Strongly norm attaining Lipschitz maps

Miguel Martín

UNIVERSIDAD DE GRANADA

Instituto de Matemáticas
Universidad de Granada

Seminario de Análisis Matemático y Matemática Aplicada

UNIVERSIDAD COMPLUTENSE MADRID

MINISTERIO DE CIENCIA, INNOVACIÓN Y UNIVERSIDADES
Cofinanciado por la Unión Europea
AGENCIA ESTATAL DE INVESTIGACIÓN
Junta de Andalucía
PID2021-122126NB-C31

Madrid, Octubre 2024

Roadmap of the talk

The talk is mainly based on. . .

- B. Cascales, R. Chiclana, L. García-Lirola, M. Martín, A. Rueda
  On strongly norm attaining Lipschitz maps
  J. Funct. Anal. (2019)

- R. Chiclana, L. García-Lirola, M. Martín, A. Rueda
  Examples and applications of the density of strongly norm attaining Lipschitz maps
  Rev. Mat. Iberoam. (2021)

- R. Chiclana, M. Martín
  The Bishop-Phelps-Bollobás property for Lipschitz maps
  Nonlinear Analysis (2019)

- R. Chiclana, M. Martín
  Some stability properties for the Bishop-Phelps-Bollobás property for Lipschitz maps
  Studia Math. (2022)

- M. Jung, M. Martín, A. Rueda
  Residuality in the set of norm attaining operators between Banach spaces
  J. Funct. Anal. (2023)

Preliminaries

Some notation

X, Y real Banach spaces

B_X closed unit ball

S_X unit sphere

X^* topological dual

$\mathcal{L}(X,Y)$ Banach space of all bounded linear operators from X to Y

Main definition and leading problem

Lipschitz function

M, N (complete) metric spaces. A map $F\colon M \longrightarrow N$ is Lipschitz if there exists a constant $k > 0$ such that

$$d(F(p), F(q)) \leqslant k\, d(p, q) \quad \forall\, p, q \in M$$

The least constant so that the above inequality works is called the Lipschitz constant of F, denoted by $L(F)$:

$$L(F) = \sup \left\{ \frac{d(F(p), F(q))}{d(p, q)} : p \neq q \in M \right\}$$

- If $N = Y$ is a normed space, then $L(\cdot)$ is a seminorm in the vector space of all Lipschitz maps from M into Y.
- F attain its Lipschitz number if the supremum defining it is actually a maximum.

Leading problem (Godefroy, 2015)

Study the metric spaces M and the Banach spaces Y such that the set of Lipschitz maps which attain their Lipschitz number is dense in the set of all Lipschitz maps.

More definitions

Pointed metric space

M is *pointed* if it carries a distinguished element called base point.

Space of Lipschitz maps

M pointed metric space, Y Banach space.
$\mathrm{Lip}_0(M,Y)$ is the Banach space of all Lipschitz maps from M to Y which are zero at the base point, endowed with the Lipschitz number as norm.

Strongly norm attaining Lipschitz map

M pointed metric space. $F \in \mathrm{Lip}_0(M,Y)$ strongly attains its norm, writing $F \in \mathrm{SNA}(M,Y)$, if there exist $p \neq q \in M$ such that

$$L(F) = \|F\| = \frac{\|F(p) - F(q)\|}{d(p,q)}.$$

Our objective is then

to study when $\mathrm{SNA}(M,Y)$ is norm dense in the Banach space $\mathrm{Lip}_0(M,Y)$

First examples

Finite sets

If M is finite, obviously every Lipschitz map attains its Lipschitz number.
★ This characterizes finiteness of M.

Example (Kadets–Martín–Soloviova, 2016)

$M = [0,1]$, $A \subseteq [0,1]$ closed with empty interior and positive Lebesgue measure.
Then, the Lipschitz function $f\colon [0,1] \longrightarrow \mathbb{R}$ given by

$$f(t) = \int_0^t \mathbb{1}_A(s)\, ds,$$

cannot be approximated by Lipschitz functions which attain their Lipschitz number.

Example (Godefroy, 2015)

M compact, $\mathrm{lip}_0(M)$ strongly separates M (e.g. M usual Cantor set, $M = [0,1]^\theta$)
$\Longrightarrow$ $\mathrm{SNA}(M,Y)$ dense in $\mathrm{Lip}_0(M,Y)$ for every finite-dimensional Y.

