Facultad de Ciencias Económicas y Empresariales
FEG WORKING PAPER SERIES
DOCUEMENTOS DE TRABAJO FEG
Universidad de Granada

FEG WORKING PAPERS SERIES

Documentos de trabajo de la Facultad de Ciencias Económicas y Empresariales de la Universidad de Granada

FEG-WP Nº 3/07

BAYESIAN ANALYSIS OF THE COMPOUND COLLECTIVE MODEL: THE NET PREMIUM PRINCIPLE WITH EXPONENTIAL POISSON AND GAMMA–GAMMA DISTRIBUTIONS

A. Hernández–Bastida, J. M. Pérez–Sánchez and E. Gómez–Déniz

Bayesian Analysis of the Compound Collective Model: the Net premium principle with Exponential Poisson and Gamma–Gamma distributions

A. Hernández–Bastida[a], J. M. Pérez–Sánchez[b] and E. Gómez–Déniz[c]

[a] *Department of Quantitative Methods in Economics, University of Granada, Spain. E-mail: bastida@ugr.es*

[b] *Department of Quantitative Methods in Economics, University of Granada, Spain. E-mail: josemag@ugr.es*

[c] *Department of Quantitative Methods in Economics, University of Las Palmas de G.C., Spain. E-mail: egomez@dmc.ulpgc.es*

ABSTRACT

This article develops a Bayesian analysis of the Compound Collective Model utilizing the Net Premium Principle, considering single-period models. With respect to likelihoods, we used a Poisson distribution for the number of claims and an Exponential distribution for the severity of the accident/event. Gamma distributions were used for the prior distributions. The robustness of the posterior premium was analyzed with respect to the prior distribution specification of the severity of the accident/event, utilizing contamination classes, these being the class of all the distributions and that of all the unimodal distributions with the same mode. Numerical applications of the results obtained were performed.

JEL Classification: C11.

Key words and phrases: Compound collective model; Bayesian analysis; Robustness analysis.

1. Introduction

In this article, we develop a Bayesian statistical analysis of the compound collective risk model, using the Net Premium Principle, which includes an analysis of robustness with respect to prior distributions.

The collective model of the Risk Theory is a sequence K, X_1, X_2,... of random variables, with the following meaning:

- K is the random variable "number of accidents or claims".
- X_i, for $i = 1, 2, \ldots$ is the random variable "cost or severity of the i–th accident".

The random variable of interest in the problem is "total cost" defined by $X = \sum_{i=1}^{K} X_i$, the probability density function of which is $\sum_{k=0}^{\infty} p(k/\theta) f^{k^*}(x/\lambda)$, where $p(k/\theta)$ denotes the probability of k claims and $f^{k^*}(x/\lambda)$ is the k–th convolution of $f(x/\lambda)$, the probability density function of the claim amount.

In some specific cases, these random variables degenerate into deterministic variables, for example, in many of the different types of life insurance, the costs of accidents are fixed amounts; such cases are not considered in the present study, in which we operate with the distribution of total cost, by means of the model for the number of accidents and the model for their severity.

The use of Bayesian analysis in the collective risk model has been studied by various authors, including Freifelder (1974), Millar and Hickman (1974) and Klugman *et al.* (1998). Schmidt (1998) presented a unifying survey of Bayesian models in different areas of actuarial mathematics. Nevertheless, relatively little has been published on the analysis of Bayesian robustness in this model; some examples in this field are Heilman and Schröter (1987), Eichenauer *et al.* (1988), Young (1998), Insua *et al.* (1999) and Gómez *et al.* (1999 a; b; 2000; 2002 a; b).

For the random variable "number of claims", we consider a Poisson distribution with a parameter θ, denominated $P(\theta)$, thus obtaining the Compound Poisson distribution. The excellent study by Panjer and Willmot (1983) showed that the Compound Poisson distribution arises in many situations in the Risk Theory, and an sample bibliography on the question was provided. This model has been well known since at least 1920 (Keffer, 1929). It has also been long known that it is difficult to obtain explicit expressions for the distribution of the total cost. Among the large volume of work on this question, let us highlight the studies by Panjer (1980; 1981), Sundt and Jewell (1981) and Willmot (1986), who presented recursive formulas by which an approximate calculation can be made of the distribution of total claims. Kozubowski and Panorska (2005) presented a set of interesting results on the sum of random variables with an Exponential distribution and a random number of summands. Nadarajah and Kotz (2006 a, b) presented a comprehensive collection of approximate forms for the Compound mixed Poisson Distribution.

