A Banach space whose set of norm attaining functionals is algebraically trivial

The 20th ILJU School of Mathematics – Banach Spaces and Related Topics

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Seoul, Republic of Korea, February 2025

Notation

X, Y real or complex Banach spaces

- $\mathbb{K}$ base field $\mathbb{R}$ or $\mathbb{C}$,
- $B_X = \{x \in X : \|x\| \leqslant 1\}$ closed unit ball of X,
- $S_X = \{x \in X : \|x\| = 1\}$ unit sphere of X,
- $\mathcal{L}(X, Y)$ bounded linear operators from X to Y,
  - $\|T\| = \sup\{\|T(x)\| : x \in S_X\}$ for $T \in \mathcal{L}(X, Y)$,
- $\mathcal{W}(X, Y)$ weakly compact linear operators from X to Y,
- $\mathcal{K}(X, Y)$ compact linear operators from X to Y,
- $\mathcal{FR}(X, Y)$ bounded linear operators from X to Y with finite rank,
- if $Y = \mathbb{K}$, $X^* = \mathcal{L}(X, Y)$ topological dual of X,

Observe that

$$\mathcal{FR}(X, Y) \subset \mathcal{K}(X, Y) \subset \mathcal{W}(X, Y) \subset \mathcal{L}(X, Y).$$

The talks will be mainly based on the paper

M. Martín.
A Banach space whose set of norm attaining functionals is algebraically trivial
J. Funct. Anal. 288 (2025), 110815

Proximinality

$M \leqslant X$ is proximinal if

$$\operatorname{dist}(x, M) = \mathbf{min}\Big\{\|x - m\| : m \in M\Big\}$$

Norm attaining functionals

$$\mathrm{NA}(X) = \Big\{f \in X^* : \|f\| = |f(x)| \text{ for some } x \in S_X\Big\}$$

Relationship

$M \leqslant X$ of finite-codimension:

- M proximinal $\implies$ $M^\perp \subset \mathrm{NA}(X)$
- The converse is not true in general
- If X/M is strictly convex, $M^\perp \subset \mathrm{NA}(X) \implies M$ proximinal

Two old questions (answered nowadays)

(S) (I. Singer, 1974) Is there always a proximinal subspace of codimension 2?

(G) (G. Godefroy, 2001) Does $\mathrm{NA}(X)$ always contain a linear subspace of dimension 2?

Read's construction and Rmoutil result

$\mathcal{R}$ is a renorming of c_0:

Enumerate $c_{00}(\mathbb{Q}) = \{u_1, u_2, \dots\}$ so that every element is repeated infinitely often.

Take a sequence of integers (a_n) such that

$$a_k > \max \operatorname{supp} u_k, \quad a_k \geqslant \|u_k\|_{\ell_1}.$$

Renorm c_0 by

$$p(x) = \|x\|_\infty + \sum_{k=1}^{\infty} 2^{-a_k^2} \big| \langle x, u_k - e_{a_k} \rangle \big|$$

where $\|\cdot\|_\infty$ is the usual norm of c_0 and $\langle \cdot, \cdot \rangle$ is the usual duality (c_0, ℓ_1).

$\mathcal{R} = (c_0, p)$ is the Read space

Theorem (Read, 2018, proved by hammer)

$\mathcal{R}$ does not contain proximinal subspaces of finite codimension greater than or equal to two.

Theorem (Rmoutil, 2017)

$\mathrm{NA}(\mathcal{R})$ *does not contain two dimensional subspaces.*

Other "Read norms"

We say that a complete norm p on X is a Read norm if $\mathrm{NA}(X, p)$ contains no two-dimensional subspace (hence (X, p) contains no proximinal subspaces of finite codimension greater than one).

Theorem (Kadets–López–Martín–Werner, 2020)

Every closed subspace X of ℓ_∞ containing an isomorphic copy of c_0 can be renormed with a Read norm.

Indeed, we may find $\{v_n^*\} \subset B_{X^*}$ such that the norm

$$p(x) = \|x\|_\infty + \rho \sum_{n=1}^{\infty} \frac{1}{2^n} v_n^*(x) \qquad \big(x \in X\big)$$

is an $(1+\rho)$-equivalent Read norm.