The Lipschitz–free space: definition

Evaluation functional, Lipschitz-free space

M pointed metric space.

- $p \in M$, $\delta_p \in \mathrm{Lip}_0(M,\mathbb{R})^*$ given by $\delta_p(f) = f(p)$ is the evaluation functional at p;
- $\delta : M \rightsquigarrow \mathrm{Lip}_0(M,\mathbb{R})^*$, $p \longmapsto \delta_p$, is an isometric embedding;
- $\mathcal{F}(M) := \overline{\mathrm{span}}\{\delta_p : p \in M\} \subseteq \mathrm{Lip}_0(M,\mathbb{R})^*$ is the Lipschitz-free space of M;
- so $\delta : M \rightsquigarrow \mathcal{F}(M)$, $p \longmapsto \delta_p$, is an isometric embedding;

Remark

- M is linearly independent in $\mathcal{F}(M)$;
- $\mathrm{dens}(\mathcal{F}(M)) = \mathrm{dens}(M)$ if M is infinite.

Molecules

- For $p \neq q \in M$, $m_{p,q} := \frac{\delta_p - \delta_q}{d(p,q)} \in \mathcal{F}(M)$ is a molecule, $\|m_{p,q}\| = 1$;
- $\mathrm{Mol}\,(M) := \{m_{p,q} : p, q \in M,\ p \neq q\}$.
- $B_{\mathcal{F}(M)} = \overline{\mathrm{conv}}(\mathrm{Mol}\,(M))$.

The Lipschitz–free space: examples

Let us describe $\mathcal{F}(M)$ for some Ms

Example 1: $M = [0,1]$

$\Phi\colon L_\infty[0,1] \longrightarrow \mathrm{Lip}_0([0,1],\mathbb{R})$ defined by $[\Phi(f)](x) = \displaystyle\int_0^x f(t)\,dt$ is an isometric isomorphism which is weak-star continuous.

★ $\Phi_*(\delta_x) = \mathbb{1}_{[0,x]}$ and $\Phi_*(\mathcal{F}([0,1])) \equiv L_1[0,1]$.

Example 2: $M = \mathbb{N}$

$\Phi\colon \ell_\infty \longrightarrow \mathrm{Lip}_0(\mathbb{N}\cup\{0\},\mathbb{R})$ defined by $[\Phi(f)](0) = 0$, $[\Phi(f)](n) = \sum_{k=0}^n f(k)$ is an isometric isomorphism which is weak-star continuous.

★ $\Phi_*(\delta_n) = \mathbb{1}_{\{1,\ldots,n\}}$ and $\Phi_*(\mathcal{F}(\mathbb{N}\cup\{0\})) \equiv \ell_1$.

The Lipschitz–free space: examples II

Example 3: a simpler example to get $\mathcal{F}(M) \equiv \ell_1$

$M = \mathbb{N} \cup \{0\}$ with $d(n,0) = 1$ and $d(n,m) = 2$.
$\Phi\colon \ell_\infty \longrightarrow \mathrm{Lip}_0(M,\mathbb{R})$ defined by $[\Phi(f)](k) = k$ is an isometric isomorphism which is weak-star continuous.

★ $\Phi_*(\delta_n) = \mathbb{1}_{\{n\}}$ and $\Phi_*(\mathcal{F}(M) \equiv \ell_1$.

Example 4: everything depends on the concrete metric

$M = \{0,1,2\}$ with $d(1,0) = d(2,0) = 1$, $1 \leqslant d(1,2) < 2$.

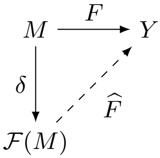
★ The unit balls of $\mathrm{Lip}_0(M,\mathbb{R})$ and of $\mathcal{F}(M)$ are hexagons.

The Lipschitz–free space: universal property

Very important property (Arens–Eells, Kadets, Godefroy–Kalton, Weaver. . .)