For the parameter θ of the distribution $P(\theta)$, we consider a prior $Gamma(a,b)$ distribution. There is significant empirical support for the use of the Gamma distribution in this model (see Dropkin, 1959; Nye and Hofflander, 1988; Ellis, Gallup and McGuire, 1990). Alternative models to the Poisson distribution have been proposed in many papers, see for example Willmot (1986; 1988), Ruohonen (1988) and Hürlimann (1990). Variations of the hypothesis of the Gamma distribution can be found in Venter (1991), who also provided a comprehensive bibliography on the matter. Tremblay (1992) presented the use of a Poisson distribution with an Inverse Gaussian distribution in bonus-malus systems.

For the amount or severity of each claim, we consider an Exponential distribution of the parameter λ, denominated $Exp(\lambda)$; our study is analogous to the method used by Frangos and Vrontos (2001), who incorporated a comprehensive bibliography on the subject and designed a bonus-malus system. For the parameter λ of the distribution $Exp(\lambda)$, we consider a $Gamma(c,d)$ distribution.

In the present paper, we do not seek to obtain explicit, simple expressions, in the knowledge that these cannot be derived in the compound collective model except in very special cases. We have also chosen not to use special functions that are obviously involved in the calculations performed (for example, the Bessel functions), unlike the practice of, for example, Nadarajah and Kotz (2006 a; b), because in our opinion such a practice does not contribute to facilitating the calculation of the results that are obtained. The emphasis in our study is placed on real, practical calculability, at a low computing cost.

The paper is organized as follows:

- Section 2 addresses the analysis of the model to be considered.
- Section 3 describes the True Individual Premium, the prior premium and the posterior premium for the Net Premium Principle.
- Section 4 analyzes the robustness of the posterior premium, on the one hand with respect to the specification of the prior distribution of λ and, on the other, with respect to the specification of the prior distribution of θ. In both cases, the hypothesis of independence between θ and λ is maintained.
- In Section 5 we draw some conclusions and comment upon questions that remain open to further study.

As was to be expected, and as commented upon above, in this study we do not obtain explicit expressions with formats similar to those commonly described, but nevertheless all the expressions used can be calculated straightforwardly, with simple computer programs, available on request from the authors.

2. Setting out the Model

Let K be the random variable “number of claims", and assume it has a Poisson distribution of parameter $\theta > 0$, denominated $P(\theta)$; therefore

$$Prob\{K = k\} = p(k/\theta) = \frac{1}{k!}e^{-\theta}\theta^k; \quad k = 0,1,2,\dots$$

Let X_i be the random variable “individual cost of the i-th claim", and assume it has an Exponential distribution of parameter $\lambda \geq 0$, denominated $Exp(\lambda)$; therefore

$$f(x_i/\lambda) = \lambda e^{-\lambda x_i}, \quad x_i.$$

In the compound collective model, we are interested in the random variable “total cost", denominated X, and which is defined by

$$X = \begin{cases} 0; & \mathrm{K} = 0 \\ \sum_{\mathrm{i}=1}^{\mathrm{k}} \mathrm{X_i}; & \mathrm{K} \geq 1 \end{cases}$$

Suppose that X represents the total claim size of a portfolio at the end of a fixed time period. The distribution of the random variable X and therefore the likelihood of the problem in the compound collective model is obtained as follows:

$$Prob\{X \leq x\} = \sum_{k=0}^{\infty} p(k|\theta) Pr\left[\left(\sum_{i=1}^{k} X_i\right) \leq x\right].$$

For the case in question, it is well known (see Gómez, 1996) that the likelihood in the compound collective model for the Poisson–Exponential pair is expressed as

$$L(x/\theta,\lambda) = \begin{cases} \sum_{n=1}^{\infty} \frac{\lambda^n x^{n-1} e^{-\lambda x}}{(n-1)!} \frac{e^{-\theta}\theta^n}{n!}, & x > 0, \\ e^{-\theta}, & x = 0. \end{cases}$$

Assume that the parameters θ and λ are independent, and let us specify a prior Gamma distribution for each of them (which in both cases is the conjugate prior distribution),

$$\pi_{10}(\theta) = \frac{b^a}{\Gamma(a)}\theta^{a-1}e^{-b\theta}; \quad \pi_{01}(\lambda) = \frac{d^c}{\Gamma(c)}\lambda^{c-1}e^{-d\lambda}.$$

Therefore, the joint prior distribution is

$$\pi_0(\theta,\lambda) = \pi_{10}(\theta) \cdot \pi_{01}(\lambda);$$

for positive θ and λ; a, b, c and d are positive, known constants.

