Physics Letters A 372 (2008) 6386–6387

Contents lists available at ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

Periodic oscillations of the relativistic pendulum with friction

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ARTICLE INFO

Article history:
Received 22 May 2008
Received in revised form 27 August 2008
Accepted 28 August 2008
Available online 2 September 2008
Communicated by C.R. Doering

MSC:
34C25

Keywords:
Relativistic pendulum
Periodic solution
Schauder fixed point theorem

ABSTRACT

It is proved that the forced pendulum equation with friction and a singular ϕ-Laplacian operator of relativistic type has periodic solutions, in contrast to what happen in the Newtonian case.

1. Introduction and main result

In this Letter we consider the existence of periodic solutions for the forced pendulum equation with relativistic effects

$$\left(\frac{x'}{\sqrt{1-\frac{x'^2}{c^2}}}\right)' + kx' + a\sin x = p(t), \tag{1.1}$$

where $c > 0$ is the speed of light in the vacuum, $k \geqslant 0$ is a possible viscous friction coefficient and p is a continuous and T-periodic forcing term with mean value $\bar{p} = \frac{1}{T}\int_0^T p(t)\,dt = 0$. This equation can be derived from an appropriate Lagrangian formulation [1]. Physically, we are assuming a basic principle of Special Relativity: the mass of a moving object is not constant but depends on its velocity. From a more mathematical perspective, the equation can be seen as a singular ϕ-Laplacian oscillator. A first related reference is [2]. More recently, the publication of [3] has renewed the interest in the study of equations with singular ϕ-Laplacian operators. If compared with the classical or Newtonian case, the relativistic pendulum has been scarcely studied, therefore at this stage it is important to point out the dynamical differences between both models. The aim of this note is to reveal a new dynamical response when relativistic effects are considered.

In the non-relativistic regime, that is, if c is assumed to be $+\infty$, we have the classical forced pendulum equation

$$x'' + kx' + a\sin x = p(t), \tag{1.2}$$

which is a paradigm in Classical Mechanics and Dynamical Systems. The long story around this equation can be found in the reviews [4,5]. Concerning the existence of periodic solutions, the first result was proved by Hamel [6] in 1922 for the conservative case $k = 0$.

Theorem 1 *(Hamel's theorem). If $k = 0$, then Eq. (1.2) has at least one T-periodic solution.*

Let us note that the original result by Hamel was proved for $p(t) = \sin t$ but the idea is easily extended to the general case. The proof is of variational type and hence the conservative nature of the problem plays the fundamental role. However, Mawhin [7] conjectured that a topological approach may be useful to prove the existence of periodic solutions in the presence of friction. The first counterexample was presented by Ortega [8]. Later, Alonso [9] provided a different counterexample, but the more general non-existence result is given in [10]. From now on, let us denote by C_T the Banach space of the continuous and T-periodic functions and by $\tilde{C}_T$ the space of the functions of C_T with zero mean value.

Theorem 2. *(See [10].) Given positive constants a, k and T, there exists $p \in \tilde{C}_T$ such that Eq. (1.2) has no T-periodic solutions.*

Our main aim is to prove that Mawhin's conjecture is partially true in the relativistic framework. Our main result is as follows.

Theorem 3. *Let us assume that $2cT \leqslant 1$. For any values a, k and for any $p \in \tilde{C}_T$, Eq. (1.1) has at least one T-periodic solution.*

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0375-9601/$ – see front matter
doi:10.1016/j.physleta.2008.08.060

The proof is an elementary application of the Schauder's Fixed Point Theorem and will be given in the next section. Of course, now an interesting open question arises: are there proper counterexamples for higher values of the period?

2. Proof of the main result

Eq. (1.1) can be written as

$$\phi(x')' + kx' + a\sin x = p(t),$$

where $\phi :]-c, c[\to \mathbb{R}$ is given by

$$\phi(u) = \frac{u}{\sqrt{1 - \frac{u^2}{c^2}}}.$$

Of course, the inverse ϕ^{-1} is a bounded operator. The first step of the proof is to make the change of variables $x = \arcsin y$. Then Eq. (1.1) is equivalent to

$$\phi\left(\frac{y'}{\sqrt{1-y^2}}\right)' + k\frac{y'}{\sqrt{1-y^2}} + ay = p(t). \tag{2.1}$$

The second step is to write the problem of finding a T-periodic solution of (2.1) as a fixed point problem for a suitable operator. A first integration of the equation gives

$$\phi\left(\frac{y'}{\sqrt{1-y^2}}\right) + k\arcsin y = \int_0^t \big(p(s) - ay(s)\big)\,ds + C,$$

where C is a constant to be fixed later. For convenience, let us define the operator

$$F[y](t) = \int_0^t \big(p(s) - ay(s)\big)\,ds - k\arcsin y.$$

Then, we get

$$y' = \sqrt{1-y^2}\,\phi^{-1}\big(F[y](t) + C\big).$$

Finally, a new integration gives

$$y(t) = \int_0^t \sqrt{1-y^2}\,\phi^{-1}\big(F[y](s) + C\big)\,ds + D.$$

Lemma 1. *For any $y \in \tilde{C}_T$, there exists a unique choice of C_y, D_y such that*

$$\mathcal{T}[y](t) \equiv \int_0^t \sqrt{1-y^2}\,\phi^{-1}\big(F[y](s) + C_y\big)\,ds + D_y \in \tilde{C}_T. \tag{2.2}$$

Proof. Periodicity is equivalent to

$$\int_0^T \sqrt{1-y^2}\,\phi^{-1}\big(F[y](s) + C_y\big)\,ds = 0.$$

As a function of C_y, the left-hand side of this equation is continuous and increasing, so the existence of a unique solution C_y for such equation follows from a basic application of the Mean Value Theorem. Once C_y is fixed, D_y is given by

$$D_y = -\frac{1}{T}\int_0^T\int_0^t \sqrt{1-y(s)^2}\,\phi^{-1}\big(F[y](s) + C_y\big)\,ds\,dt,$$

which is the unique choice such that $\mathcal{T}[y](t) \in \tilde{C}_T$. $\square$

Therefore, we have a well-defined functional $\mathcal{T} : \tilde{C}_T \to \tilde{C}_T$. Let us define the closed and convex set

$$K = \big\{y \in \tilde{C}_T:\ \|y\|_\infty \leqslant 2cT\big\}.$$

The operator $\mathcal{T}$ is well defined, continuous and compact on K. Take $y \in K$. Note that $\|\phi^{-1}[y]\|_\infty < c$ for every $y \in C_T$. Therefore,

$$\big|\mathcal{T}[y](t)\big| \leqslant 2\left|\int_0^t \sqrt{1-y(s)^2}\,\phi^{-1}\big(F[y](s) + C_y\big)\,ds\right| < 2cT$$

for all t. In consequence, by the Schauder's fixed point Theorem there exists a T-periodic solution y of (2.1). The hypothesis $2cT \leqslant 1$ enables to invert the change and hence $x = \arcsin y$ is a T-periodic solution of the original equation (1.1).

As a final note, let us remark that the same proof works if the linear friction term kx' is replaced by $h(x)x'$ without further restrictions on the continuous function h. Also, the term $\sin x$ could be extended to a more general nonlinearity, but such a generalization is out of the scope of this note and will be developed elsewhere. The method does not seem to work if the nonlinearity includes the derivative as in [2].

Acknowledgement

The author thanks two anonymous referees for pointing out Ref. [2] as well as some inaccuracies in a first version of this Letter.

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