M pointed metric space, Y Banach space.
Given $F \in \mathrm{Lip}_0(M, Y)$,
$\exists$ (a unique) $\widehat{F} \in \mathcal{L}(\mathcal{F}(M), Y)$ such that

- $F = \widehat{F} \circ \delta$ (i.e. $\widehat{F}(m_{p,q}) = \frac{F(p)-F(q)}{d(p,q)}$ for $p \neq q$),
- and so $\|\widehat{F}\| = \|F\|$.



First consequence

$$\mathcal{F}(M)^* \cong \mathrm{Lip}_0(M, \mathbb{R}).$$

Actually. . .

M metric space, Y Banach space,

$$\mathrm{Lip}_0(M, Y) \cong \mathcal{L}(\mathcal{F}(M), Y).$$

Two ways of attaining the norm

We have two ways of attaining the norm

M pointed metric space, Y Banach space, $F \in \mathrm{Lip}_0(M,Y) \cong \mathcal{L}(\mathcal{F}(M),Y)$.

- $\widehat{F} \in \mathrm{NA}(\mathcal{F}(M),Y)$ if exists $\xi \in B_{\mathcal{F}(M)}$ such that $\|F\| = \|\widehat{F}\| = \|\widehat{F}(\xi)\|$;
- $F \in \mathrm{SNA}(M,Y)$ if exists $m_{p,q} \in \mathrm{Mol}\,(M)$ such that

$$\|F\| = \|\widehat{F}\| = \|\widehat{F}(m_{p,q})\| = \frac{\|F(p) - F(q)\|}{d(p,q)}.$$

Clearly, $\mathrm{SNA}(M,Y) \subseteq \mathrm{NA}(\mathcal{F}(M),Y)$.

- Therefore, if $\mathrm{SNA}(M,Y)$ is dense in $\mathrm{Lip}_0(M,Y)$, then $\mathrm{NA}(\mathcal{F}(M),Y)$ is dense in $\mathcal{L}(\mathcal{F}(M),Y)$;
- But the opposite direction is NOT true:

Example

- $\overline{\mathrm{NA}(\mathcal{F}(M),\mathbb{R})} = \mathcal{L}(\mathcal{F}(M),\mathbb{R})$ for every M by the Bishop–Phelps theorem,
- But $\overline{\mathrm{SNA}([0,1],\mathbb{R})} \neq \mathrm{Lip}_0([0,1],\mathbb{R})$.

A drop on the geometry of the unit ball of $\mathcal{F}(M)$

Denting point

$\xi \in B_{\mathcal{F}(M)}$, TFAE:

- ξ is a denting point (of $B_{\mathcal{F}(M)}$),
- $\xi = m_{p,q}$ and for every $\varepsilon > 0$ $\exists$ $\delta > 0$ s.t. $d(p,t) + d(t,q) - d(p,q) > \delta$ when $d(p,t), d(t,q) \geqslant \varepsilon$.

★ M boundedly compact, it is equivalent to:

- $d(p,q) < d(p,t) + d(t,q)$ $\forall t \notin \{p,q\}$.

Strongly exposed point

$\xi \in B_{\mathcal{F}(M)}$, TFAE:

- ξ strongly exposed point (of $B_{\mathcal{F}(M)}$),
- $\xi = m_{p,q}$ and $\exists$ $\rho = \rho(p,q) > 0$ such that

$$\frac{d(p,t) + d(t,q) - d(p,q)}{\min\{d(p,t), d(t,q)\}} \geqslant \rho$$

when $t \notin \{p,q\}$.

Concave metric space

M is concave if all the $m_{p,q}$ are denting points.
★ Examples: $y = x^3$, S_X if X unif. convex...

Uniform Gromov rotundity

$\mathcal{M} \subset \mathrm{Mol}\,(M)$ is uniformly Gromov rotund if $\exists \rho_0 > 0$ such that

$$\frac{d(p,t) + d(t,q) - d(p,q)}{\min\{d(p,t), d(t,q)\}} \geqslant \rho_0$$

when $m_{p,q} \in \mathcal{M}$, $t \notin \{p,q\}$.
$\Longleftrightarrow$ $\mathcal{M}$ is a set of uniformly strongly exposed points (same relation $\varepsilon - \delta$)

★ $\mathrm{Mol}\,(M)$ uniformly Gromov rotund (aka M is uniformly Gromov concave) when:

- $M = ([0,1], |\cdot|^\theta)$,
- M finite and concave,
- $1 \leqslant d(p,q) \leqslant D < 2$ $\forall p,q \in M$, $p \neq q$.

