Capillary surfaces modeling liquid drops on wetting phenomena

Rafael López

Abstract The aim of the present work is to relate the shape of a liquid drop in some contexts on capillarity and wetting with the surfaces that are mathematical models of these droplets. When a liquid drop is deposited on a support substrate, we are interested whether the geometry of the support imposes restrictions to the possible configurations of the droplet. Recently there is a progress in experiments done for liquid drops deposited on (or between) spherical rigid bodies, an assembly of cylinders and on a cone that allows to consider new theoretical problems in the field of capillary surfaces. We exploit the symmetries of these supports to apply the maximum principle of elliptic equations concluding that in some cases the drop inherits part of the symmetries of the support.

Key words: capillarity, wetting, mean curvature, Delaunay surfaces, free boundary problem, tangency principle

1 Introduction

1.1 A brief approach to capillarity and wetting

Following [12], capillarity studies the interfaces between two immiscible phases and wetting refers how a liquid deposited on a solid (or liquid) substrate spreads out. Capillarity and wetting appear in a variety of industrial and engineering processes (e.g., automobile manufacturing, textile production, ink-jet printing or colloid-

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polymer mixtures) where it is of interest to understand the physical and chemical behavior of a fluid. Many experiments consist of modifying the characteristics of the liquid and the solid until to attain the desirable wetting/spreading properties [5, 12]. A simple, but illustrative example, is when a given amount of an incompressible liquid is deposited on a solid substrate. Under idealized conditions (non-roughness, constant pressure and temperature, purity or low viscosity), the only forces acting on the liquid molecules are of order of a few nanometers and are determined by the Van der Waals and electrostatic interactions. These forces are balanced except for the molecules on the liquid-air interface S of the drop which, to be in contact with the air and solid phases, are mainly attracted inward and to the sides so that the attraction energy at the interface is less than in the interior. See Fig. 1, left. Under the above physical assumptions, the total energy E of the system is $E = E_S + E_A + E_G$ where E_S is the surface tension, E_A is a wetting energy and E_G the gravitational energy (Fig. 1, right). The energy E_S is the surface energy to create the interface S and is proportional to the number of interfacial molecules, that is, the surface area of S. The energy per area of S is called the *surface tension* σ. Similarly, E_A is the energy by the adhesion of the droplet on the solid phase which is also proportional to the number of molecules of the droplet in contact with the solid. Finally, E_G represents the weight of the drop and can written as an integral $\int_V gz$, where V is the volume of the drop, g is the gravitational constant and z is the height at a point of S with respect to a reference system. In this physical system, there are present three different phases, namely, liquid-air, solid-liquid, and solid-air phase, and the three corresponding surface tensions σ, σ_{SL} and σ_{SA}, respectively: see Fig. 2, left.

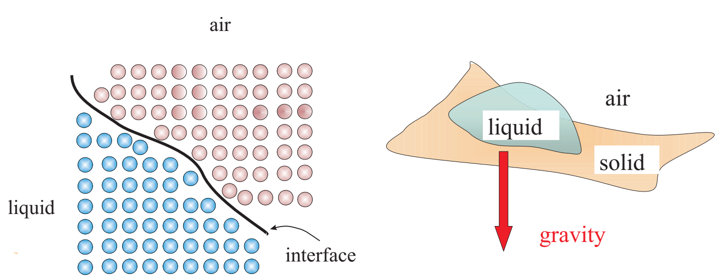

Fig. 1 Left: an interface is the boundary of two homogenous systems with different physical and chemical properties. Right: a liquid droplet deposited on a substrate under ideal physical conditions

In thermodynamic equilibrium, the interface S is free to change of shape in order to minimize its total free energy E. Assuming that the volume V of the drop remains fix (no evaporation), or in other words, if V is a Lagrange multiplier, and according to the principle of virtual work, the system will be in equilibrium if the energy E attains a critical point in the position of S. Then S satisfies the well-known Laplace equation

$$(P_L - P_A) + (\Delta d)\; g\, z = \left(\frac{1}{R_1} + \frac{1}{R_2}\right)\sigma = 2H\sigma. \tag{1}$$

