On periodic solutions of second–order differential equations with attractive–repulsive singularities

Robert Hakl, Pedro J. Torres

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$$u(0) = u(\omega), \qquad u'(0) = u'(\omega) \tag{2}$$

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Positive solution

- $u : [0, \omega] \to]0, +\infty[$

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- u satisfies (1)

Special cases of (1)

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Attractive singularity

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Repulsive singularity

$$\lim_{x \to 0_+} q(t, x) = +\infty \qquad \text{for a. e. } t \in [0, \omega]$$

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Theorem B. $f \in L\big([0,\omega]; R\big)$, $\mu \geq 1$; (7) has a positive solution iff $\int_0^\omega f(s)ds < 0$.

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Theorem C. $f \in L\big([0,\omega]; R\big)$, $\mu < 1$; (7) has a positive solution if $\int_0^\omega f(s)ds < 0$ and

$$\text{ess inf}\,\big\{f(t) : t \in [0,\omega]\big\} > -\left(\frac{\pi^2}{\omega^2\mu}\right)^{\frac{\mu}{1+\mu}} (1+\mu).$$

$$u''(t) = \frac{g(t)}{u^{\mu}(t)} - \frac{h(t)}{u^{\lambda}(t)} + f(t) \qquad \text{for a. e. } t \in [0, \omega], \tag{1}$$

Notation

For the sake of brevity we will use the following notation:

$$G = \int_0^{\omega} g(s)ds, \qquad H = \int_0^{\omega} h(s)ds,$$

$$F = \int_0^{\omega} f(s)ds, \qquad F_+ = \int_0^{\omega} [f(s)]_+ ds, \qquad F_- = \int_0^{\omega} [f(s)]_- ds.$$

Note that $F = F_+ - F_-$.

$$u''(t) = \frac{g(t)}{u^{\mu}(t)} - \frac{h(t)}{u^{\lambda}(t)} + f(t) \qquad \text{for a. e. } t \in [0, \omega], \tag{1}$$

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Theorem

Let $H > 0$, $F > 0$, functions $w, \sigma \in AC^1_{\omega}\big([0, \omega]; R\big)$ be such that the equalities

$$w''(t) = Hg(t) - Gh(t) \qquad \textit{for a. e. } t \in [0, \omega],$$

$$\sigma''(t) = -\frac{F}{H} h(t) + f(t) \qquad \textit{for a. e. } t \in [0, \omega],$$

are fulfilled, and let there exist $x_0 \in]0, +\infty[$ such that

$$x_0\big(w(t) - m_w\big) + \sigma(t) - m_{\sigma} \leq \left(\frac{H}{x_0 GH + F}\right)^{1/\lambda} - \left(\frac{1}{x_0 H}\right)^{1/\mu} \quad \textit{for } t \in [0, \omega],$$

where

$$m_w = \min\big\{w(t) : t \in [0, \omega]\big\}, \qquad m_{\sigma} = \min\big\{\sigma(t) : t \in [0, \omega]\big\}.$$

Then the equation (1), (2) has at least one positive solution.

$$u''(t) = -\frac{h(t)}{u^{\lambda}(t)} + f(t) \qquad \text{for a. e. } t \in [0, \omega], \tag{4}$$

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Corollary

Let $H > 0$, $F > 0$, $\sigma \in AC^1_{\omega}\big([0, \omega]; R\big)$ *be such that*

$$\sigma''(t) = -\frac{F}{H} h(t) + f(t) \qquad \textit{for a. e. } t \in [0, \omega],$$

and let

$$\big(M_{\sigma} - m_{\sigma}\big)^{\lambda} F < H,$$

where

$$M_{\sigma} = \max\big\{\sigma(t) : t \in [0, \omega]\big\}, \qquad m_{\sigma} = \min\big\{\sigma(t) : t \in [0, \omega]\big\}.$$

Then the problem (4), (2) *has at least one positive solution.*

$$u''(t) = -\frac{h(t)}{u^{\lambda}(t)} + f(t) \qquad \text{for a. e. } t \in [0,\omega], \tag{4}$$

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Then the problem (4), (2) has at least one positive solution.

Corollary

Let $H > 0$, $F > 0$, and let

$$\left(\frac{\omega}{4} F_+\right)^{\lambda} F \le H.$$

Then the problem (4), (2) has at least one positive solution.

