How small can be the set of norm-attaining functionals of a Banach space?

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II Encuentro de la Red de Análisis Matemático y Aplicaciones
Cullera, Valencia, Marzo de 2025

MINISTERIO DE CIENCIA, INNOVACIÓN Y UNIVERSIDADES
Cofinanciado por la Unión Europea
AGENCIA ESTATAL DE INVESTIGACIÓN
Junta de Andalucía
PID2021-122126NB-C31

Reference and objective

The talk is mainly based on the paper

M. Martín.
A Banach space whose set of norm attaining functionals is algebraically trivial
J. Funct. Anal. 288 (2025), 110815

The objective of the talk

To construct a Banach space for which its set of norm attaining functionals does not contain non-trivial (convex) cones

The Roadmap of the talk

1 Preliminaries
- Notation
- Proximinality and norm attainment
- Read's construction and Read norms
- Comments on the construction of KLMW 2020

2 The new construction

3 Remarks and questions

Preliminaries

Notation

X, Y **real** Banach spaces

- $B_X = \{x \in X \, : \, \|x\| \leqslant 1\}$ closed unit ball of X,
- $S_X = \{x \in X \, : \, \|x\| = 1\}$ unit sphere of X,
- $\mathcal{L}(X, Y)$ bounded linear operators from X to Y,
  - $\|T\| = \sup\{\|T(x)\| \, : \, x \in S_X\}$ for $T \in \mathcal{L}(X, Y)$,
- if $Y = \mathbb{R}$, $X^* = \mathcal{L}(X, Y)$ topological dual of X.

Proximinality

Proximinality

$M \leqslant X$ (closed) subspace is proximinal if

$$\operatorname{dist}(x, M) = \mathbf{min}\Big\{\|x - m\| \colon m \in M\Big\}$$

★ That is, $P_M(x) := \{m \in M \colon d(x, M) = \|x - m\|\} \neq \emptyset$ for every $x \in X$.

★ Equivalently, the quotient map $\pi_M \colon X \longrightarrow X/M$ satisfies that

$$B_{X/M} = \pi_M(B_X)$$

Related to "best approximation"...

- of lists of numbers by least square approximation (Legendre, Gauss, cent. XVIII–XIX),
- of functions by polynomials of fix degree (Chebyshev, cent. XIX)...

Some examples

1. If M is finite-dimensional or even reflexive ✓
2. If M is w^*-closed in a dual space $X = Y^*$ ✓
3. $f \in X^*$, $M = \ker f$ proximinal $\iff$ f attains its norm

Norm attainment

Norm attaining functionals

$f \in X^*$ attains its norm or it is norm attaining if

$$\|f\| = \mathbf{max}\Big\{|f(x)| \colon x \in B_X\Big\}$$

★ Write $\mathrm{NA}(X) := \Big\{f \in X^* \colon f \text{ is norm attaining}\Big\}$

Some important results

- (*Hahn–Banach*) $\mathrm{NA}(X)$ separates the points of X,
- (*James*) X is reflexive if $\mathrm{NA}(X) = X^*$,
- (*Bishop–Phelps*) $\mathrm{NA}(X)$ is norm dense in X^*,
- (*Petunin–Plichko*) Z separable, $X \subset \mathrm{NA}(Z)$ closed subspace separating the points of $Z \implies Z \equiv X^*$.

Some examples of "lineability" in norm attaining functionals

Examples

1. $\mathrm{NA}(c_0) = \left\{x \in \ell_1 \colon \#\operatorname{supp}(x) < \infty\right\}$ is a vector space
2. $\mathrm{NA}(\ell_1) = \left\{x \in \ell_\infty \colon \|x\|_\infty = \max_n\{|x(n)|\}\right\}$ contains c_0
3. Actually, if $X = Z^*$ then $\mathrm{NA}(X) \supseteq J_Z(Z)$
4. $\mathrm{NA}(L_1[0,1]) = \left\{f \in L_\infty[0,1] \colon \lambda(\{t \colon |f(t)| = \|f\|_\infty\}) > 0\right\}$ contains $\operatorname{span}\{\chi_{A_n} : n \in \mathbb{N}\}$ for any disjoint family of sets with positive measure $\{A_n\}$
5. $\mathrm{NA}(C(K)) = \left\{\mu \in M(K) \colon \mu = f|\mu| \text{ with } f \text{ continuous, } \|f\|_\infty = 1\right\}$ contains $\operatorname{span}\{\delta_{t_n} : n \in \mathbb{N}\}$ for any sequence $\{t_n\}$ of elements of K
6. If X has a monotone basis $\Longrightarrow$ $\mathrm{NA}(X)$ contains an infinite-dimensional subspace