A compilation of negative and positive results

Negative results I

First extension of the case of $[0, 1]$ (Kadets–Martín–Soloviova, 2016)

If M is metrically convex (or “geodesic”), then $\mathrm{SNA}(M, \mathbb{R})$ is not dense in $\mathrm{Lip}_0(M, \mathbb{R})$.

Definition (length space)

Let M be a metric space. M is length if $d(p, q)$ is equal to the infimum of the length of the rectifiable curves joining p and q for every pair of points $p, q \in M$.

★ Equivalently (Avilés, García, Ivankhno, Kadets, Martínez, Prochazka, Rueda, Werner)

- M is local (i.e. the Lipschitz constant of every function can be approximated in pairs of arbitrarily closed points);
- The unit ball of $\mathcal{F}(M)$ has no strongly exposed points;
- $\mathrm{Lip}_0(M, \mathbb{R})$ (and so $\mathcal{F}(M)$) has the Daugavet property.

Theorem

M length metric space $\implies$ $\overline{\mathrm{SNA}(M, \mathbb{R})} \neq \mathrm{Lip}_0(M, \mathbb{R})$

Negative results II

A different kind of example

M "fat" Cantor set in $[0,1]$, then $\overline{\mathrm{SNA}(M,\mathbb{R})} \neq \mathrm{Lip}_0(M,\mathbb{R})$ and M is totally disconnected.

Theorem

Actually, if $M \subset \mathbb{R}$ is compact and has positive measure,
then $\mathrm{SNA}(M,\mathbb{R})$ is NOT dense in $\mathrm{Lip}_0(M,\mathbb{R})$.

Possible sufficient conditions

Observation (previously commented)

$\mathrm{SNA}(M, Y)$ dense in $\mathrm{Lip}_0(M, Y) \implies \mathrm{NA}(\mathcal{F}(M), Y)$ dense in $\mathcal{L}(\mathcal{F}(M), Y)$.

Therefore, it is reasonable to discuss the known sufficient conditions for a Banach space X to have $\overline{\mathrm{NA}(X, Y)} = \mathcal{L}(X, Y)$ for every Y:

- containing a norming and uniformly strongly exposed set (Lindenstrauss, 1963),
- RNP (Bourgain, 1977),
- Property α (Schachermayer, 1983) ,
- Property quasi-α (Choi–Song, 2008).

Main result

EACH of these properties on $\mathcal{F}(M)$ implies $\overline{\mathrm{SNA}(M, Y)} = \mathrm{Lip}_0(M, Y)$ for all Y.

How to prove the positive results?

Common scheme of all the proofs

- Consider the original proof of
  "(P) on $\mathcal{F}(M)$ implies that $\overline{\mathrm{NA}(\mathcal{F}(M),Y)} = \mathcal{L}(\mathcal{F}(M),Y)$ for all Y's".
- Check, in each case, what is the set of points where the constructed set of norm attaining operators actually attain their norm.
- It happens that, in all the cases, this set is contained in the set of strongly exposed points or, maybe, in its closure.
- Strongly exposed points are molecules (Weaver, 1999), and the set of molecules is norm closed (GL-P-P-RZ, 2018).

Examples of positive results

$\mathcal{F}(M)$ has the RNP when...

$\mathcal{F}(M)$ RNP iff $\mathcal{F}(M)$ does not contains $L_1[0,1]$ iff M contains no "curve fragment" (Aliaga-Gartland-Petitjean-Prochachazka, 2022).
For instance:

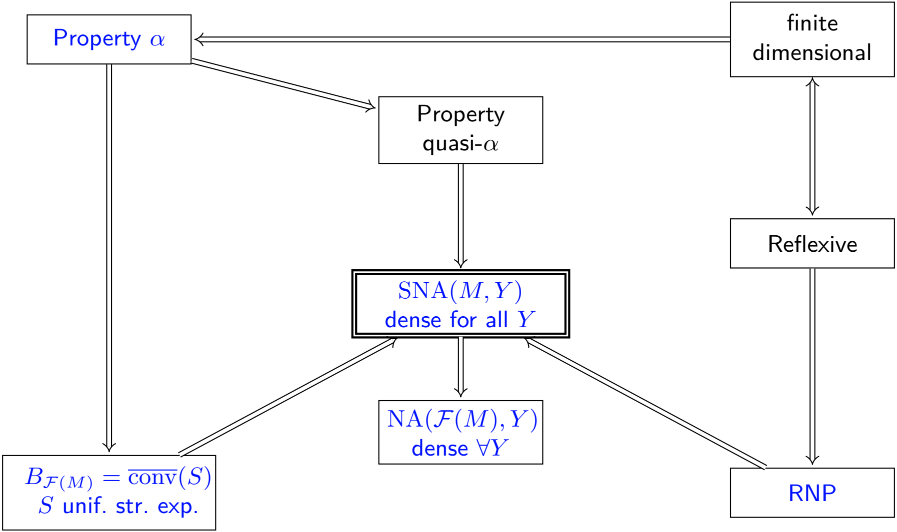
- $M = (N, d^\theta)$ with $0 < \theta < 1$ (Weaver, 1999 for the compact case);
- M is uniformly discrete (Kalton, 2004);
- M is countable (Dalet, 2015 for the boundedly compact case);
- $M \subset \mathbb{R}$ with Lebesgue measure 0 (Godard, 2010).

$\mathcal{F}(M)$ has property α when...

- M finite,
- $M \subset \mathbb{R}$ with Lebesgue measure 0,
- $1 \leqslant d(p,q) \leqslant D < 2$ for all $p, q \in M$, $p \neq q$.

For $M = (N, d^\theta)$ with $0 < \theta < 1$, we actually have a "Bishop–Phelps–Bollobás" result.

Sufficient conditions for the density of $\mathrm{SNA}(M,Y)$ for every Y: relations



Two somehow surprising examples

Some questions and an open problem

Some questions

(Q1) Does $\mathcal{F}(M)$ have the RNP if $\overline{\mathrm{SNA}(M,Y)} = \mathrm{Lip}_0(M,Y)$ $\forall Y$ **?** (Godefroy, 2015)
(recall that $\mathcal{F}(M)$ RNP iff $\mathcal{F}(M)$ does not contain $L_1[0,1]$ isomorphically)

(Q2) Is $\mathrm{SNA}(M,\mathbb{R})$ dense in $\mathrm{Lip}_0(M,\mathbb{R})$ if $B_{\mathcal{F}(M)} = \overline{\mathrm{co}}\big(\text{str-exp}\left(B_{\mathcal{F}(M)}\right)\big)$
or, even, if $\mathrm{Mol}\,(M) = \text{str-exp}\left(B_{\mathcal{F}(M)}\right)$ **?**

(Q3) Does $\mathrm{SNA}(M,\mathbb{R})$ fail to be dense in $\mathrm{Lip}_0(M,\mathbb{R})$ if $\mathcal{F}(M)$ contains an **isometric** copy of $L_1[0,1]$ **?**

In dimension one, everything works...

$M \subset \mathbb{R}$ compact, then the questions have positive answer.

but not always

We will see that the three questions have negative answer in general by using compact subsets of $\mathbb{R}^2$.

The first example: the torus

Theorem

Let $M = \mathbb{T}$ endowed with the distance inherited from the Euclidean plane.

- $\mathrm{SNA}(M, \mathbb{R})$ is not dense in $\mathrm{Lip}_0(M, \mathbb{R})$;
- M is Gromov concave (i.e. every molecule is strongly exposed).

Remarks

- It is a (negative) solution to (Q2);
- $\mathcal{F}(\mathbb{T})$ satisfies that the set of strongly exposed points generates the unit ball by closed convex hull, but strongly exposing functionals are not dense in the dual.
- (Jung–M.–Rueda, 2023) $\mathcal{F}(\mathbb{T})$ fails Lindenstrauss property A (but it satisfies Lindenstrauss' necessary condition from 1963)

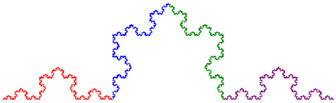
Two different curves

Koch curve

Let $M_1 = ([0,1], |\cdot|^\theta)$, $0 < \theta < 1$.