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Lemma

Let there exist positive functions $\alpha, \beta \in AC^1_\omega\big([0,\omega]; R\big)$ *such that*

$$\alpha''(t) \geq \frac{g(t)}{\alpha^\mu(t)} - \frac{h(t)}{\alpha^\lambda(t)} + f(t) \qquad \textit{for a. e. } t \in [0, \omega], \tag{8}$$

$$\beta''(t) \leq \frac{g(t)}{\beta^\mu(t)} - \frac{h(t)}{\beta^\lambda(t)} + f(t) \qquad \textit{for a. e. } t \in [0, \omega], \tag{9}$$

$$\alpha(t) \leq \beta(t) \qquad \textit{for } t \in [0, \omega].$$

Then there exists at least one positive solution to (1), (2).

$$w''(t) = Hg(t) - Gh(t), \qquad \sigma''(t) = -\frac{F}{H}h(t) + f(t),$$

$$x_0\big(w(t) - m_w\big) + \sigma(t) - m_\sigma \le \left(\frac{H}{x_0 GH + F}\right)^{1/\lambda} - \left(\frac{1}{x_0 H}\right)^{1/\mu}$$

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Proof.

$$\alpha(t) = \left(\frac{1}{x_0 H}\right)^{1/\mu} + x_0\big(w(t) - m_w\big) + \sigma(t) - m_\sigma \qquad \text{for } \ t \in [0, \omega].$$

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$$\beta''(t) = x_1 Hg(t) - \left(x_1 G + \frac{F}{H}\right) h(t) + f(t) \qquad \text{for a. e. } t \in [0, \omega],$$

$$\left(\frac{H}{x_1 GH + F}\right)^{1/\lambda} \leq \beta(t) \leq \left(\frac{1}{x_1 H}\right)^{1/\mu} \qquad \text{for } t \in [0, \omega].$$

$$\beta''(t) \leq \frac{g(t)}{\beta^\mu(t)} - \frac{h(t)}{\beta^\lambda(t)} + f(t) \qquad \text{for a. e. } t \in [0, \omega].$$

$$\alpha(t) \leq \beta(t) \qquad \text{for } t \in [0, \omega].$$

Definition

A function $\varphi \in L\big([0,\omega]; R_+\big)$ is said to verify the property (P) if the implication

$$\left.\begin{array}{l} u \in AC^1_\omega\big([0,\omega]; R\big) \\ u''(t) + \varphi(t)u(t) \geq 0 \quad \text{for a. e. } t \in [0,\omega] \end{array}\right\} \implies u(t) \geq 0 \quad \text{for } t \in [0,\omega]$$

holds.

$$u''(t) = \frac{g(t)}{u^{\mu}(t)} - \frac{h(t)}{u^{\lambda}(t)} + f(t) \qquad \text{for a. e. } t \in [0, \omega], \tag{1}$$

$$u(0) = u(\omega), \qquad u'(0) = u'(\omega) \tag{2}$$

$$u''(t) = \frac{g(t)}{u^\mu(t)} - \frac{h(t)}{u^\lambda(t)} + f(t) \qquad \text{for a. e. } t \in [0,\omega], \tag{1}$$

$$u(0) = u(\omega), \qquad u'(0) = u'(\omega) \tag{2}$$

Theorem

Let $G > 0$, $F < 0$, functions $w, \sigma \in AC^1_\omega([0,\omega]; R)$ be such that the equalities

$$w''(t) = Hg(t) - Gh(t), \qquad \sigma''(t) = \frac{|F|}{G} g(t) + f(t) \qquad \textit{for a. e. } t \in [0,\omega]$$

are fulfilled, and let there exist $x_0 \in]0, +\infty[$ such that

$$x_0\big(w(t) - m_w\big) + \sigma(t) - m_\sigma \le \left(\frac{G}{x_0 GH + |F|}\right)^{1/\mu} - \left(\frac{1}{x_0 G}\right)^{1/\lambda} \quad \textit{for } t \in [0,\omega].$$

Moreover, let us define

$$\beta(t) = \left(\frac{1}{x_0 G}\right)^{1/\lambda} + x_0\big(w(t) - m_w\big) + \sigma(t) - m_\sigma \qquad \textit{for } t \in [0,\omega]$$

and assume that $\varphi(t) = \frac{\mu g(t)}{\beta^{1+\mu}(t)}$ verifies the property (P). Then the problem (1), (2) has at least one positive solution.