Here $P_L - P_A$ is the difference between the liquid pressure P_L under S and the air pressure P_A just above S, Δd is the difference of densities between the liquid and air phases and H is the mean curvature of S. The mean curvature H at each point of S is defined by $2H = 1/R_1 + 1/R_2$, where R_i are the curvature radii. Because we are assuming ideal conditions, $P_L - P_A$ is constant, as well as, Δd, g and σ. In particular, the mean curvature H is a linear function of z, that is, for each point $(x,y,z) \in S$, we have $H(x,y,z) = \lambda z + \mu$, where $\lambda = g\Delta d/(2\sigma)$, $\mu = (P_L - P_A)/(2\sigma)$. Usually there are two extra boundary conditions. The first one supposes that the liquid-solid phase is prescribed, that is, the part that the drop wets the solid is confined in a fixed region so the boundary ∂S of S is a prescribed curve. A second (and more natural) scenario is assuming that the droplet can move freely on the substrate Π (free boundary condition). In this situation, S satisfies the so-called Young equation

$$\cos\gamma = \frac{\sigma_{SA} - \sigma_{SL}}{\sigma}, \tag{2}$$

where γ is the angle that makes S with the liquid-solid-air contact line ∂S. Because the three surfaces tensions are constant, the Young equation (2) establishes that *the contact angle γ between the drop and the substrate is constant along ∂S* [10, 17]. Here γ is the angle between the unit normal vectors N_S and N_Π of S and Π, respectively: $\cos\gamma = \langle N_S, N_\Pi \rangle$, where N_S points to the liquid drop and N_Π points outward the drop.

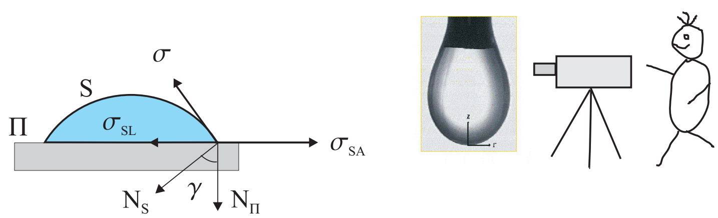

Fig. 2 Left: the contact angle γ between S and Π and the equilibrium between the three surface tensions. Right: the pendant drop method to measure the surface tension for an axisymmetric droplet

In a specific problem it is necessary the prediction of the magnitude of the capillary forces for eliminating or minimizing undesirable events, for example, an uncontrolled growth of agglomeration of particles or an abrupt change of the flow behavior of a fluid. According this, the wetting state of the fluid is determined once the three surface tensions are known. In general, it is difficult to compute all them, although the difference $\sigma_{SA} - \sigma_{SL}$ in (2) is a property of the solid and independent off the liquid used. Thus the interest focuses to compute the surface tension σ which is obtained from the Laplace equation (1) once calculated H or from the Young equation (2) if the contact angle γ is known.

1.2 The measurement of the surface tension

Among numerous measurement techniques of the surface tension σ, we describe the sessile and pendant drop method [1]. A drop is sitting (or hanging) on a horizontal plane which is taken aside-view photographs of the profile and we use a snapshot to determine the shape of S (or the angle γ) by comparing the actual shape of the drop with theoretical simulations based on the parameter σ; see Fig. 2, right. However in order to use the Young equation (2), it is actually difficult to compute explicitly the contact angle γ because the liquid is easily contaminated. The other (and more common) procedure consists to determine the mean curvature H adequating the profile shape of the drop to a well-controlled geometry and extracting σ from the Laplace equation (1). The mean curvature H of a surface $z = u(x,y)$ in Euclidean space $\mathbb{R}^3$ satisfies

$$(1+u_y^2)u_{xx} - 2u_xu_yu_{xy} + (1+u_x^2)u_{yy} = 2H(1+u_x^2+u_y^2)^{3/2}. \tag{3}$$

We observe that Eq. (3) is a PDE of order two that cannot be integrated, even if H is constant, and only be numerically approximated by analytic methods. Assuming a small scale (wetting) or that the typical size of the meniscus is much smaller that the capillary length (capillarity), the surface tension dominates the gravitational force, so the gravity can be neglected. Thus $g = 0$ in the Laplace equation (1) and we deduce that the mean curvature H is constant. As a consequence we can affirm that *the liquid-air interface S of a liquid droplet is modeled by a surface in Euclidean space where the mean curvature is the same at every point and makes a constant contact angle with the support substrate*. Surfaces with zero mean curvature everywhere ($H = 0$) are called minimal surfaces and they appear when the pressures coincide in both sides of S. Constant mean curvature surfaces are easily obtained when we dip in and out a closed wire in a container with soapy water. The soap film spanning by the wire is a minimal surface because there is not pressure difference across it. However if the wire traps air inside it, or if we blow air on it making a bubble, then there is an enclosed volume, the pressure difference is non-zero (but constant) and the surface has nonzero constant mean curvature.