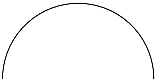
- $\mathcal{F}(M_1)$ has RNP, so $\overline{\mathrm{SNA}(M_1, Y)} = \mathrm{Lip}_0(M_1, Y)\ \forall Y$.
- Every molecule is strongly exposed,
- even more, M_1 is uniformly Gromov concave.

★ For $\theta = \log(3)/\log(4)$, M_1 is bi-Lipschitz equivalent to the Koch curve:



The unit circle

Let M_2 be the upper half of the unit circle:



- We know that $\mathrm{SNA}(M_2, \mathbb{R})$ is not dense in $\mathrm{Lip}_0(M_2, \mathbb{R})$.
- $\mathcal{F}(M_2)$ has NOT the RNP.
- However, every molecule is strongly exposed...
- but NO subset $A \subset \mathrm{Mol}\,(M_2)$ which is uniformly Gromov rotund can be norming for $\mathrm{Lip}_0(M_2, \mathbb{R})$.

Microscopically, a small piece of M_1 is equivalent to M_1 itself.

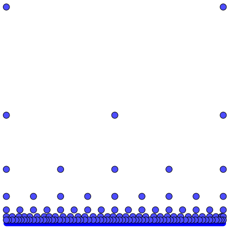
Microscopically, a small piece of M_2 is very closed to be an interval.

The second example: the "rain"

Consider the compact subset of $\mathbb{R}^2$ given by

$$M := ([0,1] \times \{0\}) \cup \bigcup_{n=0}^{\infty} \left\{ \left(\frac{k}{2^n}, \frac{1}{2^n} \right) : k \in \{0, \ldots, 2^n\} \right\}$$

Let $\mathfrak{M}_p$ be the set M endowed with the distance inherited from $(\mathbb{R}^2, \|\cdot\|_p)$ for $p = 1, 2$.



Theorem

$\mathrm{SNA}(\mathfrak{M}_p, Y)$ is dense in $\mathrm{Lip}_0(\mathfrak{M}_p, Y)$ for all Y and $p = 1, 2$. Moreover:

1. $\mathcal{F}(\mathfrak{M}_p)$ fails the RNP (actually, it contains an isometric copy of $L_1[0,1]$!);
2. $\mathcal{F}(\mathfrak{M}_1)$ has property α;
3. $\mathcal{F}(\mathfrak{M}_2)$ does not contain any norming and uniformly strongly exposed set.

Remark

This gives negative answer to both (Q1) and (Q3).

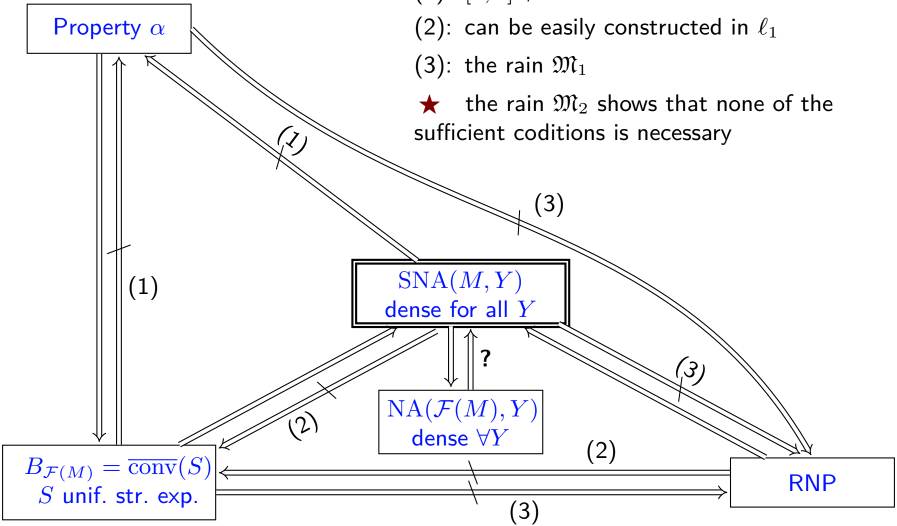
Summarizing the relations

(1): $[0,1]^\theta$, $0<\theta<1$

(2): can be easily constructed in ℓ_1

(3): the rain $\mathfrak{M}_1$

★ the rain $\mathfrak{M}_2$ shows that none of the sufficient coditions is necessary