$$u''(t) = \frac{g(t)}{u^{\mu}(t)} + f(t) \qquad \text{for a. e. } t \in [0, \omega], \tag{5}$$

$$u(0) = u(\omega), \qquad u'(0) = u'(\omega) \tag{2}$$

$$u''(t) = \frac{g(t)}{u^{\mu}(t)} + f(t) \qquad \text{for a. e. } t \in [0,\omega], \tag{5}$$

$$u(0) = u(\omega), \qquad u'(0) = u'(\omega) \tag{2}$$

Corollary

Let $G > 0$, $F < 0$, $\sigma \in AC^1_\omega\big([0,\omega]; R\big)$ *be such that*

$$\sigma''(t) = \frac{|F|}{G} g(t) + f(t) \qquad \textit{for a. e. } t \in [0,\omega],$$

and let

$$\big(M_\sigma - m_\sigma\big)^{\mu} |F| < G.$$

Let, moreover, either

$$\frac{\mu |F|^{\frac{1+\mu}{\mu}} g(t)}{\Big(G^{1/\mu} - |F|^{1/\mu}\big(M_\sigma - \sigma(t)\big)\Big)^{1+\mu}} \leq \left(\frac{\pi}{\omega}\right)^2 \qquad \textit{for a. e. } t \in [0,\omega],$$

or

$$\mu |F|^{\frac{1+\mu}{\mu}} \int_0^{\omega} \frac{g(s)}{\Big(G^{1/\mu} - |F|^{1/\mu}\big(M_\sigma - \sigma(s)\big)\Big)^{1+\mu}} ds \leq \frac{4}{\omega}\,.$$

Then the problem (5), (2) *has at least one positive solution.*

$$u''(t) = \frac{g(t)}{u^\mu(t)} + f(t) \qquad \text{for a. e. } t \in [0,\omega], \tag{5}$$

$$u(0) = u(\omega), \qquad u'(0) = u'(\omega) \tag{2}$$

$$u''(t) = \frac{g(t)}{u^\mu(t)} + f(t) \qquad \text{for a. e. } t \in [0, \omega], \tag{5}$$

$$u(0) = u(\omega), \qquad u'(0) = u'(\omega) \tag{2}$$

Corollary

Let $G > 0$ and $F < 0$. Let, moreover,

$$\left(\frac{\omega}{4}\mu G\right)^{\frac{1}{1+\mu}} |F|^{1/\mu} + \frac{\omega}{4} F_- |F|^{1/\mu} \leq G^{1/\mu}.$$

Then the problem (5), (2) has at least one positive solution.

$$u''(t) = \frac{g(t)}{u^{\mu}(t)} - \frac{h(t)}{u^{\lambda}(t)} + f(t) \qquad \text{for a. e. } t \in [0, \omega], \tag{1}$$

$$u(0) = u(\omega), \qquad u'(0) = u'(\omega) \tag{2}$$

$$u''(t) = \frac{g(t)}{u^\mu(t)} - \frac{h(t)}{u^\lambda(t)} + f(t) \qquad \text{for a. e. } t \in [0,\omega], \tag{1}$$

$$u(0) = u(\omega), \qquad u'(0) = u'(\omega) \tag{2}$$

Lemma

Let there exist positive functions $\alpha, \beta \in AC^1_\omega\big([0,\omega]; R\big)$ *such that*

$$\alpha''(t) \geq \frac{g(t)}{\alpha^\mu(t)} - \frac{h(t)}{\alpha^\lambda(t)} + f(t) \qquad \text{for a. e. } t \in [0,\omega],$$

$$\beta''(t) \leq \frac{g(t)}{\beta^\mu(t)} - \frac{h(t)}{\beta^\lambda(t)} + f(t) \qquad \text{for a. e. } t \in [0,\omega],$$

$$\beta(t) \leq \alpha(t) \qquad \text{for } t \in [0,\omega].$$

Let, moreover, there exists $\varphi \in L\big([0,\omega]; R_+\big)$ *with the property* (P) *and such that*

$$\frac{g(t)}{u^\mu(t)} - \frac{h(t)}{u^\lambda(t)} - \left(\frac{g(t)}{v^\mu(t)} - \frac{h(t)}{v^\lambda(t)}\right) \leq \varphi(t)\big(v(t) - u(t)\big) \qquad \text{for a. e. } t \in [0,\omega],$$

whenever $\beta(t) \leq u(t) \leq v(t) \leq \alpha(t)$ *for* $t \in [0,\omega]$. *Then there exists at least one positive solution to* (1), (2).

