{\mathbb{k}}(\mathscr{G})$ when $\mathscr{G}$ is transitive.

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The properties of $\mathscr{R}_{\Bbbk}(\mathscr{G})$ when $\mathscr{G}$ is transitive. Let $\mathscr{G}$ be a transitive groupoid, then its algebra of representative functions enjoys the following properties:

- $(A_0(\mathscr{G}), \mathscr{R}_{\Bbbk}(\mathscr{G}))$ is a *transitive* Hopf algebroid, in the sense that each of the fibers of its associated presheaf of groupoids is actually a transitive groupoids
  (i.e., each of the groupoids $\big(\mathscr{R}_{\Bbbk}(\mathscr{G})(C), A_0(\mathscr{G})(C)\big)$ is transitive, for any commutative algebra C). The notation is $R(C) := \mathrm{Alg}_{\Bbbk}(R, C)$.

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- The fibre functor $\boldsymbol{\omega} : \mathbf{rep}_{\Bbbk}(\mathscr{G}) \to \mathrm{proj}(A_0(\mathscr{G}))$ induces a monoidal equivalence between the category $\mathbf{rep}_{\Bbbk}(\mathscr{G})$ and the category $\mathrm{comod}_{\mathscr{R}_{\Bbbk}(\mathscr{G})}$ of comodules with finitely generated underlying module.

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- Any comodule M in $\mathrm{comod}_{\mathscr{R}_{\Bbbk}(\mathscr{G})}$ is a locally free $A_0(\mathscr{G})$-module with constant rank, in the sense that, if for some $x \in G_0$, we have $dim_{\Bbbk}(M_x) = n$, then so is the dimension of any other fibre.

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The character functor and the counit. Let $(A, \mathcal{H})$ be a commutative Hopf algebroid such that $A \neq 0$ and $Alg_{\Bbbk}(A, \Bbbk) \neq \emptyset$. The groupoid

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This establishes a contravariant functor

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Furthermore, for any groupoid $\mathscr{G}$, we have a natural transformation:

$$\mathscr{G} \longrightarrow \chi_{\Bbbk} \circ \mathscr{R}_{\Bbbk}(\mathscr{G}),$$

which is not in general a monomorphism. When it is, the groupoid $\chi_{\Bbbk}(\mathscr{R}_{\Bbbk}(\mathscr{G}))$ is sometimes called *the algebraic cover of* $\mathscr{G}$.

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- there is a natural morphism of Hopf transitive Hopf algebroids:
$$(A, \mathcal{H}) \longrightarrow \Big(A_0(\chi_{\mathbb{k}}(\mathcal{H})), \mathscr{R}_{\mathbb{k}}(\chi_{\mathbb{k}}(\mathcal{H}))\Big)$$

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As we have seen before, the represenative functions functor define a contravariant functor

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On the other hand, the character functor obviously define a contravariant functor

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There is a natural isomorphism

$$\mathit{TGrpd}\Big(-\,,\, \chi_{\Bbbk}(+)\Big) \;\cong\; \mathit{TCHAlgd}_{\Bbbk}\Big(+\,,\, \mathscr{R}_{\Bbbk}(-)\Big).$$

HAPPY BIRTHDAY STEF