Generating operators between Banach spaces

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joint work with V. Kadets, J. Merí, and A. Quero

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WILSON HALL

Some notation

X, Y real or complex Banach spaces

$\mathbb{K}$ scalar field $\mathbb{R}$ or $\mathbb{C}$

$\mathbb{T}$ modulus one scalars

B_X closed unit ball

S_X unit sphere

X^* topological dual

$\mathcal{L}(X,Y)$ bounded linear operators from X to Y

$\mathrm{conv}(\cdot)$ convex hull

$\overline{\mathrm{conv}}(\cdot)$ closed convex hull

$\mathrm{ext}(C)$ extreme points of a (closed convex bounded) set

Introduction. Spear operators

Spear vectors

Spear vector (Ardalani, 2014)

X Banach space, $u \in S_X$. We say that u is a *spear vector* if

$$\max_{\theta\in\mathbb{T}} \|u+\theta x\| = 1 + \|x\| \quad \text{for every } x \in X.$$

Examples of spear vectors:

- $\mathrm{Spear}(\mathbb{K}) = \mathbb{T}$
- $\mathrm{Spear}(\ell_1) = \mathbb{T}\{e_n : n \in \mathbb{N}\}$
- $\mathrm{Spear}(\ell_\infty) = \mathbb{T}^{\mathbb{N}}$
- $\mathrm{Spear}(L_\infty(\mu)) = \{f \in L_\infty(\mu) : |f(t)| = 1 \ \mu\text{-a.e.}\}$

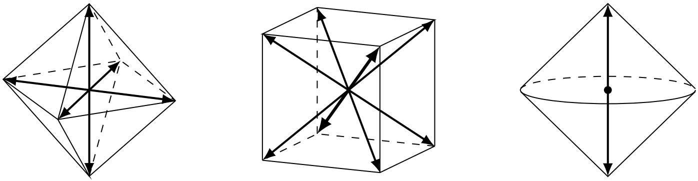

Figure: Spear vectors in the real spaces ℓ_1^3, ℓ_∞^3 and $\ell_2^2 \oplus_1 \mathbb{R}$, respectively.

Spear vector (Ardalani, 2014)

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$$\max_{\theta\in\mathbb{T}} \|u+\theta x\| = 1+\|x\| \quad \text{for every } x \in X.$$

Some properties of spear vectors:

- $z \in \operatorname{Spear}(X)$ iff $|x^*(z)| = 1$ for every $x^* \in \operatorname{ext}(B_{X^*})$
- $\operatorname{Spear}(X) \subseteq \operatorname{ext}(B_X)$ (actually, they are strongly extreme)
- $J_X(\operatorname{Spear}(X)) \subseteq \operatorname{Spear}(X^{**})$
- $z^* \in \operatorname{Spear}(X^*)$ then z^* is norm attaining. Actually,

$$B_X = \overline{\operatorname{conv}}\big\{x \in B_X : |z^*(x)| = 1\big\}$$

- X real, $\#\operatorname{Spear}(X) = \infty$, then $X \supset c_0$ or $X \supset \ell_1$
- X strictly convex, $\dim(X) \geqslant 2$, then $\operatorname{Spear}(X) = \emptyset = \operatorname{Spear}(X^*)$

Spear operator (Ardalani, 2014)

X, Y Banach spaces, $G \in \mathcal{L}(X,Y)$ with $\|G\| = 1$.

$$G \text{ *spear operator* } \overset{\text{def}}{\Longleftrightarrow} \max_{\theta\in\mathbb{T}} \|G + \theta T\| = 1 + \|T\| \text{ for every } T \in \mathcal{L}(X,Y)$$

Examples of spear operators (Kadets–M.–Merí–Pérez, 2018)

- The identity operator on $C(K)$, $L_1(\mu)$, c_0, ℓ_1, ℓ_∞...
- The inclusion $A(\mathbb{D}) \hookrightarrow C(\mathbb{T})$.
- The Fourier transforms (for instance, $\mathcal{F}\colon L_1(\mathbb{R}) \longrightarrow C_0(\mathbb{R})$ or $\mathcal{F}\colon L_1(\mathbb{T}) \longrightarrow c_0(\mathbb{Z})$).
- $G = x_0^* \otimes y_0$ spear $\iff$ $x_0^* \in \operatorname{Spear}(X^*)$ and $y_0 \in \operatorname{Spear}(Y)$.