Therefore experimentalists need to simplify Eq. (3) and the usual idea is assuming symmetric shapes so the discrete computational procedures developed to simulate the mathematical behavior of these processes can be fast and manageable. In this sense, it would be useful to reduce this equation into an ODE if, owing to symmetries, the equation depends only on one coordinate. The most common situation is assuming axisymmetric solutions of (3), that is, S is a surface of revolution. If $u = u(r)$ is the distance to the rotation axis, a first integration of (3) is

$$Hu^2 - \frac{u}{\sqrt{1+u'^2}} = c \tag{4}$$

for some $c \in \mathbb{R}$. From this equation, we can solve some cases: if $c = 0$, the solution of (4) is the circle $u(r) = \sqrt{1-H^2r^2}/H$ and S is a sphere of radius $1/|H|$; if u is a constant function, then the solution is $u(r) = 1/(2H)$ (for $1+4Hc = 0$) and S is a cylinder of radius $1/(2|H|)$; if $H = 0$, then $c = m^2 > 0$, $u(r) = m\cosh(r/m)$ and S is a catenoid. However for arbitrary c, the solutions of (4) can not integrate completely and they can only be represented by elliptic integrals. The profile curves of the solutions of (4) are mathematically characterized to be the roulettes of the focus of a conic and the surfaces are called Delaunay surfaces [8, 9]. Besides spheres and cylinders, they are unduloids, nodoids and catenoids: see Fig. 3. Usually, experiments utilize symmetric devices to sit or hang a droplet from a circular opening where *the observed interface is assumed to be a surface revolution*. In general, it is utilized pendant drops that the sessiles one because they are easily controllable. Once that we know that the interface is rotational, determining the geometry of the drop consists to capture and digitalize its image, extracting its contour, smoothing the profile and comparing the shape with the theoretical Delaunay surfaces (Fig. 4). Finally, a software (for example, a Runge-Kutta method, a technique based on finite elements or the Surface Evolver) works to compute the mean curvature H. This measurement method is simple and it does not require a sophisticated machinery or any special cleanliness of the solid substrate.

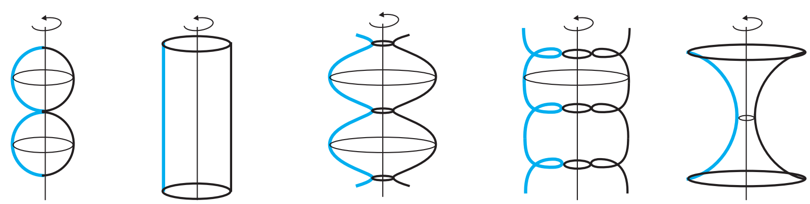

Fig. 3 Delaunay surfaces. From left to right: sphere, cylinder, unduloid, nodoid and catenoid

In contrast to the assumption that a droplet hanging for a circular opening is axisymmetric (independently with or without gravity), and from the theoretical viewpoint, the shape of a surface with constant mean curvature (cmc surface in short) in Euclidean space spanning a circle $\mathbb{S}^1$ is not well known up today and only some partial results ensure that a compact cmc surface in $\mathbb{R}^3$ spanning $\mathbb{S}^1$ is a planar disk or a spherical cap. For the state-of-the-art in this topic, see [19]. In the free boundary problem, it is unknown whether the geometry of the substrate affects to the geometry of a cmc surface supported on it, for example, if it inherits its symmetries. First mathematical results were obtained by Wente in [34] assuming embeddedness of the surface.

2 Capillary surfaces supported on spheres, cylinders, cones and wedges

Recently there is a great interest in the study of liquid drops deposited on (or between) configurations formed by spherical rigid bodies, an assembly of cylinders,

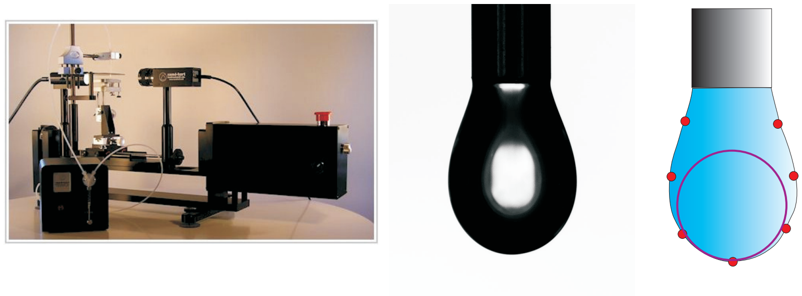

Fig. 4 Left: description of the typical apparatus of the pendant drop method. Right: a pendant drop is modeled by an axisymmetric surface by adjusting its contour