Relationship and two old problems

Relationship

$M \leqslant X$ of finite-codimension:

- M proximal $\implies M^\perp \subset \mathrm{NA}(X)$
- The converse is not true in general (examples in $X = C(K)$ for any K)
- If X/M is strictly convex, $M^\perp \subset \mathrm{NA}(X) \implies M$ proximal

Two old questions

(S) (I. Singer, 1972) Is there always a proximinal subspace of codimension 2?

(G) (G. Godefroy, 2001) Does $\mathrm{NA}(X)$ always contain a linear subspace of dimension 2?

Observation

A positive answer to (S) would give a positive answer to (G)

Answer

The answer to both questions is negative, as shown by Read and Rmoutil:

Read's construction and Rmoutil result

$\mathcal{R}$ is a renorming of c_0:

Enumerate $c_{00}(\mathbb{Q}) = \{u_1, u_2, \dots\}$ so that every element is repeated infinitely often.

Take a sequence of integers (a_n) such that

$$a_k > \max \operatorname{supp} u_k, \quad a_k \geqslant \|u_k\|_{\ell_1}.$$

Renorm c_0 by

$$p(x) = \|x\|_\infty + \sum_{k=1}^{\infty} 2^{-a_k^2} \left|\langle x, u_k - e_{a_k}\rangle\right|$$

where $\|\cdot\|_\infty$ is the usual norm of c_0 and $\langle \cdot, \cdot \rangle$ is the usual duality (c_0, ℓ_1).

$\mathcal{R} = (c_0, p)$ is the Read space

Theorem (Read, 2018, proved by hammer)

$\mathcal{R}$ does not contain proximinal subspaces of finite codimension greater than or equal to two.

Theorem (Rmoutil, 2017)

$\mathrm{NA}(\mathcal{R})$ does not contain two dimensional subspaces.

Other "Read norms"

Read norm

We say that a complete norm p on X is a Read norm if $\mathrm{NA}(X,p)$ contains no two-dimensional subspace (hence (X,p) contains no proximinal subspaces of finite codimension greater than one).

Theorem (Kadets–López–Martín–Werner, 2020)

Every closed subspace X of ℓ_∞ containing an isomorphic copy of c_0 can be renormed with a Read norm.

Indeed, we may find $\{v_n^*\} \subset B_{X^*}$ such that the norm

$$p(x) = \|x\|_\infty + \rho \sum_{n=1}^{\infty} \frac{1}{2^n} v_n^*(x) \qquad \big(x \in X\big)$$

is an $(1+\rho)$-equivalent Read norm.

- (X,p) is strictly convex
- If X^* is separable, $(X,p)^{**}$ is strictly convex

Read norm and the norms of the further examples

★ ALL those norms are of the form

$$p(x) = \|x\| + \rho \sum_{n=1}^{\infty} \frac{1}{2^n} |v_n^*(x)| = \|x\| + \|R(x)\|_1 \qquad (x \in X)$$

where

- $\|\cdot\|$ is the original norm of X,
- $R\colon X \longrightarrow \ell_1$ is compact and one-to-one,
- it is very important that the range of R is ℓ_1.

★ All these norms are NOT smooth, hence $\mathrm{NA}(X, p)$ contains non-trivial cones.

★ We may construct *a posteriori* smooth Read norms, but they also satisfy that their sets of norm attaining functionals contain non-trivial cones!!