Some properties (Kadets–M.–Merí–Pérez, 2018)

- $G\colon \ell_1 \longrightarrow Y$ spear $\iff$ $G(e_n) \in \operatorname{Spear}(Y)$ $\forall n \in \mathbb{N}$
- $G\colon X \longrightarrow c_0$ spear $\iff$ $G^*(e_n) \in \operatorname{Spear}(X^*)$ $\forall n \in \mathbb{N}$
- If $\dim(X) < \infty$, $G\colon X \longrightarrow Y$ spear $\iff$ $G(x) \in \operatorname{Spear}(Y)$ $\forall x \in \operatorname{ext}(B_X)$
- If $\dim(Y) < \infty$, $G\colon X \longrightarrow Y$ spear $\iff$ $G^*(y^*) \in \operatorname{Spear}(X^*)$ $\forall y^* \in \operatorname{ext}(B_{Y^*})$

Isomorphic and isometric consequences (Kadets–M.–Merí–Pérez, 2018)

X, Y Banach spaces, $G \in \mathcal{L}(X,Y)$ spear operator.

- If X is real and $\dim G(X) = \infty$, then $X^* \supset \ell_1$.
- If X^* is strictly convex or smooth, then $X = \mathbb{K}$.
- If Y^* is strictly convex, then $Y = \mathbb{K}$.
- If X contains strongly exposed points, Y smooth, then $Y = \mathbb{K}$.
  (M.-Merí-Quero-Sain-Roy, 2023)

Numerical radius with respect to G

X, Y Banach spaces, $G \in \mathcal{L}(X,Y)$ with $\|G\| = 1$, $T \in \mathcal{L}(X,Y)$

$$v_G(T) := \inf_{\delta>0} \sup\Big\{|y^*(Tx)| \colon y^* \in S_{Y^*},\, x \in S_X,\, \operatorname{Re} y^*(Gx) > 1-\delta\Big\}.$$

- $v_G(\cdot)$ is a seminorm on $\mathcal{L}(X,Y)$.
- $v_G(T) \leqslant \|T\|$ for every $T \in \mathcal{L}(X,Y)$.

G spear operator $\iff$ $v_G(T) = \|T\|$ for every $T \in \mathcal{L}(X,Y)$.

Leading idea of the manuscript (and the talk)

Introduce a new seminorm on $\mathcal{L}(X,Y)$ between $v_G(\cdot)$ and the usual operator norm to get a weaker property than being spear:

$$v_G(T) = \inf_{\delta>0} \sup\{|y^*(Tx)| \colon y^* \in S_{Y^*},\, x \in S_X,\, \operatorname{Re} y^*(Gx) > 1-\delta\}.$$

$$\|T\| = \inf_{\delta>0} \sup\{|y^*(Tx)| \colon y^* \in S_{Y^*},\, x \in S_X,\, \cancel{\operatorname{Re} y^*(Gx) > 1-\delta}\}.$$

$$\begin{aligned}\|T\|_G &:= \inf_{\delta>0} \sup\{|y^*(Tx)| \colon y^* \in S_{Y^*},\, x \in S_X,\, \|Gx\| > 1-\delta\} \\ &= \inf_{\delta>0} \sup\{\|Tx\| \colon y^* \in S_{Y^*}, x \in S_X,\, \|Gx\| > 1-\delta\}.\end{aligned}$$

Generating operators

Norm relative to an operator

X, Y Banach spaces, $G \in \mathcal{L}(X,Y)$ with $\|G\| = 1$, $T \in \mathcal{L}(X,Y)$.

$$\|T\|_G := \inf_{\delta>0} \sup\{\|Tx\| : x \in S_X,\ \|Gx\| > 1-\delta\}.$$

Relative norm (Kadets–M.–Merí–Quero, 2023)

X, Y, Z Banach spaces and $G \in \mathcal{L}(X,Y)$ with $\|G\| = 1$. The *(semi-)norm of* $T \in \mathcal{L}(X,Z)$ *relative to* G is

$$\|T\|_G := \inf_{\delta>0} \sup\left\{\|Tx\| : x \in \operatorname{att}(G,\delta)\right\},$$

where, for each $\delta > 0$, $\operatorname{att}(G,\delta) := \{x \in S_X : \|Gx\| > 1-\delta\}$.