cones or planes because this variety of systems may be found like a crystallization, agglomeration, phase sintering, liquid foams and emulsions [14, 16, 18, 24, 28]. Moreover, the improvement of the numerical analysis methods as well as the modeling software allows to consider new theoretical problems in capillarity and wetting. When the size of the liquid drop is very small, the effect of gravity is negligible and no other force is considered. In such a case, the interface S has the same mean curvature H everywhere. We need again to model the liquid bridges as Delaunay surfaces where the geometry associated is relatively simple or at least giving conditions that ensure that S is rotational. In this section, by a *capillary surface* we mean a cmc surface S with free boundary on a substrate Π and S makes a constant contact angle with Π along its boundary ∂S. Since we are considering bounded droplets, we also suppose that S is compact. The symmetry of the mentioned supports in this section allows to get (at least theoretically) explicit examples of pieces of Delaunay surfaces that are capillary surfaces. Some examples appear in Fig. 5, where the support Π is a sphere, a circular cylinder and a circular cone and the rotation axis of S coincides with the one of Π.

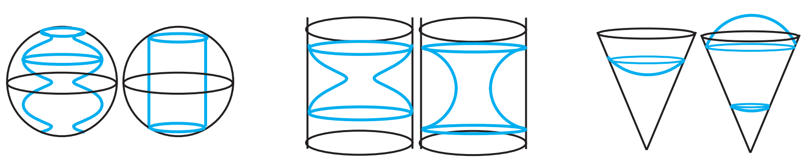

Fig. 5 Pieces of Delaunay surfaces that make a constant contact angle with a sphere, a circular cylinder and a cone

In what follows, we show some results on the symmetry of a capillary surface when the support substrate is a sphere, a right cylinder, a cone and a wedge. See [20, 21, 22, 23].

2.1 Droplets on a sphere

Consider a cmc surface S whose boundary lies on a sphere, which we suppose to be the unit sphere $\mathbb{S}^2$ and denote by $\mathbb{B}^3$ the unit ball enclosed by $\mathbb{S}^2$. Previous results on capillary surfaces on $\mathbb{S}^2$ included in $\mathbb{B}^3$ were obtained assuming that the contact angle is constant [11, 30, 31].

Theorem 1. *Let S be an embedded cmc surface on $\mathbb{S}^2$ whose boundary ∂S is included in a hemisphere $\mathbb{S}^2_+$. Suppose S is a capillary surface. Let W be the 3-domain bounded by $S \cup \Omega$, where $\Omega \subset \mathbb{S}^2_+$ is the domain bounded by ∂S. If $W \subset \overline{\mathbb{R}^3 - \mathbb{B}^3}$ or $S \subset \mathbb{B}^3$, then S is part of a sphere.*

This result extends if we replace the capillary condition by assuming that the boundary ∂S is a circle. In such a case and when $W \subset \overline{\mathbb{R}^3 - \mathbb{B}^3}$, we add the hypothesis that the mean curvature H satisfies $|H| \geq 1$.

2.2 Droplets on a right cylinder

By a right cylinder we mean $\Sigma = C \times \mathbb{R}$, where $C \subset \mathbb{R}^2$ is a simple planar closed curve. The cylinder is said to be circular if C is a circle. The cylinder Σ determines two domains in $\mathbb{R}^3$, namely, the inside and the outside, that is, $\Omega \times \mathbb{R}$ and $\mathbb{R}^3 \setminus \overline{\Omega \times \mathbb{R}}$, where $\Omega \subset \mathbb{R}^2$ is the bounded domain by C. Consider a capillary surface S on Σ that lies in one side of Σ. A first question to elucidate is if the boundary ∂S is a curve nullhomotopic in Σ or if ∂S is homotopic to C. For example, the first setting could occurs if the volume of S is very small, and the second one when a cylindrical tube is introduced in a container of liquid and the liquid rises up by capillarity. In the latter one, we ask if S is a graph $z = u(x,y)$ on Ω.

Theorem 2. *Let Σ be a right cylinder and let S be an embedded capillary surface on Σ such that $S \subset inside(\Sigma)$.*

1. *If ∂S is homotopic to C, then S is a graph on Ω. If Σ is a circular cylinder, then S is a planar disk or a spherical cap.*
2. *If $\partial S = C_1 \cup C_2$ and each C_i, $(i = 1,2)$ is homotopic to C, then S has a symmetry with respect to a plane orthogonal to the axis.*
3. *If Σ is a circular cylinder and ∂S is contained in a half cylinder of Σ, then S has two mutually planes of symmetry and S is a topological disk.*

In the item 3, by a half cylinder of Σ we mean one of the two components remaining when we intersect Σ by a plane containing the rotation axis.