Referring the notation to $\pi_{10}(\theta)$, the value of mathematical expectation is a/b, and that of the variance is a/b^2; the distribution is unimodal when $a > 1$ and in this case, the value of the mode is $\theta_0 = (a-1)/b$; Pearson's coefficient of asymmetry is never annulled and the central moment of order 3 is only annulled when $a = 0$.

By direct integration, it is straightforward to find that the marginal distribution of x is expressed as follows:

$$m(x/\pi_0) = \int_\theta \int_\lambda L(x/,\lambda)\pi_0(\theta,\lambda)d\theta d\lambda$$

$$= \frac{b^a}{\Gamma(a)} \frac{d^c}{\Gamma(c)} \frac{1}{(b+1)^a} \frac{1}{(x+d)^c} \sum_{n=1}^{\infty} \frac{x^{n-1}\Gamma(a+n)\Gamma(c+n)}{(n-1)!n!(b+1)^n(x+d)^n}, \quad x > 0,$$

$$m(0/\pi_0) = \int_\theta \int_\lambda e^{-\theta}\pi_0(\theta,\lambda)d\theta d\lambda = \left(\frac{b}{b+1}\right)^a;$$

and therefore,

$$m(x/\pi_0) = \begin{cases} \frac{b^a d^c}{\Gamma(a)\Gamma(c)(b+1)^a(x+d)^c} \sum_{n=1}^{\infty} T_n, & x > 0 \\ \left(\frac{b}{b+1}\right)^a, & x = 0 \end{cases}$$

denoting,

$$T_n \equiv T(n;x,a,b,d) = \frac{x^{n-1}\Gamma(a+n)\Gamma(c+n)}{(n-1)!n!(b+1)^n(x+d)^n}.$$

It is straightforward to show that the series in the first row of expression ref: 3 is a convergent series of positive terms for any positive value of a, b, c, d and x.

We now provide an example in which the marginal distribution $m(x|\pi_0)$ is determined.

Calculation and graphical representation of the Marginal Distribution.

Assume the following prior distributions are specified:

$\theta \prec Gamma(2,7)$ *and* $\lambda \prec Gamma(5,3)$. Then,

$$m(x|\pi_0) = \begin{cases} \frac{49 \cdot 243}{1! \cdot 4! \cdot 64(x+3)^5} \sum_{n=1}^{\infty} T_n, & x > 0 \\ \left(\frac{49}{64}\right), & x = 0 \end{cases},$$

the values of which are incorporated into the Figure ref: Fig1.

The posterior distribution of (θ,λ) given the sampling observation x, is obtained as follows:

$$\pi_0(\theta,\lambda/x) = \begin{cases} \dfrac{\theta^{a-1}\lambda^{c-1}e^{-(b+1)\theta}e^{-(x+d)\lambda}\frac{1}{x}\sum_n \frac{(\lambda x\theta)^n}{(n-1)!n!}}{\frac{1}{(b+1)^a}\frac{1}{(x+d)^c}\sum_n T_n}, & x > 0 \\ \dfrac{(b+1)^a d^c}{\Gamma(a)\Gamma(c)}\theta^{a-1}\lambda^{c-1}e^{-(b+1)\theta}e^{-d\lambda}, & x = 0 \end{cases}$$

It is straightforward to show that the series of the numerator in the first row in expression ref: 5 is a convergent series of positive terms for any positive value of θ, a, b, c, d and x. The series of the denominator are the same as that in expression ref: 3.

3. The Net Premium Principle

The following Lemmas 1 and 2 are well known (see, for example, Gómez, 1996). We reproduce them here for the sake of completeness, and merely sketch out the proof.