- $\|\cdot\|_G$ is a seminorm on $\mathcal{L}(X,Z)$.
- If $Z = Y$, then for every $T \in \mathcal{L}(X,Y)$,

$$v_G(T) \leqslant \|T\|_G \leqslant \|T\|.$$

Generating operator (Kadets–M.–Merí–Quero, 2023)

Let X, Y be Banach spaces and let $G \in \mathcal{L}(X,Y)$ be a norm-one operator. We say that G is *generating* if $\|T\|_G = \|T\|$ for every $T \in \mathcal{L}(X,Y)$.

Proposition

X, Y Banach spaces and $G \in \mathcal{L}(X,Y)$ with $\|G\| = 1$. Then, TFAE:

1. G is generating.
2. $\|T\|_G = \|T\|$ for every $T \in \mathcal{L}(X,Z)$ and every Banach space Z.
3. There is a (non null) Banach space Z such that $\|T\|_G = \|T\|$ for every *rank-one* operator $T \in \mathcal{L}(X,Z)$.
4. $B_X = \overline{\operatorname{conv}}(\operatorname{att}(G,\delta))$ for every $\delta > 0$.

If $\dim(X) < \infty$, G generating $\iff$ $B_X = \operatorname{conv}(\{x \in S_X : \|Gx\| = 1\})$.

Examples:

- The identity operator on every Banach space and every isometric embedding.
- Spear operators.
- The natural inclusion $G\colon \ell_1 \hookrightarrow c_0$ is generating since

$$\mathbb{T}\{e_n\colon n\in\mathbb{N}\} \subset \operatorname{att}(G,\delta)\ \forall \delta>0 \implies B_{\ell_1} = \overline{\operatorname{conv}}(\operatorname{att}(G,\delta))\ \forall\delta>0.$$

- The natural inclusion $G\colon L_\infty[0,1] \hookrightarrow L_1[0,1]$ is generating since

$$\{f\in L_\infty[0,1]\colon |f(t)|=1 \text{ a.e.}\} \subset \operatorname{att}(G,\delta)\ \forall\delta>0$$
$$\implies B_{L_\infty[0,1]} = \overline{\operatorname{conv}}(\operatorname{att}(G,\delta))\ \forall\delta>0.$$

Lemma

X, Y Banach spaces and $G \in \mathcal{L}(X,Y)$ with $\|G\| = 1$. If G is generating, then $\|Gx\| = 1$ for every denting point x of B_X.

Proposition

X, Y Banach spaces and $G \in \mathcal{L}(X,Y)$ with $\|G\| = 1$. Suppose that X has the RNP. Then, G is generating if and only if $\|Gx\| = 1$ for every denting point x of B_X.

Particular cases: X, Y Banach spaces

- $G\colon \ell_1 \longrightarrow Y$ generating $\iff$ $\|Ge_n\| = 1$ for every $n \in \mathbb{N}$.
- If $\dim(X) < \infty$, $G\colon X \longrightarrow Y$ generating $\iff$ $\|Gx\| = 1$ for every $x \in \operatorname{ext}(B_X)$.

Spear sets (Kadets–M.–Merí–Pérez, 2018)

Let X be a Banach space. We say that $F \subset B_X$ is a *spear set* if

$$\max_{\theta\in\mathbb{T}} \sup_{z\in F} \|z+\theta x\| = 1+\|x\| \qquad \text{for every } x\in X.$$

- $F = B_X$ and $F = S_X$ are spear sets
- $F = \{z\}$ is a spear set $\iff$ z is a spear vector
- $F \subset B_{L_\infty[0,1]}$ spear set $\iff$ $\forall A \subset [0,1]$ with $|A|>0$ and $\forall \varepsilon>0\ \exists B\subset A$ with $|B|>0$ and $f\in F$ such that $|f(t)|>1-\varepsilon\ \forall t\in B$.
- $F \subset B_{X^*}$ spear set iff $\forall \varepsilon>0$,

$$B_X = \overline{\operatorname{conv}}\left\{x\in B_X\colon \sup_{x^*\in F} \operatorname{Re} x^*(x) > 1-\varepsilon\right\}.$$

Theorem

X, Y Banach spaces and $G\in\mathcal{L}(X,Y)$ with $\|G\|=1$.
Then, G is generating if and only if $G^*(B_{Y^*})$ is a spear set of X^*.