- *(X, p) is strictly convex*
- *If X^* is separable, $(X, p)^{**}$ is strictly convex*

Norm attaining operators I (All operators)

X, Y Banach spaces,
$\mathrm{NA}(X,Y) = \big\{T \in \mathcal{L}(X,Y) \colon \|T\| = \max\{\|Tx\| \colon x \in S_X\}\big\}.$

When is $\mathrm{NA}(X,Y)$ dense in $\mathcal{L}(X,Y)$?

- $Y = \mathbb{K}$ ✓ (Bishop–Phelps)
- $X = L_1[0,l]$ or $X = C(K)$ or $X = c_0$ ✗ (Lindenstrauss)
- X reflexive ✓ (Lindenstrauss), X RNP ✓ (Bourgain)
- X up to renorming in the separable case ✓ (Schachermayer)
- $c_0 \subset Y \subset \ell_\infty$ canonical copies ✓ (Lindenstrauss)
- Y finite-dimensional polyhedral ✓ (Lindenstrauss)
- $Y = \ell_p$ $1 < p < \infty$ ✗ (Gowers)
- $Y = \ell_1$ or Y strictly convex infinite-dimensional ✗ (Acosta)
- Y infinite ℓ_p-sum of finite-dimensional spaces ✗ (Fovelle)
- Y up to renorming ✓ (Partington)
- Y polyhedral with $Y^* \equiv \ell_1$ or $Y \leqslant c_0(\Gamma)$ ✓ (Jung–Martín–Rueda)

Norm attaining operators II (Compact operators)

X, Y Banach spaces,
$\mathrm{NA}(X,Y) = \big\{T \in \mathcal{L}(X,Y) \colon \|T\| = \max\{\|Tx\| \colon x \in S_X\}\big\}$.

When is $\mathrm{NA}(X,Y) \cap \mathcal{K}(X,Y)$ dense in $\mathcal{K}(X,Y)$?

★ Very often:

- All the positive cases in the previous slide: X RNP, $c_0 \subset Y \subset \ell_\infty$ canonical copies, up to renorming for X separable or for Y arbitrary ✓ (adapting proofs)
- $X = C(K)$ or $X = L_1(\mu)$ or $Y^* \equiv L_1(\mu)$ or $Y = L_1(\mu)$ ✓ (Johnson–Wolfe)
- If X^* has the metric π-property with w^*-projections ✓ (Johnson–Wolfe)
- Y polyhedral with the AP ✓ (Johnson–Wolfe)
- Y uniform algebra ✓ (Cascales–Guirao–Kadets)
- $X \leqslant c_0$ with monotone basis or $X^* \equiv \ell_1$ ✓ (Martín)

★ But not always:

- Exists $X \leqslant c_0$, $Y \leqslant \ell_p$ ✗ (Martín)
- X can be taken with basis (but X^* fails the AP) ✗ (Martín)

Norm attaining operators III (Finite-rank operators)

X, Y Banach spaces,
$\mathrm{NA}(X,Y) = \big\{T \in \mathcal{L}(X,Y) \colon \|T\| = \max\{\|Tx\| \colon x \in S_X\}\big\}$.

When is $\mathcal{F}(X,Y)$ contained in $\overline{\mathrm{NA}(X,Y) \cap \mathcal{F}(X,Y)}$ **?**

★ Very often:

- All the positive cases in the two previous slides: X RNP, $c_0 \subset Y \subset \ell_\infty$ canonical copies, up to renorming for X separable or for Y arbitrary, $X = C(K)$, $X = L_1(\mu)$, $Y^* \equiv L_1(\mu)$, $Y = L_1(\mu)$, Y polyhedral... ✓ (adapting proofs)
- If $\mathrm{NA}(X)$ contains a dense linear subspace ✓ (Kadets–López–Martín–Werner)
  - This covers all the previous results with some strong form of the AP in the dual.
  - But also gives: X proximinal subspace of c_0 or $X = \mathcal{K}(\ell_2, \ell_2)$ or X c_0-sum of reflexive spaces (maybe without the AP)

★ We are not aware of any negative example!

★ Is there X such that $\mathrm{NA}(X, \ell_2^2)$ consists of rank-one operators **?**

★ How can we get rank-two operators in $\mathrm{NA}(X, \ell_2^2)$ **?**

Read norm and the norms of the further examples

★ ALL those norms are of the form

$$p(x) = \|x\| + \sum_{n=1}^{\infty} \rho_n |v_n^*(x)| = \|x\| + \|R(x)\|_1 \qquad (x \in X)$$

where

- $\|\cdot\|$ is the original norm of X,
- $R\colon X \longrightarrow \ell_1$ is compact and one-to-one,
- it is very important that the range of R is ℓ_1.