Lemma *The True Individual Premium, P, is equal to the product of the expected number of claims and the expected cost; in symbolic form, this is expressed as:*

$$P = \theta \cdot \frac{1}{\lambda}.$$

Lemma *The prior premium, P', is obtained with the following expression:*

$$P'[\pi_0(\theta,\lambda)] = \frac{ad}{b(c-1)}.$$

Lemma *The posterior premium, P^*, is expressed by*

$$P^* = \begin{cases} \dfrac{x+d}{b+1}\dfrac{\sum_{n=1}^{\infty}\frac{a+n}{c+n-1}T_n}{\sum_{n=1}^{\infty}T_n}, & x > 0 \\ \dfrac{ad}{(b+1)(c-1)}, & x = 0 \end{cases}$$

where x is the total claim amount generated by n claims produced in a single–period of time.

Calculation and graphical representation of the posterior premium. We continue to consider the prior procedure used in Example 1, such that

$$P^*[\pi_0(\theta,\lambda|x)] = \begin{cases} \dfrac{(x+3)}{8}\dfrac{\sum_{n=1}^{\infty}\frac{2+n}{4+n}T_n}{\sum_{n=1}^{\infty}T_n}, & x > 0 \\ \dfrac{6}{32}, & x = 0 \end{cases}$$

Figure ref: Fig2 extracts the values of the posterior premium.

4. Analysis of Robustness

In this section, we examine, independently, the analysis of Bayesian robustness for each of the two parameters θ and λ, and of the likelihood, with respect to the specified prior distribution.

The analysis carried out is based on contamination classes (see Sivaganesan and Berger, 1987, 1989; Sivaganesan, 1988, 1989, 1991; and Berger, 1994), in which it is assumed that the prior distribution of the parameter, denominated ϕ, belongs to a class of possible distributions of probability defined by the contamination of a singular prior distribution, considering various contaminant classes. Specifically, this approach consists in assuming that a singular prior distribution $\pi(\phi)$ is specified for the parameter ϕ, but that there exists a degree of uncertainty concerning this specification, this uncertainty being quantified by the amount ε; in other words, it can only be specified that the prior distribution of ϕ belongs to a class of probability distributions taking the following form:

$$G_{\phi}(\pi,\varepsilon) = \{\pi^{c}(\phi) = (1-\varepsilon)\pi(\phi) + \varepsilon q(\phi); q \in Q\},$$

where

$\pi(\phi)$ is the singular prior distribution specified for ϕ;

$\varepsilon \in [0,1]$ is the degree of contamination; and

Q is the class of contaminant distributions of probability, the definition of which incorporates non-renounceable aspects of the prior distribution of ϕ.

An extreme case would be: $Q_1 =$. Another case we will examine is that of $Q_2 =$. We write $G_{\phi}^{(i)}(\pi,\varepsilon)$, with $i = 1,2$, to indicate that the contaminant class is Q_i.

The aim of the present study is to analyze the range of variation of the magnitude of interest, which in this case is the posterior premium:

- on the one hand, when the prior distribution of λ varies within a class of contamination distributions, for different degrees of contamination, i.e. for different values of ε. The corresponding prior distributions are expressed as
  $\pi_0^{2c}(\theta,\lambda) = \pi_{10}(\theta)\pi^{c}(\lambda)$, with $\pi^{c}(\lambda) \in G_{\lambda}^{(i)}(\pi_{01},\varepsilon)$.
- on the other hand, when the prior distribution of θ varies within a class of contamination distributions, for different degrees of contamination, i.e. for different values of ε. The corresponding prior distributions are expressed as
  $\pi_0^{1c}(\theta,\lambda) = \pi^{c}(\theta)\pi_{01}(\lambda)$, with $\pi^{c}(\theta) \in G_{\theta}^{(i)}(\pi_{10},\varepsilon)$.

Throughout the analysis, we maintain the hypothesis that λ and θ are independent. For the purposes of the present study, the following results are useful:

Lemma *If $A > 0$, $f(x)$ and $g(x)$ are continuous functions with $g(x) \geq 0$, then,*

$$_{dF(x)}\left[_{dF(x)}\right]\frac{B + \int f(x)dF(x)}{A + \int g(x)dF(x)} = {}_{x}\left[_{x}\right]\frac{B + f(x)}{A + g(x)},$$

where the upper (lower) is taken for all the probability distributions $dF(x)$, and where A, B, $f(x)$, $g(x)$ are such that the upper (lower) of $\frac{B+f(x)}{A+g(x)}$ is obtained for any value of x.