$$\varphi(t) = \frac{\mu g(t)}{\beta^{1+\mu}(t)} \qquad \text{for } \ t \in [0,\omega],$$

$$\frac{g(t)}{u^\mu(t)} - \frac{h(t)}{u^\lambda(t)} - \left(\frac{g(t)}{v^\mu(t)} - \frac{h(t)}{v^\lambda(t)}\right) \le \varphi(t)\big(v(t) - u(t)\big) \qquad \text{for a. e. } \ t \in [0,\omega],$$

whenever $\beta(t) \le u(t) \le v(t) \le \alpha(t)$ for $t \in [0,\omega]$,

$$\varphi(t) = \frac{\mu g(t)}{\beta^{1+\mu}(t)} \qquad \text{for } t \in [0, \omega],$$

$$\frac{g(t)}{u^{\mu}(t)} - \frac{h(t)}{u^{\lambda}(t)} - \left(\frac{g(t)}{v^{\mu}(t)} - \frac{h(t)}{v^{\lambda}(t)} \right) \leq \varphi(t)\big(v(t) - u(t)\big) \qquad \text{for a. e. } t \in [0, \omega],$$

whenever $\beta(t) \leq u(t) \leq v(t) \leq \alpha(t)$ for $t \in [0, \omega]$,

Proof.

$$\psi(y) = \frac{\mu}{\beta^{1+\mu}} y + \frac{1}{y^{\mu}}$$

is nondecreasing for $y \geq \beta$.

$$\varphi(t) = \frac{\mu g(t)}{\beta^{1+\mu}(t)} \qquad \text{for } t \in [0, \omega],$$

$$\frac{g(t)}{u^\mu(t)} - \frac{h(t)}{u^\lambda(t)} - \left(\frac{g(t)}{v^\mu(t)} - \frac{h(t)}{v^\lambda(t)}\right) \le \varphi(t)\big(v(t) - u(t)\big) \qquad \text{for a. e. } t \in [0, \omega],$$

whenever $\beta(t) \le u(t) \le v(t) \le \alpha(t)$ for $t \in [0, \omega]$,

Proof.

$$\psi(y) = \frac{\mu}{\beta^{1+\mu}} y + \frac{1}{y^\mu}$$

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$$g(t)\left(\frac{\mu}{\beta^{1+\mu}(t)} u(t) + \frac{1}{u^\mu(t)}\right) - \frac{h(t)}{u^\lambda(t)} \le g(t)\left(\frac{\mu}{\beta^{1+\mu}(t)} v(t) + \frac{1}{v^\mu(t)}\right) - \frac{h(t)}{v^\lambda(t)} \quad \text{for } t \in [0, \omega]$$

whenever $\beta(t) \le u(t) \le v(t)$ for $t \in [0, \omega]$.

$$\varphi(t) = \frac{\mu g(t)}{\beta^{1+\mu}(t)} \qquad \text{for } t \in [0, \omega],$$

$$\frac{g(t)}{u^{\mu}(t)} - \frac{h(t)}{u^{\lambda}(t)} - \left(\frac{g(t)}{v^{\mu}(t)} - \frac{h(t)}{v^{\lambda}(t)}\right) \le \varphi(t)\big(v(t) - u(t)\big) \qquad \text{for a. e. } t \in [0, \omega],$$

whenever $\beta(t) \le u(t) \le v(t) \le \alpha(t)$ for $t \in [0, \omega]$,

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$$\psi(y) = \frac{\mu}{\beta^{1+\mu}} y + \frac{1}{y^{\mu}}$$

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$$g(t)\left(\frac{\mu}{\beta^{1+\mu}(t)} u(t) + \frac{1}{u^{\mu}(t)}\right) - \frac{h(t)}{u^{\lambda}(t)} \le g(t)\left(\frac{\mu}{\beta^{1+\mu}(t)} v(t) + \frac{1}{v^{\mu}(t)}\right) - \frac{h(t)}{v^{\lambda}(t)} \quad \text{for } t \in [0, \omega]$$

whenever $\beta(t) \le u(t) \le v(t)$ for $t \in [0, \omega]$.

$$\frac{g(t)}{u^{\mu}(t)} - \frac{h(t)}{u^{\lambda}(t)} - \left(\frac{g(t)}{v^{\mu}(t)} - \frac{h(t)}{v^{\lambda}(t)}\right) \le \frac{\mu g(t)}{\beta^{1+\mu}(t)}(v(t) - u(t)).$$