★ We may construct *a posteriori* smooth Read norms, but they also satisfy that their sets of norm attaining functionals contain non-trivial cones!!

Comments on the construction of KLMW 2020

The construction of KLMW 2020: A tentative calculation

$p(x) = \|x\| + \sum 2^{-n}|v_n^*(x)|$. Then $B_{(X^*,p^*)} = B_{X^*} + \sum 2^{-n}[-v_n^*, v_n^*]$ (Minkowski sum)

Let $x^* \in \mathrm{NA}(X,p)$, $\|x^*\| = 1$, be norm attaining at x; then

$$x^* = x_0^* + \sum 2^{-n} t_n v_n^*$$

for some $x_0^* \in \mathrm{NA}(X)$, $\|x_0^*\| = 1$, and $t_n = \operatorname{sign} v_n^*(x)$ whenever $v_n^*(x)$ is nonzero.
Write the same decomposition for
$y^* \in \mathrm{NA}(X,p)$, $\|y^*\| = 1$, norm attaining at y:

$$y^* = y_0^* + \sum 2^{-n} t'_n v_n^*.$$

Let's try to prove that $x^* + y^* \notin \mathrm{NA}(X,p)$: Otherwise we would have a similar decomposition for $z^* = (x^* + y^*)/\|x^* + y^*\|$:

$$z^* = z_0^* + \sum 2^{-n} s_n v_n^*.$$

Sort the items, setting $\lambda = \|x^* + y^*\|$:

$$0 = x^* + y^* - \lambda z^* = [x_0^* + y_0^* - \lambda z_0^*] + \Big[\sum (t_n + t'_n - \lambda s_n) v_n^*\Big]$$

The construction of KLMW 2020: Wish list

$$0 = [x_0^* + y_0^* - \lambda z_0^*] + \left[\sum (t_n + t_n' - \lambda s_n) v_n^*\right]$$

We now **wish** to select the v_n^* to be sort of "orthogonal" to $\mathrm{span}(\mathrm{NA}(X))$ (which contains the first bracket) so that both brackets vanish.

In addition we **wish** the v_n^* to have some Schauder basis character so that we can deduce from $\sum (t_n + t_n' - \lambda s_n) v_n^* = 0$ that all $t_n + t_n' - \lambda s_n = 0$.

Finally we **wish** the support points x and y to be distinct, and we **wish** the span of the v_n^* to be dense enough to separate x and y for many n, i.e., $v_n^*(x) < 0 < v_n^*(y)$ and thus $t_n + t_n' = 0$ fairly often, while at the same time $s_n \neq 0$ for at least one of those n.

This contradiction would show that $x^* + y^* \notin \mathrm{NA}(X, p)$.

If $x = y$, then $x \neq -y$, getting $x^* - y^* \notin \mathrm{NA}(X, p)$.

The new construction

Theorem

There is a Banach space X such that $NA(X)$ contains no non-trivial cones

- Actually, given $f, g \in NA(X)$ linearly independent, $tf + (1-t)g \notin NA(X)$ whenever $0 < t < 1$
- Equivalently, $\forall E \leq X^*$ of dimension two, $E \cap NA(X)$ is contained in the union of two lines

The needed tools

Operator ranges

X Banach space, Y linear subspace of X. Y is an operator range if there exists Z infinite-dimensional Banach space, $T\colon Z \longrightarrow X$ one-to-one continuous with $T(Z) = Y$.

★ Equivalently, there is a complete norm $\|\cdot\|_1$ on Y dominating $\|\cdot\|$.

Example (Important example in ℓ_1)

Exists $Y \subset \ell_1$ operator range such that

$$\#\big(\mathbb{N} \setminus \operatorname{supp}(y)\big) < \infty \qquad \big(y \in Y \setminus \{0\}\big)$$

Theorem (Main property for us)

Y separable operator range. Exists $T\colon \ell_1(\mathbb{N}\times\mathbb{N}) \longrightarrow Y$ one-to-one, $\|T\| = 1$, with

$$\overline{\left\{ \frac{T(e_{n,m})}{\|T(e_{n,m})\|} : n \in \mathbb{N} \right\}} = S_Y \qquad (m \in \mathbb{N}).$$

An smooth norm on c_0 with small NA set

Theorem (Debs–Godefroy–Saint-Raymond, 1995)

X separable, $S\colon \ell_2 \longrightarrow X$ with dense range. We define an equivalent norm $|\cdot|$ on X such that

$$B_{(X,|\cdot|)} = B_X + S(B_{\ell_2}).$$

Then

- $(X,|\cdot|)$ *is smooth,*
- $\mathrm{NA}\big((X,|\cdot|)\big) = \mathrm{NA}(X)$.