The construction of KLMW 2020: A tentative calculation

$p(x) = \|x\| + \sum 2^{-n}|v_n^*(x)|$. Then $B_{(X^*,p^*)} = B_{X^*} + \sum 2^{-n}[-v_n^*, v_n^*]$ (Minkowski sum)

Let $x^* \in \mathrm{NA}(X,p)$, $\|x^*\| = 1$, be norm attaining at x; then

$$x^* = x_0^* + \sum 2^{-n} t_n v_n^*$$

for some $x_0^* \in \mathrm{NA}(X)$, $\|x_0^*\| = 1$, and $t_n = \operatorname{sign} v_n^*(x)$ whenever $v_n^*(x)$ is nonzero.
Write the same decomposition for $y^* \in \mathrm{NA}(X,p)$, $\|y^*\| = 1$, norm attaining at y:

$$y^* = y_0^* + \sum 2^{-n} t_n' v_n^*.$$

Let's try to prove that $x^* + y^* \notin \mathrm{NA}(X,p)$: Otherwise we would have a similar decomposition for $z^* = (x^* + y^*)/\|x^* + y^*\|$:

$$z^* = z_0^* + \sum 2^{-n} s_n v_n^*.$$

Sort the items, setting $\lambda = \|x^* + y^*\|$:

$$0 = x^* + y^* - \lambda z^* = [x_0^* + y_0^* - \lambda z_0^*] + \Big[\sum (t_n + t_n' - \lambda s_n) v_n^*\Big]$$

The construction of KLMW 2020: Wish list

$$0 = [x_0^* + y_0^* - \lambda z_0^*] + \left[\sum (t_n + t_n' - \lambda s_n) v_n^*\right]$$

We now **wish** to select the v_n^* to be sort of "orthogonal" to $\mathrm{span}(\mathrm{NA}(X))$ (which contains the first bracket) so that both brackets vanish.

In addition we **wish** the v_n^* to have some Schauder basis character so that we can deduce from $\sum (t_n + t_n' - \lambda s_n) v_n^* = 0$ that all $t_n + t_n' - \lambda s_n = 0$.

Finally we **wish** the support points x and y to be distinct, and we **wish** the span of the v_n^* to be dense enough to separate x and y for many n, i.e., $v_n^*(x) < 0 < v_n^*(y)$ and thus $t_n + t_n' = 0$ fairly often, while at the same time $s_n \neq 0$ for at least one of those n.

This contradiction would show that $x^* + y^* \notin \mathrm{NA}(X, p)$.

If $x = y$, then $x \neq -y$, getting $x^* - y^* \notin \mathrm{NA}(X, p)$.

The new construction

The main result

Theorem

There is a Banach space $\mathfrak{X}$ such that $\mathrm{NA}(\mathfrak{X})$ contains no non-trivial cones.

★ Actually:

- Given $f, g \in \mathrm{NA}(\mathfrak{X})$ linearly independent and $0 < t < 1$,

$$t\,f + (1-t)\,g \notin \mathrm{NA}(\mathfrak{X}).$$

- Equivalently, if $E \leqslant \mathfrak{X}^*$ has dimension two, then $E \cap \mathrm{NA}(\mathfrak{X})$ is contained in the union of two lines.

The first tool: smooth version of c_0 with small NA set

Theorem (Debs–Godefroy–Saint-Raymond, 1995)

X separable, $S\colon \ell_2 \longrightarrow X$ with dense range. We define an equivalent norm $|\cdot|$ on X such that

$$B_{(X,|\cdot|)} = B_X + S(B_{\ell_2}).$$

Then

- $(X, |\cdot|)$ *is smooth,*
- $\mathrm{NA}\big((X, |\cdot|)\big) = \mathrm{NA}(X)$.

The smooth version of c_0 with small NA set

There is a **smooth** norm $|\!|\!|\cdot|\!|\!|$ on c_0 such that

$$\mathrm{NA}\big((c_0, |\!|\!|\cdot|\!|\!|)\big) = \big\{x \in \ell_1 \colon \#\,\mathrm{supp}(x) < \infty\big\}.$$

The second tool: using operator ranges

Operator ranges

X Banach space, Y linear subspace of X. Y is an operator range if there exists Z infinite-dimensional Banach space, $T\colon Z \longrightarrow X$ one-to-one continuous with $T(Z) = Y$.

★ Equivalently, there is a complete norm $\|\cdot\|_1$ on Y dominating $\|\cdot\|$.