In case that S has zero mean curvature, we have a strong result under the hypothesis that the surface is immersed.

Theorem 3. *Let S be capillary minimal surface on Σ such that $S \subset inside(\Sigma)$. If ∂S is a graph on C, then S is a horizontal planar domain.*

2.3 Droplets on a cone

Consider $\Omega \subset \mathbb{S}^2$ a simply connected domain of $\mathbb{S}^2$ and included in a hemisphere of $\mathbb{S}^2$. If $\Gamma = \partial\Omega$, the cone determined by Γ is defined as $\Sigma = \{\lambda p : \lambda > 0, p \in \Gamma\}$, that is, the set of all rays starting from the origin O through all points of Γ. If Γ is a circle, we say that Σ is a circular cone. The inside of the cone Σ is the corresponding 3-domain $\{\lambda p : \lambda > 0, p \in \Omega\}$.

We consider a capillary surface S whose boundary lies on Σ and contained in the inside of Σ. As in the case of a right cylinder, we do not know whether ∂S is nullhomologous in $\Sigma - \{O\}$ or if ∂S is homotopic to Γ in $\Sigma - \{O\}$ and S has a one-to-one central projection on Ω (a radial graph), that is, each ray starting from the vertex intersects S one point at most. See Fig. 6, left. We obtain:

Theorem 4. *Let S be an embedded capillary surface supported on a cone Σ and let us fix N the unit normal vector field of S pointing towards the liquid drop. If $H \leq 0$, then S is a radial graph and the boundary ∂S has only one connected component which is homologous to Γ in $\Sigma - \{O\}$. In the particular case that the cone is circular, then S is a planar disk or a spherical cap.*

In other words, Theorem 4 says that the non-positivity of H implies that S is a topological disk and that there are no capillary bridges between the walls of Σ. As a consequence, and dropping the assumption on the sign of H, we have (Fig. 6, right):

Corollary 1. *If S is a capillary surface on a circular cone Σ such that the contact angle γ satisfies $\gamma \leq (\pi - \varphi)/2$, being φ the amplitude of Σ, then S is a planar disk or a spherical cap.*

In this case, the hypothesis on γ implies $H \leq 0$: this is a consequence of comparing S with spherical caps or planar disks having the same mean curvature and the same contact angle with Σ.

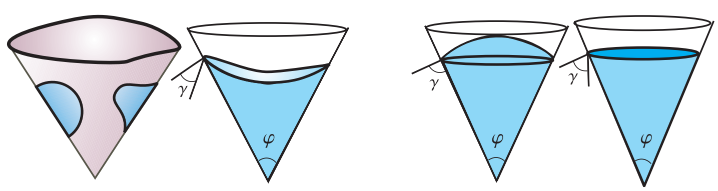

Fig. 6 Left: possible configurations of a liquid drop deposited on a cone. Right: spherical caps and planar disk are examples of capillary surfaces on a circular cone

2.4 Droplets on a wedge

If we intersect appropriately a Delaunay surface by two orthogonal planes $\Pi_1 \cup \Pi_2$ to the rotation axis, we obtain a capillary surface contacting $\Pi_1 \cup \Pi_2$ with the same contact angle. It is known by experiments that only some pieces of Delaunay surfaces are physically realized, that is, only some surfaces are stable in the sense that the second variation of the energy E is non-negative. Early results of Vogel and Athanassenas prove that the only stable capillary surfaces connecting two parallel planes are rotational surfaces and that if the contact angle γ is $\pi/2$, then the half-sphere and the cylinder are the only possibilities [4, 33]. However, the problem is far to be completely known for a general contact angle or other assumptions replacing stability [2, 6, 15, 25, 35].

A similar situation occurs when Π_1 and Π_2 are not parallel planes. In this case, the 3-domain determined by $\Pi_1 \cup \Pi_2$ is called a wedge. For this support, there are explicit examples of capillary surfaces when we place a sphere centered in the plane bisecting the wedge ($\gamma > \pi/2$), or if the center lies in the axes of the wedge ($\gamma = \pi/2$). See Fig. 7. A first question posed is on the existence of capillary surfaces with cylindrical topology connecting Π_1 and Π_2: see [7, 26, 27]. Under this context and $\gamma = \pi/2$ (Fig. 7, right), we prove:

Theorem 5. *Consider a cmc surface S on a wedge with contact angle $\gamma = \pi/2$. If S is stable or it is embedded, then S is part of a sphere centered at the vertex.*