Theorem (Godefroy–Yahdi–Kaufman, 2001)

There is an equivalent norm $|\cdot|$ on c_0 such that

$$|u|^* = \left(\|u\|_1^2 + \|u\|_2^2\right)^{\frac{1}{2}} \qquad (u \in \ell_1).$$

Then

- $(c_0,|\cdot|)^*$ *is LUR, hence* $(c_0,|\cdot|)$ *is Fréchet smooth,*
- $\mathrm{NA}\big((c_0,|\cdot|)\big) = \big\{u \in \ell_1 \colon \#\operatorname{supp}(u) < \infty\big\}$.

The first tool

The first tool

There is an **smooth** equivalent norm $\|\!|\cdot|\!\|$ on c_0 and a **one-to-one** norm-one operator $T\colon \ell_1(\mathbb{N}\times\mathbb{N}) \longrightarrow (c_0, \|\!|\cdot|\!\|)^*$ such that

- $\big(\operatorname{span}\mathrm{NA}(c_0, \|\!|\cdot|\!\|)\big) \cap T\big(\ell_1(\mathbb{N}\times\mathbb{N})\big) = \{0\}$,
- $\left\{\frac{T(e_{n,m})}{\|\!|T(e_{n,m})|\!\|} : n \in \mathbb{N}\right\}$ is dense in $S_{(c_0,\|\!|\cdot|\!\|)^*}$ for every $m \in \mathbb{N}$.

Calculating the set of norm-attaining functionals when adding two norms

Lemma A

X, Y Banach spaces, $R\colon X \longrightarrow Y$, define

$$q(x) = \|x\|_X + \|Rx\|_Y \qquad (x \in X).$$

Then:

1. $B_{(X,q)^*} = B_{(X,\|\cdot\|)^*} + R^*(B_{Y^*})$.
2. If $x \in X$ and $x^* \in X^*$ satisfy

$$q(x^*) = 1, \;\; x^*(x) = q(x),$$

then $x^* = x_0^* + R^*(y^*)$ where

$$x_0^* \in S_{(X,\|\cdot\|)^*},\; y^* \in S_{Y^*},\; x_0^*(x) = \|x\|_X \;\text{ and }\; y^*(Rx) = \|Rx\|_Y$$

Smooth versions of ℓ_1 whose NA sets behave similar to ℓ_1

Lemma B

Φ sequence in $\mathbb{R}^+$, $\|\Phi\| < 1$, $S_\Phi \colon \ell_2 \longrightarrow \ell_1$, $S_\Phi(a) = \Phi \cdot a$. Consider an equivalent norm $|\cdot|_\Phi$ in ℓ_1 such that

$$B_{(\ell_1, |\cdot|_\Phi)} = B_{\ell_1} + S_\Phi(B_{\ell_2}).$$

Then

- $|f|_\Phi^* = \|f\|_\infty + \|S_\Phi^*(f)\|_2$.
- $(\ell_1, |\cdot|_\Phi)^*$ is strictly convex, so $(\ell_1, |\cdot|_\Phi)$ is smooth.
- $x \in \ell_1$, $f \in \ell_\infty$, $|f|_\Phi^* = 1$, $\langle f, x \rangle = |x|_\Phi$, and $|x(n)| > \Phi(n)|x|_\Phi$

$$\Longrightarrow \begin{cases} f(n) = \operatorname{sign}(x(n))\|f\|_\infty \\ \|f\|_\infty \geqslant 1 - \|\Phi\|_1. \end{cases}$$

Smooth versions of ℓ_1 whose NA sets behave similar to ℓ_1 II

Observation C

$\{Z_m\colon m\in\mathbb{N}\}$ smooth spaces, $Z=\left[\bigoplus_{m\in\mathbb{N}} Z_m\right]_{\ell_1}$, $Z^*=\left[\bigoplus_{m\in\mathbb{N}} Z_m^*\right]_{\ell_\infty}$.