Example (Important example in ℓ_1 (KLMW, 2020))

Exists $Y \subset \ell_1$ dense operator range such that

$$\#\big(\mathbb{N} \setminus \operatorname{supp}(y)\big) < \infty \qquad \big(y \in Y \setminus \{0\}\big)$$

Theorem (Main property for us (adapting KLMW, 2020))

Y separable operator range. Exists $T\colon \ell_1(\mathbb{N}\times\mathbb{N}) \longrightarrow Y$ ***one-to-one****, $\|T\| = 1$, with*

$$\overline{\left\{\frac{T(e_{n,m})}{\|T(e_{n,m})\|} : n \in \mathbb{N}\right\}} = S_Y \qquad (m \in \mathbb{N}).$$

Mixing the first and second tools

Lemma A.
The dense sequence with "basis character" which is orthogonal to NA

There is a **smooth** equivalent norm $|||\cdot|||$ on c_0 and a **one-to-one** norm-one operator $T\colon \ell_1(\mathbb{N}\times\mathbb{N}) \longrightarrow (c_0, |||\cdot|||)^*$ such that

- $\big(\operatorname{span} \mathrm{NA}(c_0, |||\cdot|||)\big) \cap T\big(\ell_1(\mathbb{N}\times\mathbb{N})\big) = \{0\}$,
- Writing $v^*_{n,m} := T(e_{n,m})$, we have that $\left\{\frac{v^*_{n,m}}{|||v^*_{n,m}|||} : n \in \mathbb{N}\right\}$ is dense in $S_{(c_0,|||\cdot|||)^*}$ for every $m \in \mathbb{N}$.
- If $(\alpha_{n,m})$ is absolutely sumable, then

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \alpha_{n,m}\, v^*_{n,m} = T\left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \alpha_{n,m}\, e_{n,m}\right).$$

In particular,

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \alpha_{n,m}\, v^*_{n,m} = 0 \quad\Longrightarrow\quad \alpha_{m,m} = 0 \ \forall n,m \in \mathbb{N}.$$

Smooth versions of ℓ_1 whose NA sets behave similar to ℓ_1

Lemma B.
The smooth versions of ℓ_1

Fix $m \in \mathbb{N}$. Write $S_m \colon \ell_2 \longrightarrow \ell_1$

$$[S_m(a)](n) = \frac{1}{2^m}\frac{1}{2^n}\|v_{n,m}^*\|a(n) \qquad (n \in \mathbb{N}).$$

Consider the equivalent norm $|\cdot|_m$ in ℓ_1 such that

$$B_{(\ell_1,|\cdot|_m)} = B_{\ell_1} + S_m(B_{\ell_2}).$$

Then:

1. $|f|_m^* = \|f\|_\infty + \|S_m^*(f)\|_2$ for every $f \in (\ell_1, |\cdot|_m)^*$;
2. $(\ell_1, |\cdot|_m)^*$ is strictly convex, so $(\ell_1, |\cdot|_m)$ is **smooth**;
3. if $x \in (\ell_1, |\cdot|_m)$, $f \in (\ell_1, |\cdot|_m)^*$, and $n \in \mathbb{N}$ satisfy that

$$|f|_m^* = 1, \quad \langle f, x\rangle = |x|_m, \quad |x(n)| > \frac{1}{2^m}\frac{1}{2^n}\|v_{n,m}^*\||x|_m,$$

then

$$f(n) = \operatorname{sign}(x(n))\|f\|_\infty \quad \text{and} \quad \|f\|_\infty \in \left[1 - \frac{1}{2^m}, 1\right].$$

Calculating the set of norm-attaining functionals when adding two norms

Lemma C.
Calculating the set NA when adding two norms

X, Y Banach spaces, $R\colon X \longrightarrow Y$, define

$$q(x) = \|x\|_X + \|Rx\|_Y \qquad (x \in X).$$

Then:

1. $B_{(X,q)^*} = B_{(X,\|\cdot\|)^*} + R^*(B_{Y^*})$.
2. If $x \in X$ and $x^* \in X^*$ satisfy

$$q(x^*) = 1, \quad x^*(x) = q(x),$$

then $x^* = x_0^* + R^*(y^*)$ where

$$x_0^* \in S_{(X,\|\cdot\|)^*},\ y^* \in S_{Y^*},$$

$$x_0^*(x) = \|x\|_X \ \text{ and } \ [R^* y^*](x) = y^*(Rx) = \|Rx\|_Y$$

The operator "R"