Lemma *Let $q(\phi)$ be a unimodal distribution with mode in ϕ_0 and let $h(\phi)$ be a function of ϕ; then*

$$\int_{\phi} h(\phi)q(\phi)d\phi = \int_{z} h^{*}(z)dF(z),$$

where F(z) is a distribution function and $h^*(z) = \begin{cases} \frac{1}{z}\int_{\phi_0}^{\phi_0+z} h(\phi)d\phi; & z \neq 0 \\ h(\phi_0); & z = 0 \end{cases}$

Lemma *For any pair of real numbers a and b such that a<b and* $\forall n \in Z^+$, *the following equality is found:*

$$\int_a^b e^{-\lambda x}\lambda^n d\lambda = \sum_{k=0}^{n} \frac{n!}{(n-k)!}\frac{1}{x^{k+1}}[a^{n-k}e^{-ax} - b^{n-k}e^{-bx}].$$

Lemma *For* $\pi_0^{2c}(\theta,\lambda) = \pi_{10}(\theta)\pi^c(\lambda)$, *we have* $\pi^c(\lambda) \in G_\lambda^{(i)}(\pi_{01},\varepsilon)$ *and it is found that*

$$P^*[\pi_0^{2c}(\theta,\lambda/x)] = \begin{cases} \dfrac{A_0 + \int_\lambda A_1(\lambda)q(\lambda)d\lambda}{A_2 + \int_\lambda A_3(\lambda)q(\lambda)d\lambda}; & x \neq 0 \\ \dfrac{\frac{1-\varepsilon}{\varepsilon}\frac{ad}{(c-1)(b+1)} + \frac{a}{b+1}\int_\lambda \frac{1}{\lambda}q(\lambda)d\lambda}{\frac{1-\varepsilon}{\varepsilon}+1}; & x = 0 \end{cases}$$

where,

$$A_0 \equiv A_0(a,b,c,d,x,\varepsilon) = \frac{1-\varepsilon}{\varepsilon}\frac{d^c}{\Gamma(c)(b+1)(x+d)^{c-1}}\sum_{n=1}^{\infty}\frac{a+n}{n+c-1}T_n;$$

$$A_1(\lambda) \equiv A_1(a,b,c,d,x,\lambda) = \frac{1}{b+1}\frac{e^{-\lambda x}}{\lambda}\sum_{n=1}^{\infty}\frac{x^{n-1}\Gamma(a+n+1)\lambda^n}{(n-1)!n!(b+1)^n};$$

$$A_2 \equiv A_2(a,b,c,d,x,\varepsilon) = \frac{1-\varepsilon}{\varepsilon}\frac{d^c}{\Gamma(c)(x+d)^c}\sum_{n=1}^{\infty}T_n;$$

$$A_3(\lambda) \equiv A_3(a,b,c,d,x,\lambda) = e^{-\lambda x}\sum_{n=1}^{\infty}\frac{x^{n-1}\Gamma(a+n)\lambda^n}{(n-1)!n!(b+1)^n}.$$

Lemma *For* $\pi_0^{1c}(\theta,\lambda) = \pi^c(\theta)\pi_{01}(\lambda)$, *with* $\pi^c(\theta) \in G_\theta^{(i)}(\pi_{10},\varepsilon)$ *we find that,*

$$P^*[\pi_0^{1c}(\theta,\lambda/x)] = \begin{cases} \dfrac{B_0 + \int_\theta B_1(\theta)q(\theta)d\theta}{B_2 + \int_\theta B_3(\theta)q(\theta)d\theta}; & x \neq 0 \\ \dfrac{\frac{1-\varepsilon}{\varepsilon}\frac{adb^a}{(c-1)(b+1)^{a+1}} + \frac{d}{c-1}\int_\theta \theta e^{-\theta}q(\theta)d\theta}{\frac{1-\varepsilon}{\varepsilon}\left(\frac{b}{b+1}\right)^a + \int_\theta e^{-\theta}q(\theta)d\theta}; & x = 0 \end{cases}$$

where,

$$B_0 \equiv B_0(a,b,c,d,x,\varepsilon) = \frac{1-\varepsilon}{\varepsilon}\frac{b^a}{\Gamma(a)(b+1)^{a+1}}\sum_{n=1}^{\infty}\frac{a+n}{c+n-1}T_n;$$

$$B_1(\theta) \equiv B_1(c,d,x,\theta) = \theta e^{-\theta} \sum_{n=1}^{\infty} \frac{x^{n-1}\Gamma(c+n-1)\theta^n}{(n-1)!n!(x+d)^n};$$

$$B_2 \equiv B_2(a,b,c,d,x,\varepsilon) = \frac{1-\varepsilon}{\varepsilon} \frac{b^a}{\Gamma(a)(b+1)^a(x+d)} \sum_{n=1}^{\infty} T_n;$$

$$B_3(\theta) \equiv B_3(c,d,x,\theta) = \frac{e^{-\theta}}{x+d} \sum_{n=1}^{\infty} \frac{\Gamma(n+c)x^{n-1}\theta^n}{(n-1)!n!(x+d)^n}.$$