$z=(z_m)\in Z$, $f=(f_m)\in Z^*$ with $\|f\|_\infty=1$, $\langle f,z\rangle=\|z\|_1$,
$\implies f_m(z_m)=\|z_m\|$ for every $m\in\mathbb{N}$, hence $\|f_m\|=1$ when $z_m\neq 0$.

As a consequence, the norm of Z is smooth at every element $z=(z_m)\in X$ such that ALL coordinates z_m are not null.

Proof of The new construction

Theorem

There is a Banach space X such that $NA(X)$ contains no non-trivial cones

- Actually, given $f, g \in NA(X)$ linearly independent, $tf + (1-t)g \notin NA(X)$ whenever $0 < t < 1$

- Equivalently, $\forall E \leq X^*$ of dimension two, $E \cap NA(X)$ is contained in the union of two lines

Before defining the norm

★ Use the first tool to take an smooth equivalent norm $\|\!|\cdot|\!\|$ on c_0 and a **one-to-one** norm-one operator $T\colon \ell_1(\mathbb{N}\times\mathbb{N}) \longrightarrow (c_0, \|\!|\cdot|\!\|)^*$ such that

- $\big(\operatorname{span} \mathrm{NA}(c_0, \|\!|\cdot|\!\|)\big) \cap T\big(\ell_1(\mathbb{N}\times\mathbb{N})\big) = \{0\}$,
- Writing $v^*_{n,m} := T(e_{n,m})$, we have that $\left\{\frac{v^*_{n,m}}{\|\!|v^*_{n,m}|\!\|} : n \in \mathbb{N}\right\}$ is dense in $S_{(c_0,\|\!|\cdot|\!\|)^*}$ for every $m \in \mathbb{N}$.

★ Fix $m \in \mathbb{N}$. Let $|\cdot|_m$ be the equivalent norm on ℓ_1 given by Lemma B for the sequence $\Phi_m(n) = \dfrac{1}{2^m}\dfrac{1}{2^n}\|\!|v^*_{n,m}|\!\|$ for every $n \in \mathbb{N}$. Then:

1. $(\ell_1, |\cdot|_m)^*$ is strictly convex, so $(\ell_1, |\cdot|_m)$ is smooth;
2. for $f \in (\ell_1, |\cdot|_m)^*$, $\|f\|_\infty \leqslant |f|^*_m$;
3. if $x \in (\ell_1, |\cdot|_m)$, $f \in (\ell_1, |\cdot|_m)^*$, and $n \in \mathbb{N}$ satisfy that

$$|f|^*_m = 1, \quad \langle f, x\rangle = |x|_m, \quad |x(n)| > \frac{1}{2^m}\frac{1}{2^n}\|\!|v^*_{n,m}|\!\||x|_m,$$

then

$$f(n) = \operatorname{sign}(x(n))\|f\|_\infty \quad \text{and} \quad \|f\|_\infty \in \left[1 - \frac{1}{2^m}, 1\right].$$

Before defining the norm II

★ Define $R_m\colon (c_0, \|\!|\cdot|\!\|) \longrightarrow (\ell_1, |\cdot|_m)$ by

$$[R_m(x)](n) = \frac{m}{2^m}\frac{1}{2^n} v^*_{n,m}(x) \qquad \big(n \in \mathbb{N},\ x \in X\big).$$

Then:

1. R_m is one-to-one since $\{v^*_{n,m}\colon n \in \mathbb{N}\}$ separates the points of $(X, \|\!|\cdot|\!\|)$.
2. $|R_m(x)|_m \leqslant \|R_m(x)\|_1 \leqslant \frac{m}{2^m} \|\!|x|\!\|$ for every $x \in X$.

★ Define $R\colon (c_0, \|\!|\cdot|\!\|) \longrightarrow W := \left[\bigoplus_{m\in\mathbb{N}} (\ell_1, |\cdot|_m)\right]_{\ell_1}$ by

$$R(x) = \big(R_m(x)\big)_{m\in\mathbb{N}} \qquad (x \in X).$$

1. For $x \in X$, $\|R(x)\|_W = \displaystyle\sum_{m=1}^{\infty} |R_m(x)|_m \leqslant \left(\sum_{m=1}^{\infty} \frac{m}{2^m}\right) \|\!|x|\!\|$.
2. If $x \neq 0$, ALL coordinates of $R(x)$ are non zero.
3. Hence, the function $x \longmapsto \|R(x)\|_W$ is smooth at $c_0 \setminus \{0\}$.