★ Define $R_m \colon (c_0, \|\!| \cdot |\!\|) \longrightarrow (\ell_1, |\cdot|_m)$ by

$$[R_m(x)](n) = \frac{m}{2^m}\frac{1}{2^n} v^*_{n,m}(x) \qquad \big(n \in \mathbb{N},\ x \in X\big).$$

Then:

1. R_m is one-to-one since $\{v^*_{n,m} \colon n \in \mathbb{N}\}$ separates the points of $(X, \|\!| \cdot |\!\|)$.
2. $|R_m(x)|_m \leqslant \|R_m(x)\|_1 \leqslant \frac{m}{2^m} \|\!|x|\!\|$ for every $x \in X$.

★ Define $R \colon (c_0, \|\!| \cdot |\!\|) \longrightarrow W := \left[\bigoplus_{m\in\mathbb{N}} (\ell_1, |\cdot|_m)\right]_{\ell_1}$ by

$$R(x) = \big(R_m(x)\big)_{m\in\mathbb{N}} \qquad (x \in X).$$

1. For $x \in X$, $\|R(x)\|_W = \sum_{m=1}^{\infty} |R_m(x)|_m \leqslant \left(\sum_{m=1}^{\infty} \frac{m}{2^m}\right) \|\!|x|\!\|$.
2. If $x \neq 0$, ALL coordinates of $R(x)$ are non zero.
3. Hence, the function $x \longmapsto \|R(x)\|_W$ is **smooth** at $c_0 \setminus \{0\}$.

The operator "R" II

★ $R_m\colon (c_0, \|\!| \cdot \|\!|) \longrightarrow (\ell_1, |\cdot|_m)$

$$[R_m(x)](n) = \frac{m}{2^m}\frac{1}{2^n} v^*_{n,m}(x) \qquad \big(n \in \mathbb{N},\ x \in X\big).$$

★ $R\colon (c_0, \|\!| \cdot \|\!|) \longrightarrow W := \left[\bigoplus_{m\in\mathbb{N}} (\ell_1, |\cdot|_m)\right]_{\ell_1}$

$$R(x) = \big(R_m(x)\big)_{m\in\mathbb{N}} \qquad (x \in X).$$

★ Let us calculate R^*:

Given $(w^*_m)_{m\in\mathbb{N}} \in W^* = \left[\bigoplus_{m\in\mathbb{N}} (\ell_1, |\cdot|_m)^*\right]_{\ell_\infty}$

$$R^*\big((w^*_m)\big) = T\left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} w^*_m(n)\frac{m}{2^m}\frac{1}{2^n}\, e_{n,m}\right) \in \boldsymbol{T(\ell_1(}\mathbb{N}\times\mathbb{N}\boldsymbol{))}\ .$$

Besides,

$$R^*\big((w^*_m)\big) = 0 \quad\Longrightarrow\quad w^*_m(n) = 0\ \forall n, m \in \mathbb{N}.$$

Defining the norm

★ For $x \in c_0$, define
$$p(x) := \|x\| + \|R(x)\|_W$$
and write $\mathfrak{X} := (c_0, p)$. $\mathfrak{X}$ **is smooth**.

★ Consider $x \in \mathfrak{X}$, $x^* \in \mathfrak{X}^*$, $p(x) = p(x^*) = x^*(x) = 1$, then
$$x^* = x_0^* + R^*(w^*)$$
with $x_0^* \in \mathrm{NA}(c_0, \|\cdot\|)$ with $\|x_0^*\| = 1$, $w^* = (w_m^*) \in S_{W^*}$ satisfying
$$x_0^*(x) = \|x\|$$
$$\langle R^*((w_m^*)), x\rangle = \langle (w_m^*), Rx\rangle = \sum_{m=1}^{\infty} w_m^*(R_m x) = \|Rx\|_W = \sum_{m=1}^{\infty} |R_m x|_m.$$
Since $R_m(x) \neq 0$, $|w_m^*|_m^* = 1$ and $w_m^*(R_m x) = |R_m x|_m$ for all $m \in \mathbb{N}$.