Note that the serie that appears in expression ref: 18 is the same as the one in expression ref: 8 and that the serie in expression ref: 20 is the same as the one in expression ref: 3. It can be shown that the series in expressions ref: 19 and ref: 21 are convergent series of positive terms for any value of c, d, x and θ.

5. Analysis of robustness for the prior distribution of the parameter "individual cost of each claim"

The following results show that the range of variation of the posterior premium P^*, when the prior distribution of λ varies within a contamination class as $G_\lambda^{(1)}(\pi_{01},\varepsilon)$, can be obtained by calculating the upper and the lower of a real function of a real variable.

Theorem *The range of variation of the posterior premium when the prior distribution of λ belongs to the class $G_\lambda^{(1)}(\pi_{01},\varepsilon)$ can be calculated by determining the range of variation of a function of λ. Specifically, the following equality is confirmed, and the equality is also valid when the upper is replaced by the lower.*

$$\pi^c(\lambda)\in G_\lambda^{(1)}(\pi_{01},\varepsilon) \; P^*[\pi^{2c}(\theta,\lambda/x)] = \lambda \begin{cases} \dfrac{A_0+A_1(\lambda)}{A_2+A_3(\lambda)}; & x \neq 0 \\[2ex] \dfrac{\frac{1-\varepsilon}{\varepsilon}\frac{ad}{(c-1)(b+1)}+\frac{a}{b+1}\frac{1}{\lambda}}{\frac{1-\varepsilon}{\varepsilon}+1}; & x = 0 \end{cases}$$

where A_0, $A_1(\lambda)$, A_2 and $A_3(\lambda)$ are as in Lemma 7.

Calculation of the range of variation of the posterior premium when the prior distribution of the parameter "distribution of the severity of the accident" belongs to a class of contamination in which the contaminant class is that of all the probability distributions. In this example, we use the prior data derived in Examples 1 and 2 to illustrate the result of Theorem 1.

In order to measure the Bayesian sensitivity, or the robustness of the intervals calculated, we use a normalized measure of relative sensitivity, defined in Sivaganesan (1991), the R.S. sensitivity factor, which is expressed as:

$$RS = \frac{(Sup(P^*) - Inf(P^*))}{2P^*} \times 100.$$

Table ref: tab1 shows the minima, maxima and the R.S. sensitivity factor for the application considered.

We now address the analysis of the robustness for the contamination class $G_\lambda^{(2)}(\pi_{01},\varepsilon)$, and obviously in this case it must be assumed that c is greater than 1 and that the mode is $\lambda_0 = (c-1)/d$. The following result shows that the problem of searching for the upper and lower of the posterior premium when the prior distribution of λ belongs to the class $G_\lambda^{(2)}(\pi_{01},\varepsilon)$ can be transformed into the search for the upper and lower, respectively, of a real function of a real variable.

Theorem *The range of variation of the posterior premium when the prior distribution of λ belongs to the class $G_\lambda^{(2)}(\pi_{01},\varepsilon)$ can be calculated by determining the range of variation of a function of a real variable. Specifically, the following equality is confirmed, and the equality is also valid when the upper is replaced by the lower.*

$$\pi^c(\lambda)\in G_\lambda^{(2)}(\pi_{01},\varepsilon)\quad P^*[\pi^{2c}(\theta,\lambda/x)] = z\begin{cases}\frac{A_0+A_1^*(z)}{A_2+A_3^*(z)}; & x \neq 0\\ A_4^*(z); & x = 0\end{cases}$$

where A_0 and A_2 are as in Lemma 7 and $A_1^*(z)$, $A_3^*(z)$ and $A_4^*(z)$ are given by