□

$$u''(t) = \frac{g(t)}{u^{\mu}(t)} - \frac{h(t)}{u^{\lambda}(t)} + f(t) \qquad \text{for a. e. } t \in [0,\omega], \tag{1}$$

$$u(0) = u(\omega), \qquad u'(0) = u'(\omega) \tag{2}$$

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Theorem

Let $\lambda > \mu$, $H > 0$, $G > 0$, $F = 0$, functions $w, \sigma \in AC^1_\omega\big([0,\omega]; R\big)$ be such that

$$w''(t) = Hg(t) - Gh(t) \qquad \textit{for a. e. } t \in [0,\omega],$$
$$\sigma''(t) = f(t) \qquad \textit{for a. e. } t \in [0,\omega],$$

and let there exist $x_0 \in]0,+\infty[$ such that

$$x_0\big(w(t) - m_w\big) + \sigma(t) - m_\sigma \leq \left(\frac{1}{x_0 G}\right)^{1/\lambda} - \left(\frac{1}{x_0 H}\right)^{1/\mu} \quad \textit{for } t \in [0,\omega],$$

where

$$m_w = \min\big\{w(t) : t \in [0,\omega]\big\}, \qquad m_\sigma = \min\big\{\sigma(t) : t \in [0,\omega]\big\}.$$

Then the problem (1), (2) has at least one positive solution.

$$u''(t) = \frac{g(t)}{u^{\mu}(t)} - \frac{h(t)}{u^{\lambda}(t)} \qquad \text{for a. e. } t \in [0, \omega], \tag{3}$$

$$u(0) = u(\omega), \qquad u'(0) = u'(\omega) \tag{2}$$

$$u''(t) = \frac{g(t)}{u^{\mu}(t)} - \frac{h(t)}{u^{\lambda}(t)} \qquad \text{for a. e. } t \in [0, \omega], \tag{3}$$

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Corollary

Let $\lambda > \mu$, $H > 0$, $G > 0$, and let $w \in AC^1_{\omega}\big([0,\omega]; R\big)$ be such that

$$w''(t) = Hg(t) - Gh(t) \qquad \textit{for a. e. } t \in [0, \omega]$$

is fulfilled. Let, moreover,

$$M_w - m_w \le \frac{H^{\frac{1+\lambda}{\lambda-\mu}}}{G^{\frac{1+\mu}{\lambda-\mu}}} \left(\frac{(1+\lambda)\mu}{(1+\mu)\lambda} \right)^{\frac{(1+\lambda)\mu}{\lambda-\mu}} \frac{\lambda - \mu}{(1+\mu)\lambda},$$

where

$$M_w = \max\big\{w(t) : t \in [0,\omega]\big\}, \qquad m_w = \min\big\{w(t) : t \in [0,\omega]\big\}.$$

Then the problem (3), (2) has at least one positive solution.

$$u''(t) = \frac{g(t)}{u^{\mu}(t)} - \frac{h(t)}{u^{\lambda}(t)} \qquad \text{for a. e. } t \in [0, \omega], \tag{3}$$

$$u(0) = u(\omega), \qquad u'(0) = u'(\omega) \tag{2}$$

$$u''(t) = \frac{g(t)}{u^{\mu}(t)} - \frac{h(t)}{u^{\lambda}(t)} \qquad \text{for a. e. } t \in [0,\omega], \tag{3}$$

$$u(0) = u(\omega), \qquad u'(0) = u'(\omega) \tag{2}$$

Corollary

Let $\lambda > \mu$, $H > 0$, and $G > 0$. Let, moreover,

$$\frac{G^{1+\lambda}}{H^{1+\mu}} \le \left(\frac{4}{\omega}\right)^{\lambda-\mu} \left(\frac{(1+\lambda)\mu}{(1+\mu)\lambda}\right)^{(1+\lambda)\mu} \left(\frac{\lambda-\mu}{(1+\mu)\lambda}\right)^{\lambda-\mu}.$$

Then the problem (3), (2) has at least one positive solution.

$$u''(t) = \frac{g(t)}{u^{\mu}(t)} - \frac{h(t)}{u^{\lambda}(t)} + f(t) \qquad \text{for a. e. } t \in [0, \omega], \tag{1}$$

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Theorem

Let $\mu > \lambda$, $H > 0$, $G > 0$, $F = 0$, *functions* $w, \sigma \in AC^1_{\omega}\big([0,\omega]; R\big)$ *be such that*

$$w''(t) = Hg(t) - Gh(t) \qquad \textit{for a. e. } t \in [0,\omega],$$
$$\sigma''(t) = f(t) \qquad \textit{for a. e. } t \in [0,\omega],$$

and let there exist $x_0 \in]0,+\infty[$ *such that*

$$x_0\big(w(t) - m_w\big) + \sigma(t) - m_\sigma \leq \left(\frac{1}{x_0 H}\right)^{1/\mu} - \left(\frac{1}{x_0 G}\right)^{1/\lambda} \qquad \textit{for } t \in [0,\omega].$$