The main part of the proof

★ Pick $x^*, z^* \in \mathrm{NA}(\mathfrak{X}, \mathbb{R})$ with $p(x^*) = p(z^*) = 1$ which are linearly independent and that attain their norms at $x, z \in S_{\mathfrak{X}}$, respectively.

★ Hence, x, z **are linearly independent** as the norm of $\mathfrak{X}$ is smooth.

★ Suppose, for the sake of getting a contradiction, that $u^* := tx^* + (1-t)z^* \in \mathrm{NA}(\mathfrak{X}, \mathbb{R})$ for some $0 < t < 1$ and that it attains its norm at $u \in S_{\mathfrak{X}}$.

★ There are x_0^*, z_0^*, u_0^* in $S_{(X,\|\cdot\|)^*} \cap \mathrm{NA}(X, \|\cdot\|)$, sequences $\Xi = (\xi_m^*)_{m\in\mathbb{N}}$, $\Theta = (\zeta_m^*)_{m\in\mathbb{N}}$, $\Upsilon = (u_m^*)_{m\in\mathbb{N}}$ in S_{W^*} satisfying

$$x^* = x_0^* + R^*(\Xi), \quad z^* = z_0^* + R^*(\Theta), \quad \frac{tx^* + (1-t)z^*}{p(tx^* + (1-t)z^*)} = u_0^* + R^*(\Upsilon),$$

and

$$\xi_m^*(R_m x) = |R_m x|_m, \quad \zeta_m^*(R_m z) = |R_m z|_m, \quad u_m^*(R_m u) = |R_m u|_m.$$

The main part of the proof II

★ There are x_0^*, z_0^*, u_0^* in $S_{(X,\|\cdot\|)^*} \cap \mathrm{NA}(X, \|\cdot\|)$, sequences $\Xi = (\xi_m^*)_{m\in\mathbb{N}}$, $\Theta = (\zeta_m^*)_{m\in\mathbb{N}}$, $\Upsilon = (u_m^*)_{m\in\mathbb{N}}$ in S_{W^*} satisfying

$$x^* = x_0^* + R^*(\Xi), \quad z^* = z_0^* + R^*(\Theta), \quad \frac{tx^* + (1-t)z^*}{p(tx^* + (1-t)z^*)} = u_0^* + R^*(\Upsilon),$$

and

$$\xi_m^*(R_m x) = |R_m x|_m, \quad \zeta_m^*(R_m z) = |R_m z|_m, \quad u_m^*(R_m u) = |R_m u|_m.$$

★ We have that

$$t\,x_0^* + (1-t)z_0^* - p(tx^* + (1-t)z^*)u_0^* = -R^*\Big(t\,\Xi + (1-t)\Theta - p(tx^* + (1-t)z^*)\Upsilon\Big)$$

Therefore,

$$R^*\Big(t\,\Xi + (1-t)\Theta - p(tx^* + (1-t)z^*)\Upsilon\Big) = 0,$$

hence

$$t\,\xi_m^*(n) + (1-t)\zeta_m^*(n) - p(tx^* + (1-t)z^*)u_m^*(n) = 0 \qquad (n, m \in \mathbb{N}).$$

Getting the contradiction

$$t\,\xi_m^*(n) + (1-t)\zeta_m^*(n) = p(tx^* + (1-t)z^*)u_m^*(n) \qquad (n, m \in \mathbb{N}).$$

★ $p(tx^* + (1-t)z^*) < 1$ (otherwise, $x^*(u) = z^*(u) = 1$)

★ Find $\delta > 0$ and $\phi \in (c_0, \|\cdot\|)^*$ with $\|\phi\| = 1$, $\phi(x) > \delta$, $\phi(y) > \delta$.