Before defining the norm III

★ $R_m : (c_0, \|\!|\cdot|\!\|) \longrightarrow (\ell_1, |\cdot|_m)$

$$[R_m(x)](n) = \frac{m}{2^m}\,\frac{1}{2^n}\,v^*_{n,m}(x) \qquad \big(n \in \mathbb{N},\ x \in X\big).$$

★ $R\colon (c_0, \|\!|\cdot|\!\|) \longrightarrow W := \left[\bigoplus_{m\in\mathbb{N}} (\ell_1, |\cdot|_m)\right]_{\ell_1}$

$$R(x) = \big(R_m(x)\big)_{m\in\mathbb{N}} \qquad (x \in X).$$

★ Let us calculate R^*:

Given $(w^*_m)_{m\in\mathbb{N}} \in W^* = \left[\bigoplus_{m\in\mathbb{N}} (\ell_1, |\cdot|_m)^*\right]_{\ell_\infty}$

$$R^*\big((w^*_m)\big) = T\left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} w^*_m(n)\,\frac{m}{2^m}\,\frac{1}{2^n}\,e_{n,m}\right) \in \boldsymbol{Y}\ .$$

Besides,

$$R^*\big((w^*_m)\big) = 0 \quad \Longrightarrow \quad w^*_m(n) = 0\ \forall n, m \in \mathbb{N}.$$

Defining the norm

★ For $x \in c_0$, define
$$p(x) := \|x\| + \|R(x)\|_W$$
and write $\mathfrak{X} := (c_0, p)$. $\mathfrak{X}$ **is smooth**.

★ Consider $x \in \mathfrak{X}$, $x^* \in \mathfrak{X}^*$, $p(x) = p(x^*) = x^*(x) = 1$, then
$$x^* = x_0^* + R^*(w^*)$$
with $x_0^* \in \mathrm{NA}(c_0, \|\cdot\|)$ with $\|x_0^*\| = 1$, $w^* = (w_m^*) \in S_{W^*}$ satisfying
$$x_0^*(x) = \|x\|$$
$$\langle R^*((w_m^*)), x\rangle = \langle (w_m^*), Rx\rangle = \sum_{m=1}^{\infty} w_m^*(R_m x) = \|Rx\|_W = \sum_{m=1}^{\infty} |R_m x|_m.$$
Since $R_m(x) \neq 0$, $|w_m^*|_m^* = 1$ and $w_m^*(R_m x) = |R_m x|_m$ for all $m \in \mathbb{N}$.

The main part of the proof

★ Pick $x^*, z^* \in \mathrm{NA}(\mathfrak{X}, \mathbb{R})$ with $p(x^*) = p(z^*) = 1$ which are linearly independent and that attain their norms at $x, z \in S_{\mathfrak{X}}$, respectively.

★ Hence, x, z are linearly independent as the norm of $\mathfrak{X}$ is smooth.

★ Suppose, for the sake of getting a contradiction, that
$u^* := tx^* + (1-t)z^* \in \mathrm{NA}(\mathfrak{X}, \mathbb{R})$ for some $0 < t < 1$
and that it attains its norm at $u \in S_{\mathfrak{X}}$.

★ There are x_0^*, z_0^*, u_0^* in $S_{(X,\|\cdot\|)^*} \cap \mathrm{NA}(X, \|\cdot\|)$,
sequences $\Xi = (\xi_m^*)_{m\in\mathbb{N}}$, $\Theta = (\zeta_m^*)_{m\in\mathbb{N}}$, $\Upsilon = (u_m^*)_{m\in\mathbb{N}}$ in S_{W^*} satisfying

$$x^* = x_0^* + R^*(\Xi), \quad z^* = z_0^* + R^*(\Theta), \quad \frac{tx^* + (1-t)z^*}{p(tx^* + (1-t)z^*)} = u_0^* + R^*(\Upsilon),$$

and

$$\xi_m^*(R_m x) = |R_m x|_m, \quad \zeta_m^*(R_m z) = |R_m z|_m, \quad u_m^*(R_m u) = |R_m u|_m.$$