Some consequences of the denseness of strongly norm attaining Lipschitz functions

An useful lemma and a first consequence

Lemma

M metric space, $f \in \mathrm{SNA}(M,Y)$, $\widehat{f}(m_{p,q}) = \|f\|$. Then:

1. either $\exists x, y \in M$ such that

$$m_{x,y} \in \operatorname{ext}\left(B_{\mathcal{F}(M)}\right) \quad \text{and} \quad d(p,q) = d(p,x) + d(x,y) + d(y,q)$$

   (so, in particular, $\widehat{f}(m_{x,y}) = \|f\|$);
2. or there is an isometric embedding $\phi\colon [0, d(p,q)] \longrightarrow M$ with $\phi(0) = p$ and $\phi(d(p,q)) = q$.

Consequence

M metric space such that $\mathrm{SNA}(M,\mathbb{R})$ is dense. Then

$$B_{\mathcal{F}(M)} = \overline{\operatorname{conv}}\left(\operatorname{ext}\left(B_{\mathcal{F}(M)}\right)\right).$$

Remark (Lindenstrauss, 1963)

X subspace of ℓ_∞ such that $\overline{\mathrm{NA}(X,Y)} = \mathcal{L}(X,Y)$ $\forall Y$.
Then, $B_X = \overline{\operatorname{conv}}\left(\operatorname{ext}(B_X)\right)$.

The main consequence

Theorem

M compact, $M \not\supset [0,1]$, Y Banach space, $\mathrm{SNA}(M,Y)$ dense in $\mathrm{Lip}_0(M,Y)$.
Then, $\mathrm{SNA}(M,Y)$ (and so $\mathrm{NA}(\mathcal{F}(M),Y)$) contains an **open** dense subset.

Corollary

M compact, Y Banach space, $\mathcal{F}(M)$ RNP.
Then, $\mathrm{SNA}(M,Y)$ (and so $\mathrm{NA}(\mathcal{F}(M),Y)$) contains an **open** dense subset.

Remarks

The presence of dense open subset of norm attaining things is a rare phenomenon:

- (Acosta–Kadets, 2011): if X is not reflexive, then there is X' isomorphic to X such that $\mathrm{NA}(X',\mathbb{R})$ has empty interior.
- (Acosta–Aizpuru–Aron–GarcíaPacheco, 2007): $\mathrm{NA}(L_1[0,1],\mathbb{R})$ has empty interior since $L_\infty[0,1] \setminus \mathrm{NA}(L_1[0,1],\mathbb{R})$ is dense in $L_\infty[0,1]$.

The main consequence

Theorem

M compact, $M \not\supseteq [0,1]$, Y Banach space, $\mathrm{SNA}(M,Y)$ dense in $\mathrm{Lip}_0(M,Y)$.
Then, $\mathrm{SNA}(M,Y)$ (and so $\mathrm{NA}(\mathcal{F}(M),Y)$) contains an **open** dense subset.

Corollary

M compact, Y Banach space, $\mathcal{F}(M)$ RNP.
Then, $\mathrm{SNA}(M,Y)$ (and so $\mathrm{NA}(\mathcal{F}(M),Y)$) contains an **open** dense subset.

Question

Is it possible to remove the condition of $M \not\supseteq [0,1]$ in the theorem ?

Example

$\mathrm{SNA}(\mathfrak{M}_p,Y)$ contains an open dense subset for $p = 1, 2$ and for every Y.

Another consequence

Proposition (consequence of the proof of the main theorem)

M compact, $\mathrm{SNA}(M,\mathbb{R})$ dense in $\mathrm{Lip}_0(M,\mathbb{R})$. Then,

$$B_{\mathcal{F}(M)} = \overline{\mathrm{conv}}\Big(\text{str-exp}\left(B_{\mathcal{F}(M)}\right)\Big).$$

Remark (Lindenstrauss, 1963)

X separable Banach space such that $\overline{\mathrm{NA}(X,Y)} = \mathcal{L}(X,Y)\ \forall Y$.
Then, $B_X = \overline{\mathrm{conv}}\,(\text{str-exp}\,(B_X))$.