$$A_1^*(z) = \begin{cases} k_1\frac{1-e^{-xz}}{z} + \frac{1}{z}\frac{1}{(b+1)^2}\sum_{n=1}^{\infty}\frac{x^n\Gamma(a+n+2)}{(n+1)!(b+1)^n}[U_n(x,\lambda_0) - V_n(x,\lambda_0,z)]; & z \neq 0\\ \frac{1}{b+1}\frac{e^{-\lambda_0 x}}{\lambda_0}\sum_{n=1}^{\infty}\frac{x^{n-1}\Gamma(a+n+1)\lambda_0^n}{(n-1)!n!(b+1)^n}; & z = 0\end{cases}$$

$$A_3^*(z) = \begin{cases} \frac{1}{z}\sum_{n=1}^{\infty}\frac{x^{n-1}\Gamma(a+n)}{(n-1)!(b+1)^n}[U_n(x,\lambda_0) - V_n(x,\lambda_0,z)]; & z \neq 0\\ e^{-\lambda_0 x}\sum_{n=1}^{\infty}\frac{x^{n-1}\Gamma(a+n)\lambda_0^n}{(n-1)!n!(b+1)^n}; & z = 0\end{cases}$$

$$A_4^*(z) = \begin{cases} \frac{1-\varepsilon}{\varepsilon}\frac{ad}{(c-1)(b+1)} + \frac{a}{b+1}\frac{1}{z}\ln\frac{\lambda_0+z}{\lambda_0}; & z \neq 0\\ \frac{\frac{1-\varepsilon}{\varepsilon}\frac{ad}{(c-1)(b+1)}+\frac{a}{b+1}\frac{1}{\lambda_0}}{\frac{1-\varepsilon}{\varepsilon}+1}; & z = 0\end{cases}$$

using the notation

$$k_1 \equiv k_1(a,b,x,\lambda_0) = \frac{\Gamma(a+2)e^{-x\lambda_0}}{(b+1)^2 x}.$$

$$U_n(x,\lambda_0) = \frac{e^{-\lambda_0 x}}{x}\sum_{k=0}^{n}\frac{\lambda_0^{n-k}}{(n-k)!}\frac{1}{x^k}.$$

$$V_n(x,\lambda_0,z) = \frac{e^{-(\lambda_0+z)x}}{x}\sum_{k=0}^{n}\frac{(\lambda_0+z)^{n-k}}{(n-k)!}\frac{1}{x^k}.$$

It can be shown that the series in the first rows in expressions ref: 25 and ref: 26 are convergent series of positive terms for any value of a, b, x, λ_0 and z. The series in the second rows of these expressions are analogous to those in the series for expression ref: 11.

Calculation of the range of variation of the posterior premium when the prior distribution of the parameter “distribution of the severity of the accident´´ belongs to a contamination class in which the contaminant is that of all the unimodal probability distributions with the same mode. The data elicited are the same as in the previous examples. Table ref: tab2 shows ranges of variation and sesitivity factor for Theorem 2. To illustrate Theorem 1 and 2 results, Figure ref: Fig3 analyzes the range of variation of the premium for $x = 1$.

6. Robustness analysis for the prior distribution of the parameter “number of claims"

In this section, we analyze the Bayesian robustness for the parameter θ of likelihood, with respect to the specified prior distribution. As in the previous section, the analysis carried out is based on contamination classes.

In the following result, parallel to Theorem 1, it is apparent that the problem of searching for the upper and the lower of the posterior premium when the prior distribution of θ belongs to the class $G_\theta^{(1)}(\pi_{10}, \varepsilon)$ can be transformed into the search for the upper and the lower, respectively, of a real function of the real variable θ.

Theorem *The range of variation of the posterior premium when the prior distribution of θ belongs to the class $G_\theta^{(1)}(\pi_{10}, \varepsilon)$ can be calculated by determining the range of variation of a function of θ. Specifically, the following equality is found, which is also valid when the upper is replaced by the lower.*

$$\underset{\pi^c(\theta)\in G_\theta^{(1)}(\pi_{10},\varepsilon)}{} P^*[\pi^{1c}(\theta, \lambda/x)] = \underset{\theta}{} \begin{cases} \frac{B_0+B_1(\theta)}{B_2+B_3(\theta)}; & x \neq 0 \\ \frac{\frac{1-\varepsilon}{\varepsilon}\frac{adb^a}{(c-1)(b+1)^{a+1}}+\frac{d}{c-1}\theta e^{-\theta}}{\frac{1-\varepsilon}{\varepsilon}\left(\frac{b}{b+1}\right)^a+e^{-\theta}}; & x = 0 \end{cases}$$

where B_0, $B_1(\theta)$, B_2 and $B_3(\theta)$ are as in Lemma 8.