The main part of the proof II

★ There are x_0^*, z_0^*, u_0^* in $S_{(X,\|\cdot\|)^*} \cap \mathrm{NA}(X, \|\cdot\|)$, sequences $\Xi = (\xi_m^*)_{m\in\mathbb{N}}$, $\Theta = (\zeta_m^*)_{m\in\mathbb{N}}$, $\Upsilon = (u_m^*)_{m\in\mathbb{N}}$ in S_{W^*} satisfying

$$x^* = x_0^* + R^*(\Xi), \quad z^* = z_0^* + R^*(\Theta), \quad \frac{tx^* + (1-t)z^*}{p(tx^* + (1-t)z^*)} = u_0^* + R^*(\Upsilon),$$

and

$$\xi_m^*(R_m x) = |R_m x|_m, \quad \zeta_m^*(R_m z) = |R_m z|_m, \quad u_m^*(R_m u) = |R_m u|_m.$$

★ We have that

$$t\,x_0^* + (1-t)z_0^* - p(tx^* + (1-t)z^*)u_0^* = -R\Big(t\,\Xi + (1-t)\Theta - p(tx^* + (1-t)z^*)\Upsilon\Big)$$

Therefore,

$$R\Big(t\,\Xi + (1-t)\Theta - p(tx^* + (1-t)z^*)\Upsilon\Big) = 0,$$

hence

$$t\,\xi_m^*(n) + (1-t)\zeta_m^*(n) - p(tx^* + (1-t)z^*)u_m^*(n) = 0 \qquad (n, m \in \mathbb{N}).$$

- As x, z are linearly independent $\exists \phi \in S_{(X, \|\cdot\|)^*}$ and $\delta > 0$: $\phi(x) > \delta, \phi(z) > \delta$

Proof $x \notin \mathbb{R}(x-z)$

$\Rightarrow \exists \phi \in S_{(X, \|\cdot\|)^*}$ $\phi(x) = d(x, \mathbb{R}(x-z))$

$\phi(x-z) = 0$

$\Rightarrow \phi(x) = \phi(z) > 0$

- $p(tx^* + (1-t)z^*) < 1$ as $tx^* + (1-t)z^* \in NA(X, p)$

otherwise, $[tx^* + (1-t)z^*](v) = 1$

$\Rightarrow x^*(v) = z^*(v) = 1$!!

- Consider $m \in \mathbb{N}$ such that $\frac{1}{m} < \delta$ and $p(tx^* + (1-t)z^*) < 1 - \frac{1}{2m}$

As $\left\{ \frac{v_{n,m}^*}{|||v_{n,m}^*|||} : n \in \mathbb{N} \right\}$ is dense in $S_{(\mathfrak{X}, |||\cdot|||)^*}$

$\exists n \in \mathbb{N}$ such that

$$v_{n,m}^*(x) > \delta |||v_{n,m}^*||| \text{ and } v_{n,m}^*(z) > \delta |||v_{n,m}^*|||$$

Then,

$$[R_m(x)](n) = \frac{m}{2^m} \frac{1}{2^n} v_{n,m}^*(x) >$$

$$> \frac{m}{2^m} \frac{1}{2^n} \delta |||v_{n,m}^*||| > \frac{1}{2^m} \frac{1}{2^n} |||v_{n,m}^*|||$$

$$\geq \frac{1}{2^m} \frac{1}{2^n} |||v_{n,m}^*||| \, |R_m x|_m$$

$$\Longrightarrow \xi_m^*(n) = \|\xi_m^*\|_\infty \in \left[1 - \frac{1}{2^m}, 1\right]$$

- As $\left\{ \frac{u^*_{n,m}}{|||u^*_{n,m}|||} : n \in \mathbb{N} \right\}$ is dense in $S_{(\mathcal{X}, |||\cdot|||)^*}$

$$\Longrightarrow \xi^*_m(n) = \|\xi^*_m\|_\infty \in \left[1 - \frac{1}{2^m}, 1\right]$$

- Analogously, $\zeta^*_m(n) = \|\zeta^*_m\|_\infty \in [1 - 1/2^m, 1]$

Since

$$\underbrace{t\,\xi^*_m(n) + (1-t)\,\zeta^*_m(n)}_{\in [1-1/2^m, 1]} = \underbrace{p(tx^* + (1-t)z^*)\,u^*_m(n)}_{\text{modulus} < 1 - 1/2^m}$$

We get a contradiction!