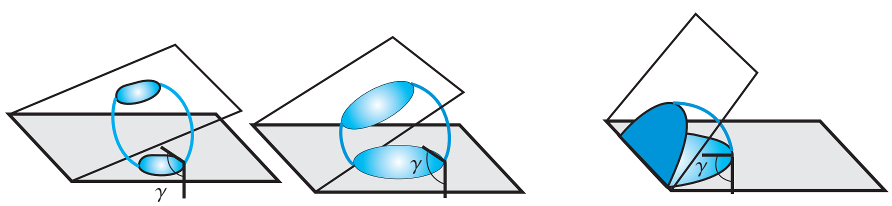

Fig. 7 Left: capillary surfaces on a wedge. Right: a spherical cap meeting orthogonally the walls of a wedge

3 The proof methods

Motivated by experiments on wetting and capillarity, we assume that the interface of a droplet is an embedded surface. In our context, and since our surfaces are compact, embeddedness is equivalent to say that the surface has not self-intersections. In the theory of embedded cmc surfaces, one of the main ingredients in the proofs is the Alexandrov reflection principle. Alexandrov proved that the sphere is the only embedded closed cmc surface [3]. Although this result was expected, the novelty came from the proof, where the very surface is utilized as a barrier with itself to obtain the desired result. This idea has been extensively utilized not only in geometry but also in PDE theory, starting with the breaking paper of Serrin [32]. We briefly explain the Alexandrov method. The mean curvature equation (3) is elliptic but not linear. However if u_1 and u_2 are two solutions of (3), the difference function $u = u_1 - u_2$ satisfies a *linear* elliptic equation $Lu = 0$ and we can apply the maximum principle [13]. In the context of cmc surfaces, this result is known as the tangency principle which asserts that if S_1 and S_2 are two surfaces with the same constant mean curvature, which are tangent at a point $p \in S_1 \cap S_2$ and S_1 lies in one side of S_2 around p, then S_1 and S_2 coincide in a neighborhood of p, and by extension of the argument, S_1 and S_2 coincide in a common open and closed set [19]. For the proof, let S be an embedded closed cmc surface and let us fix a direction $\mathbf{a} \in \mathbb{R}^3$. Consider a plane coming from infinity and orthogonal to $\mathbf{a}$ until arriving the first contact point with S: Fig. 8, left. Next, we follow moving the plane and reflecting the surface that lies behind the plane until the first time that the reflected surface (with respect to a plane $P_{\mathbf{a}}$) reaches the initial surface. In the touching point between both surfaces, the tangency principle implies that the reflected surface and the part of the surface in that side of $P_{\mathbf{a}}$ must coincide, proving that $P_{\mathbf{a}}$ is a plane of symmetry of S. Doing the same argument for all spatial directions $\mathbf{a}$, we conclude that S must be a round sphere.

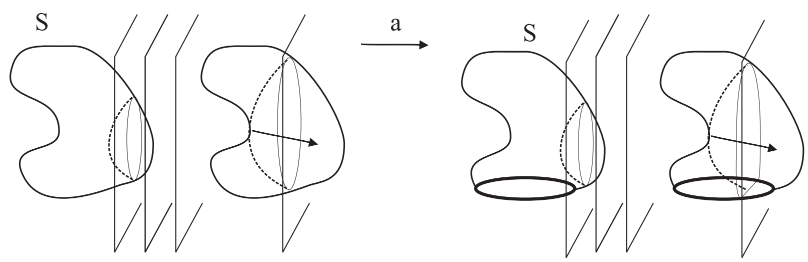

Fig. 8 The Alexandrov reflection method. Left: the closed embedded case. Right: the circular boundary case

In case that the boundary of S is a circle, we need to assume that S lies in one side of the plane containing ∂S: see Fig. 8, right. This prevents that the first contact point may occur between an interior point with the boundary ∂S because in such a case, the reflected surface and S are not tangent at the first touching point and we can not utilize the tangency principle.

In each one of the support substrates considered in Section 2, we have different possibilities of choices of planes to start with the reflection method. We explain in each case [20, 21, 22, 23].