Spear sets (Kadets–M.–Merí–Pérez, 2018)

Let X be a Banach space. We say that $F \subset B_X$ is a *spear set* if

$$\max_{\theta \in \mathbb{T}} \sup_{z \in F} \|z + \theta x\| = 1 + \|x\| \qquad \text{for every } x \in X.$$

Theorem

X, Y Banach spaces and $G \in \mathcal{L}(X, Y)$ with $\|G\| = 1$.
Then, G is generating if and only if $G^*(B_{Y^*})$ is a spear set of X^*.

Corollary

X, Y Banach spaces, $x_0^* \in S_{X^*}$, and $y_0 \in S_Y$.
$G = x_0^* \otimes y_0$ is generating if and only if x_0^* is a spear vector of X^*.

- X has denting points, $G \in \mathcal{L}(X,Y)$ generating $\Longrightarrow$ G attains its norm.
- **[Kadets–M.–Merí–Pérez, 2018]**: Every spear $x^* \in X^*$ attains its norm $\Longrightarrow$ G rank-one and generating attains its norm.

Question: Does every generating operator attain its norm? No, there are counterexamples even of rank two.

Example: Consider $g\colon [0,1] \longrightarrow \ell_2^2$ given by $g(t) = (\cos t, \sin t)$ and $G\colon L_1[0,1] \longrightarrow \ell_2^2$ defined by:

$$G(x) = \int_0^1 x(t)g(t)\,dt \qquad (x \in L_1[0,1]).$$

Then, G is generating but does not attain its norm.

- G generating $\iff$ $\{y^* \circ g\colon y^* \in B_{Y^*}\}$ is a spear set of $B_{L_\infty[0,1]}$ $\iff$ $\|g(t)\| = 1$ a.e. ✓
- Suppose that there is a non-zero $x \in L_1[0,1]$ such that $\|Gx\| = \|x\|$.

$$\left|\int_0^1 x(t)g(t)dt\right| = \left\|\int_0^1 x(t)g(t)dt\right\| = \int_0^1 |x(t)|dt = \int_0^1 |x(t)g(t)|dt.$$

$\Longrightarrow$ $\exists \lambda \in \mathbb{T}$ such that $xg = \lambda|xg| = \lambda|x|$ a.e. which is a contradiction

Theorem

Let X be a Banach space. Then, TFAE:

1. There exists a Banach space Y and a norm-one operator $G \in \mathcal{L}(X,Y)$ such that G is generating but does not attain its norm.
2. There exists a spear set $\mathcal{B} \subset B_{X^*}$ such that $\sup_{x^* \in \mathcal{B}} |x^*(x)| < 1$ for every $x \in S_X$.

1)⇒2) Take $\mathcal{B} = G^*(B_{Y^*})$ spear set of B_{X^*} and

$$\sup_{x^* \in \mathcal{B}} |x^*(x)| = \sup_{y^* \in B_{Y^*}} |(G^*y^*)(x)| = \sup_{y^* \in B_{Y^*}} |y^*(Gx)| = \|Gx\| < 1 \quad \forall x \in S_X.$$

2)⇒1) Define $G\colon X \longrightarrow \ell_\infty(\mathcal{B})$ by

$$(Gx)(x^*) = x^*(x) \qquad (x^* \in \mathcal{B},\ x \in X).$$

- $\|G\| = 1$ but $\|Gx\| = \sup_{x^* \in \mathcal{B}} |(Gx)(x^*)| = \sup_{x^* \in \mathcal{B}} |x^*(x)| < 1$ for every $x \in S_X$.
- $\mathcal{B} \subset G^*(B_{\ell_1(\mathcal{B})}) \subset G^*(B_{\ell_\infty(\mathcal{B})^*}) \implies G^*(B_{\ell_\infty(\mathcal{B})^*})$ is a spear set. □

The set of all generating operators

X, Y Banach spaces

$$\mathrm{Gen}(X,Y) := \{G \in \mathcal{L}(X,Y)\colon \|G\| = 1,\ G \text{ is generating}\}.$$

Questions:

- Are there Banach spaces Y such that $\mathrm{Gen}(X,Y) \neq \emptyset$ for every Banach space X?
- Are there Banach spaces X such that $\mathrm{Gen}(X,Y) \neq \emptyset$ for every Banach space Y?