Moreover, assume that $\varphi(t) = \frac{\mu g(t)}{\beta^{1+\mu}(t)}$ *verifies the property* (P), *where* β *is given by*

$$\beta(t) = \left(\frac{1}{x_0 G}\right)^{1/\lambda} + x_0\big(w(t) - m_w\big) + \sigma(t) - m_\sigma \qquad \textit{for } t \in [0,\omega].$$

Then the problem (1), (2) *has at least one positive solution.*

$$u''(t) = \frac{g(t)}{u^{\mu}(t)} - \frac{h(t)}{u^{\lambda}(t)} \qquad \text{for a. e. } t \in [0, \omega], \tag{3}$$

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Corollary

Let $\mu > \lambda$, $H > 0$, $G > 0$, $w \in AC^1_{\omega}\big([0, \omega]; R\big)$ be such that

$$w''(t) = Hg(t) - Gh(t) \qquad \textit{for a. e. } t \in [0, \omega]$$

is fulfilled, and let

$$M_w - m_w \leq \frac{G^{\frac{1+\mu}{\mu-\lambda}}}{H^{\frac{1+\lambda}{\mu-\lambda}}} \left(\frac{(1+\mu)\lambda}{(1+\lambda)\mu} \right)^{\frac{(1+\mu)\lambda}{\mu-\lambda}} \frac{\mu - \lambda}{(1+\lambda)\mu} \, .$$

Moreover, let us define

$$\beta(t) = \left(\frac{(1+\mu)\lambda}{(1+\lambda)\mu} \right)^{\frac{\mu}{\mu-\lambda}} \left(\frac{G}{H} \right)^{\frac{1}{\mu-\lambda}} + \left(\frac{(1+\lambda)\mu}{(1+\mu)\lambda} \right)^{\frac{\lambda\mu}{\mu-\lambda}} \frac{H^{\frac{\lambda}{\mu-\lambda}}}{G^{\frac{\mu}{\mu-\lambda}}} \big(w(t) - m_w \big) \quad \textit{for } t \in [0, \omega],$$

and assume that $\varphi(t) = \frac{\mu g(t)}{\beta^{1+\mu}(t)}$ verifies the property (P). Then the problem (3), (2) has at least one positive solution.

$$u''(t) = \frac{g(t)}{u^{\mu}(t)} - \frac{h(t)}{u^{\lambda}(t)} \qquad \text{for a. e. } t \in [0, \omega], \tag{3}$$

$$u(0) = u(\omega), \qquad u'(0) = u'(\omega) \tag{2}$$

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Corollary

Let $\mu > \lambda$, $H > 0$, *and* $G > 0$. *Let, moreover,*

$$\frac{H^{1+\mu}}{G^{1+\lambda}} \leq \left(\frac{4}{\omega}\right)^{\mu-\lambda} \left(\frac{(1+\mu)\lambda}{(1+\lambda)\mu}\right)^{(1+\mu)\lambda} \left(\frac{\mu-\lambda}{(1+\lambda)\mu}\right)^{\mu-\lambda} \times$$

$$\times \min\left\{1, \left(\frac{1+\lambda}{\mu-\lambda}\right)^{\mu-\lambda} \left(\frac{(1+\mu)\lambda}{(1+\lambda)\mu}\right)^{(\mu-\lambda)(1+\mu)}\right\}.$$

Then the problem (3), (2) *has at least one solution.*

Open problem:

$$u''(t) = \frac{g(t)}{u^{\mu}(t)} - \frac{h(t)}{u^{\lambda}(t)} + f(t) \qquad \text{for a. e. } t \in [0, \omega], \tag{1}$$

$\lambda = \mu$ and $F = 0$.

Open problem:

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- J. L. Bravo, P. J. Torres (preprint)

$$u'' = \frac{a(t)}{u^3}; \qquad u(0) = u(\omega), \qquad u'(0) = u'(\omega) \tag{8}$$

$$a(t) = \begin{cases} a_+ & \text{for } t \in [0, t_0[\\ -a_- & \text{for } t \in [t_0, \omega] \end{cases}$$

with $a_+, a_- > 0$.

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with $a_+, a_- > 0$.

Theorem. (8) has a positive solution iff $\int_0^{\omega} a(s)ds < 0$.