1. Suppose that the support is a sphere $\mathbb{S}^2$ and that ∂S is a circle in $\mathbb{S}^2_+$ (Th. 1). Recall that in this case, we are assuming $|H| \geq 1$. Let Π be the horizontal plane containing the center O of $\mathbb{S}^2$. A first step is proving that if W lies outside $\mathbb{B}^3$,

then $O \notin W$. On the contrary, consider the uniparametric family of spherical caps of radius bigger than 1 below Π and with the common boundary to be the circle $\Pi \cap \mathbb{S}^2$. Starting from the radius $r = 1$, we increase the radius of these caps until the first contact with S: Fig. 9, left. At the contact point, the mean curvature of the cap, namely $1/r$, must be bigger than $|H|$, which it is not possible because $|H| \geq 1$. As a conclusion, if W lies outside $\mathbb{B}^3$, then $O \notin W$. The reflection process starts with horizontal planes coming from below until we reach S (Fig. 9, right). Next, we follow moving up the plane and reflecting. Since ∂S lies in the upper hemisphere, there is not a touching point before arriving to the plane Π since, on the contrary, there would be a horizontal plane of symmetry: a contradiction because $\partial S \subset \mathbb{S}^2_+$. Once arrived to the origin, we fix a horizontal straight line $L \subset \Pi$ passing through O. Let us replace the above planes by a family of planes all containing L (Fig. 9, right). Next we are going to rotate the plane and we follow the reflection method until the first touching point p. If p is an interior point, a standard argument implies that the plane is a plane of symmetry, so of ∂S. If p is a boundary point, then the plane is a plane of symmetry of ∂S. Repeating this argument for any horizontal straight line L through the center of $\mathbb{S}^2$, we conclude that S is a spherical cap.
In case that S is a capillary surface, the only difference in the above argument is that if the first touching point p is a boundary point (necessarily with respect to a plane containing L), the condition on the constancy of the contact angle implies that the reflected surface and the initial one are tangent at p. Thus we apply the (boundary version) tangency principle [13] concluding that the plane is a plane of symmetry of the surface.

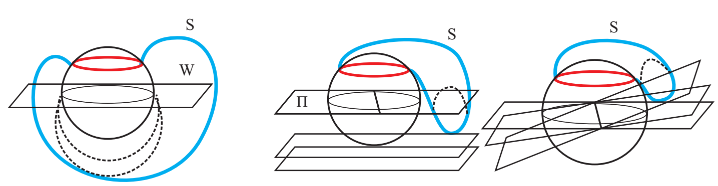

Fig. 9 Reflection method for a capillary surface droplet supported on a sphere

2. Suppose that the support is a right cylinder $\Sigma = C \times \mathbb{R}$. In the item 1 of Theorem 2, the reflection method uses a family of orthogonal planes to a vertical line and coming from infinity (Fig. 10, left). In case of existence of a horizontal plane where the reflected surface touches the first time with the initial surface at some interior point, then this plane is a plane of symmetry. This is a contradiction with the fact that ∂S is a curve homotopic to C. If the first contact point occurs at a boundary point, the condition on the constancy of the contact angle implies that the initial and the reflected surface are tangent at that point, and the proof works. For the item 2, the argument is similar.
   For the item 3, and because ∂S lies in a half cylinder, then ∂S is nullhomotopic in Σ. Thus S together a domain of Σ bounds a 3-domain $W \subset \mathbb{R}^3$. A first step

consists to apply the reflection method with a uniparametric family of planes orthogonal to the rotation axis. Hence we obtain a first plane of symmetry P_1 of S. Let Π be the plane containing the rotation axis that leaves in one side ∂S. We now use a uniparametric family of planes parallel to Π and all them lie in the other side of Π (not containing ∂S). The reflection method works until we arrive to the very plane Π (Fig. 10, right). The hypothesis on ∂S to be contained in a half cylinder prevents the existence of a first contact point. At this position, we replace the planes by a family of planes containing the axis. We follow the reflection method by rotating these planes until the first (interior or boundary) contact point, obtaining a new plane of symmetry P_2 of S. The plane P_2 contains the axis so P_2 is orthogonal to P_1. Because S is symmetric by these orthogonal planes P_1 and P_2, then S is a topological disk.

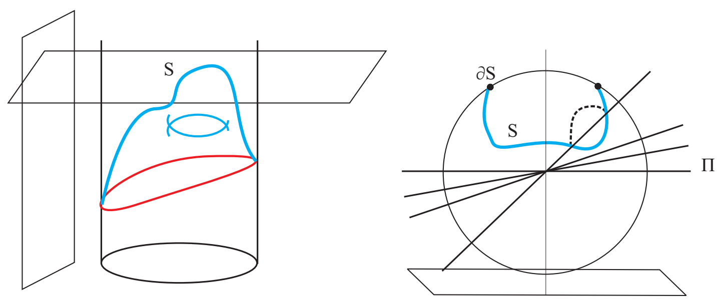

Fig. 10 Reflection method for a capillary surface supported on a right cylinder. Right: a top view of Σ