Proposition

For every Banach space Y there is a Banach space X such that $\mathrm{Gen}(X,Y) = \emptyset$.

Proposition

X Banach space. Then,

$$\mathrm{Gen}(X,Y) \neq \emptyset \text{ for every Banach space } Y \iff \mathrm{Spear}(X^*) \neq \emptyset.$$

The set of all generating operators

Questions: X Banach space

- When $\operatorname{Gen}(X,Y) = S_{\mathcal{L}(X,Y)}$ for all Banach spaces Y?

Proposition

X Banach space. Then,

$$\exists\, Y \text{ Banach space such that } \operatorname{Gen}(X,Y) = S_{\mathcal{L}(X,Y)} \implies X = \mathbb{K}.$$

In this case, $\operatorname{Gen}(\mathbb{K},Y) = S_{\mathcal{L}(\mathbb{K},Y)}$ for every Banach space Y.

- When $B_{\mathcal{L}(X,Y)} = \overline{\operatorname{conv}}(\operatorname{Gen}(X,Y))$ for all Banach spaces Y?

Theorem

(Ω,Σ,μ) finite measure space, Y Banach space. Then,

$$\{T \in \mathcal{L}(L_1(\mu),Y) \colon \|T\| \leqslant 1,\ T \text{ is representable}\} \subset \overline{\operatorname{conv}}\left(\operatorname{Gen}(L_1(\mu),Y)\right).$$

As a consequence, if Y has the RNP with respect to μ, then

$$B_{\mathcal{L}(L_1(\mu),Y)} = \overline{\operatorname{conv}}\left(\operatorname{Gen}(L_1(\mu),Y)\right).$$

The set of all generating operators

Corollary

- $B_{\mathcal{L}(\ell_1,Y)} = \overline{\text{conv}}\,(\text{Gen}(\ell_1,Y))$ for every Banach space Y.
- $B_{\mathcal{L}(\ell_1^n,Y)} = \text{conv}\,(\text{Gen}(\ell_1^n,Y))$ for every Banach space Y and every $n \in \mathbb{N}$.

Proposition

X *real* Banach space, $\dim(X) = n$.

$$B_{\mathcal{L}(X,Y)} = \overline{\text{conv}}(\text{Gen}(X,Y)) \text{ for every Banach space } Y \implies X = \ell_1^n.$$

Proposition

X *real* Banach space, $\dim(X) = n$.

$$B_{\mathcal{L}(X,Y)} = \overline{\operatorname{conv}}(\operatorname{Gen}(X,Y)) \text{ for every Banach space } Y \implies X = \ell_1^n.$$

Sketch of the proof.

- $B_{\mathcal{L}(X,\mathbb{K})} = \overline{\operatorname{conv}}(\operatorname{Gen}(X,\mathbb{K})) \implies X^*$ is almost CL-space and so $n(X) = 1$.
- **[McGregor, 1971]**: X real finite-dimensional, $n(X) = 1 \implies \operatorname{ext}(B_X)$ is finite.
- The set $\operatorname{ext}(B_X)$ has exactly $2n$ elements and so $X = \ell_1^n$.
  If $\operatorname{ext}(B_X)$ has more than $2n$ elements, we can take $\{e_1, \ldots, e_n\} \subset \operatorname{ext}(B_X)$ linearly independent and $\mathrm{e}_{n+1} \in \operatorname{ext}(B_X) \setminus \{\pm e_j : j = 1, \ldots, n\}$.
  Consider Y Banach space with

  $$B_Y = \operatorname{conv}\left(\operatorname{ext}(B_X) \cup \{\pm(1+\varepsilon)e_{n+1}\}\right)$$

  so that $e_1, \ldots, e_n \operatorname{ext}(B_Y)$.
  The natural inclusion $G\colon X \hookrightarrow Y$ is not generating as $\|Ge_{n+1}\|_Y < 1$ and $G \notin \operatorname{conv}(\operatorname{Gen}(X,Y))$. □

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