★ Consider $m \in \mathbb{N}$ such that $\frac{1}{m} < \delta$ and $p(tx^* + (1-t)z^*) < 1 - \frac{1}{2^m}$

★ As $\left\{ \frac{v_{n,m}^*}{\|v_{n,m}^*\|} : n \in \mathbb{N} \right\}$ is dense in $S_{(c_0,\|\cdot\|)^*}$, we may find $n \in \mathbb{N}$ such that

$$v_{n,m}^*(x) > \delta \|v_{n,m}^*\| \quad \text{and} \quad v_{n,m}^*(z) > \delta \|v_{n,m}^*\|.$$

Hence, $\begin{cases} [R_m(x)](n) = \frac{m}{2^m}\frac{1}{2^n} v_{n,m}^*(x) \geqslant \cdots \geqslant \frac{1}{2^m}\frac{1}{2^n}\|v_{n,m}^*\| |R_m x|_m \\ [R_m(z)](n) = \frac{m}{2^m}\frac{1}{2^n} v_{n,m}^*(z) \geqslant \cdots \geqslant \frac{1}{2^m}\frac{1}{2^n}\|v_{n,m}^*\| |R_m z|_m \end{cases}$

So we get that

$$\xi_m^*(n) = \|\xi_m^*\|_\infty \in \left[1 - \frac{1}{2^m}, 1\right] \qquad \zeta_m^*(n) = \|\zeta_m^*\|_\infty \in \left[1 - \frac{1}{2^m}, 1\right]$$

A clear contradiction!!

Remarks and questions

Some geometric properties of $\mathfrak{X}$

Minimal proximinal behaviour of two-codimensional subspaces of $\mathfrak{X}$

$M \leqslant \mathfrak{X}$ of codimension two. Then

- $\pi_M(B_{\mathfrak{X}}) \cap S_{\mathfrak{X}/M}$ contains, at most, two distinct elements and their opposites.
- There are $x_1, x_2 \in \mathfrak{X}$ such that

$$\Big\{x \in \mathfrak{X} \colon P_M(x) \neq \emptyset\Big\} \subseteq \Big(\mathbb{R}x_1 + M\Big) \cup \Big(\mathbb{R}x_2 + M\Big)$$

Some properties of the norm

- $\mathfrak{X}^{**}$ is strictly convex,
- hence $\mathfrak{X}^*$ is smooth and $\mathfrak{X}$ is strictly convex (and smooth)

Extensions of the renorming

Other spaces

The process can be repeated in every separable Banach spaces containing an isomorphic copy of c_0 (e.g. in $C[0,1]$).

Limitation (KLMW, 2020)

If Z has the RNP, then $\mathrm{NA}(Z)$ contains the $\mathbb{Q}$ linear span of an $\mathbb{R}$-linearly independent infinite set.

A question

Does $\mathfrak{X}$ satisfies the analogous property for three (or more!) linearly independent norm attaining functionals**?**

★ That is, given $f, g, h \in \mathrm{NA}(\mathfrak{X})$ linearly independent,

$$t_1\, f + t_2\, g + t_3\, h \notin \mathrm{NA}(\mathfrak{X}) \quad \text{when } t_1, t_2, t_3 > 0\textbf{?}$$

Two results of Veselý, 2009

- For every Z, $\mathrm{NA}(Z) \cap S_{Z^*}$ is $\mathfrak{c}$-dense in S_{Z^*}.
- If Z admits an equivalent F-differentiable norm (e.g. $Z \simeq c_0$), then $\mathrm{NA}(Z) \cap S_{Z^*}$ is pathwise connected and locally pathwise connected.

Relation with norm attaining finite-rank operators

X, Y Banach spaces,
$\mathrm{NA}(X,Y) := \big\{T \in \mathcal{L}(X,Y)\colon \|T\| = \max\{\|Tx\|\colon x \in S_X\}\big\}$.

Open problem since the 1970's (actually, since the 1960's)

Is $\mathrm{NA}(X,\ell_2^2)$ dense in $\mathcal{L}(X,\ell_2^2)$ for all Xs**?**

★ Actually, is there a Banach space X such that $\mathrm{NA}(X,\ell_2^2)$ does not contain rank-two operators**?**

How to get rank-two norm attaining operators from X into ℓ_2^2?

- (obvious) If X contains a two-codimensional proximinal subspace
- (folklore) If $\mathrm{NA}(X)$ contains a two-dimensional subspace
- (KLMW, 2020) If $\mathrm{NA}(X)$ contains a non-trivial cone

Hence, by now, $\mathfrak{X}$ is the only possible counterexample to the second question

Open problem

Is there any rank-two operator in $\mathrm{NA}(\mathfrak{X},\ell_2^2)$**?**

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