3. In the case of a cone, the reflection process with respect to planes is substituted by a spherical reflection method, which appeared firstly in [26] replacing by inversions about a one parameter family of spheres all centered at the center O of $\mathbb{S}^2$. Although an inversion does not preserve H, there is a certain control of the mean curvature of the inverted surface in order to use the tangency principle. Exactly if $\mathbb{S}^2_r \subset \mathbb{R}^3$ is the sphere of radius r centered at O, the spherical reflection about $\mathbb{S}^2_r$ is the inversion mapping defined by

$$\phi_r : \mathbb{R}^3 \setminus \{O\} \to \mathbb{R}^3 \setminus \{O\}, \ \ \hat{p} := \phi_r(p) = \frac{r^2}{|p|^2} p.$$

Let H be the mean curvature of S with respect to a unit normal vector field N. Denote by $\hat{S}_r$ the spherical reflection of S about ϕ_r and consider on $\hat{S}_r$ the orientation

$$\hat{N}(\hat{p}) = N(p) - \frac{2\langle N(p), p\rangle}{|p|^2} p.$$

Then the mean curvature of $\hat{S}_r$ is

$$\hat{H}(\hat{p}) = \frac{H|p|^2 + 2\langle N(p), p\rangle}{r^2}. \quad (5)$$

We start the spherical reflection method from spheres $\mathbb{S}^2_r$ with r sufficiently big until the first contact point p_0 with S. Because N points inside the liquid, then $\langle N(p_0), p_0\rangle < 0$. We have from (5) that $\hat{H}(p_0) \leq H|p_0|^2/r^2 < H$, where we use $H \leq 0$. Following with the reflection across inversions and using the assumption on the non-positivity of H, we conclude that there is not a contact point between the inverted surface with the part of S inside $\mathbb{S}^2_r$, which proves that S is a radial graph on Ω. For the second part of Theorem 4, we use that the the only cmc surface in $\mathbb{R}^3$ that is invariant by an inversion about a sphere is an open set of a sphere or a plane.

4. We only prove Theorem 5 when S is an embedded surface. First we extend the Ros formula [29] proving that if a compact embedded surface with not necessarily constant mean curvature H meets orthogonally a wedge, then

$$\int_S \frac{1}{H}\, dM \geq 3V, \quad (6)$$

where V is the volume of S and the equality holds if and only if S is part of a sphere. The proof of (6) involves the Reilly formula for a solution of PDE with Dirichlet and Neumann boundary conditions and the classical Minkowski formula

$$\int_S (1 + H\langle N, x\rangle)\, dS = -\frac{1}{2}\int_{\partial S} \langle \nu, x\rangle\, ds,$$

where ν is the inward unit conormal along ∂S. After a rigid motion, the orthogonality intersection condition means that $\langle \nu, x\rangle = 0$ and as H is constant, we get $A - 3HV = 0$, where A is the area of S. This implies equality in (6) and the result follows.

4 Conclusions

In the present paper we have discussed under what conditions some geometric configurations of a liquid droplet in thermodynamic equilibrium is a surface of revolution. Our motivation comes from the fact that experiments devoted to compute the surface tension σ of a liquid (e.g. the pendant drop method) assume previously that if the boundary of the air-liquid interface is symmetric, or if the drop is supported on a highly symmetric substrate, the liquid drop receives the same symmetries. In recent years there is a great progress in the creation of new materials and experimentation at nanometer and microscopic scales of fluids deposited between configurations of spheres, cylinders and planes. In some industrial experiments, there exist processes of crystallization and agglomeration which require the knowledge of the effects of the capillary forces of the liquids bridges connecting these solids and avoiding an abrupt change in the liquid shape, or preventing undesirable overflowing events. To quantify and estimate these forces, the mathematical models for droplets and liquid bridges are cmc surfaces which are assumed to be surfaces of revolution because the analytic expression of the mean curvature equation (3) in the axisymmetric case (4) is easier. Recent progress in experiments with a wider variety of morphologies on the substrate has given a new boost in the theoretical study that it was not previously considered.

Our results show that if these drops are modeled by a surface with constant mean curvature and under assumptions of embeddedness, then the droplet inherits some symmetries of the support substrate Π when Π is a sphere, a right cylinder, a cone or a wedge. This allows to provide a mathematical understanding of why the shapes of these drops are axisymmetric. These results provide us new directions of investigation, for example, assuming that the droplet has self-intersections which means that in the fluid may appear empty chambers of liquid or that the droplet does not lie completely in one side of the substrate.

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