Question

Is it possible to remove the compactness condition on M ?

Some open problems

Some open problems

Two old results

- (Bourgain, 1977; Huff, 1980): X has the RNP iff $\overline{\mathrm{NA}(X',Y)} = \mathcal{L}(X',Y)$ for every Y and every renorming X' of X.
- (Schachermayer, 1983): X separable (WCG) $\Longrightarrow$ exists X' renorming of X such that $\overline{\mathrm{NA}(X',Y)} = \mathcal{L}(X',Y)$ for every Y.

Open problem 1

M (compact) metric space such that $\overline{\mathrm{SNA}(M',Y)} = \mathrm{Lip}_0(M',Y)$ for every Y and every M' bi-Lipschitz equivalent to M.

★ Does $\mathcal{F}(M)$ have the RNP?

Open problem 2

M (compact) metric space.

★ Does there exists M' bi-Lipschitz equivalent to M such that $\overline{\mathrm{SNA}(M',Y)} = \mathrm{Lip}_0(M',Y)$ for every Y?

Some open problems II

A related result

M metric such that for every $N \subset M$ and every N' bi-Lipschitz equivalent to N one has that $\overline{\mathrm{SNA}(N',\mathbb{R})} = \mathrm{Lip}_0(N',\mathbb{R})$. Then, $\mathcal{F}(M)$ has the RNP.

Indeed, otherwise M contains a "curve fragment" C and C is bi-Lipschitz equivalent to a subset of $[0,1]$ with positive measure.

Open problem 3

(M,d) metric space such that $\overline{\mathrm{SNA}((M,d),\mathbb{R})} = \mathrm{Lip}_0((M,d),\mathbb{R})$ and let d' equivalent metric on M. Is $\mathrm{SNA}((M,d'),\mathbb{R})$ dense?

★ We do not know even for $M = [0,1]$.

Open problem 4 (Aliaga at a conference in Murcia, 2024)

Suppose $\mathrm{SNA}(M,\mathbb{R}) = \mathrm{NA}(\mathcal{F}(M),\mathbb{R})$.

★ Does $\mathcal{F}(M)$ have the RNP?

Lineability and spaceability

Some lineability and spaceability related results

Subspaces in the set of strongly norm-attaining functions

M infinite pointed metric space

- (Kadets–Roldán, 2022) $\mathrm{SNA}(M, \mathbb{R})$ is lineable.
- (Avilés–MartínezCervantes–RuedaZoca–Tradacete, 2024) $\mathrm{SNA}(M, \mathbb{R})$ is spaceable: it always contains **isomorphic** copies of c_0.
- (Dantas–Medina–Quilis–Roldán, 2023) $\mathrm{SNA}(M, \mathbb{R})$ contains **isometric** copies of c_0 if M is not uniformly discrete.
- (Dantas–Medina–Quilis–Roldán, 2023) $\mathrm{SNA}(M, \mathbb{R})$ does not always contain **isometric** copies of c_0.

Subspaces in the set of non strongly norm-attaining functions

M infinite pointed metric space

- (Choi–Jung–Lee–Roldán, 2024) $\big(\mathrm{Lip}_0(M, \mathbb{R}) \setminus \mathrm{SNA}(M, \mathbb{R})\big) \cup \{0\}$ contains an isometric copy of ℓ_∞.
- Many similar results concerning other kind of norm attainment for Lipschitz maps. . .

The main bibliography of the talk

B. Cascales, R. Chiclana, L. García-Lirola, M. Martín, A. Rueda
On strongly norm attaining Lipschitz maps
J. Funct. Anal. (2019)

R. Chiclana, L. García-Lirola, M. Martín, A. Rueda
Examples and applications of the density of strongly norm attaining Lipschitz maps
Rev. Mat. Iberoam. (2021)

R. Chiclana, M. Martín
The Bishop-Phelps-Bollobás property for Lipschitz maps
Nonlinear Analysis (2019)

R. Chiclana, M. Martín
Some stability properties for the Bishop-Phelps-Bollobás property for Lipschitz maps
Studia Math. (2022)

M. Jung, M. Martín, A. Rueda
Residuality in the set of norm attaining operators between Banach spaces
J. Funct. Anal. (2023)