Calculation of the range of variation of the posterior premium when the prior distribution of the parameter “number of accidents" belongs to a class of contamination in which the contaminant class is that of all the probability distributions. Calculation of minima, maxima and R.S. sensitivity factors, taking the same prior assumptions as in the previous examples, i.e. that the prior parameters are a=2, b=7, c=5, d=3 , and applying Theorem 3. Table ref: tab3 describes the minimas, maximas and R.S. factors considering this Theorem.

Let us now consider the robustness analysis for $G_\theta^{(2)}(\pi_{10}, \varepsilon)$, assuming that $a > 1$ and that $\theta_0 = (a-1)/b$ is the value of the mode. In the following result, parallel to that of Theorem 2, it is clear that the problem of seeking the upper and the lower of the posterior premium when the prior distribution of θ belongs to the class $G_\theta^{(2)}(\pi_{10}, \varepsilon)$ can be transformed into the search for the upper and the lower, respectively, of a real function of a real variable.

Theorem *The range of variation of the posteriori premium when the prior distribution of θ belongs to the class $G_\theta^{(2)}(\pi_{10}, \varepsilon)$ can be calculated by determining the range of variation of a real function of a real variable. Specifically, the following equality is confirmed, and this is equally valid when the upper is replaced by the lower.*

$$\underset{\pi^c(\theta)\in G_\theta^{(2)}(\pi_{10},\varepsilon)}{} P^* = \underset{z}{} \begin{cases} \frac{B_0+B_1^*(z)}{B_2+B_3^*(z)}; & x \neq 0 \\ B_4^*(z); & x = 0 \end{cases}$$

where B_0 and B_2 are as in Lemma 8 and $B_1^*(z)$, $B_3^*(z)$ and $B_4^*(z)$ are given by

$$B_1^*(z) = \begin{cases} \frac{1}{z}\sum_{n=1}^{\infty} \frac{x^{n-1}\Gamma(c+n-1)(n+1)!}{(n-1)!n!(x+d)^n}\left[U_{n+1}(1,\theta_0) - V_{n+1}(1,\theta_0,z)\right]; & z \neq 0 \\ \theta_0 e^{-\theta_0}\sum_{n=1}^{\infty} \frac{x^{n-1}\Gamma(c+n-1)\lambda^n}{(n-1)!n!(x+d)^n}; & z = 0 \end{cases}$$

$$B_3^*(z) = \begin{cases} \frac{1}{z}\sum_{n=1}^{\infty} \frac{x^{n-1}\Gamma(n+c)}{(n-1)!(x+d)^{n+1}}\left[U_n(1,\theta_0) - V_n(1,\theta_0,z)\right]; & z \neq 0 \\ \frac{e^{-\theta_0}}{x+d}\sum_{n=1}^{\infty} \frac{x^{n-1}\Gamma(n+c)\theta_0^n}{(n-1)!n!(x+d)^n}; & z = 0 \end{cases}$$

$$B_4^*(z) = \begin{cases} \frac{k_2+\frac{d}{c-1}\left[(\theta_0+1)e^{-\theta_0}-(\theta_0+z+1)e^{-(\theta_0+z)}\right]}{k_3+\left[e^{-\theta_0}-e^{-(\theta_0+z)}\right]}; & z \neq 0 \\ \frac{k_2+\frac{d}{c-1}\theta_0 e^{-\theta_0}}{k_3+e^{-\theta_0}}; & z = 0 \end{cases}$$

using the notation,

$$k_2 \equiv k_2(a,b,c,d,\varepsilon) = \frac{1-\varepsilon}{\varepsilon}\frac{adb^a}{(c-1)(b+1)^{a+1}};$$

$$k_3 \equiv k_3(a,b,\varepsilon) = \frac{1-\varepsilon}{\varepsilon}\left(\frac{b}{b+1}\right)^a.$$

The proof of the convergence of the series that appear in expressions ref: 31 and ref: 32 is analogous to that performed for the convergence of the series in expressions ref: 25 and ref: 26.

Calculation of the range of variation of the posterior premium when the prior distribution of the parameter "number of accidents" belongs to a class of contamination in which the contaminant class is that of all the unimodal distributions with the same mode. Calculation of the minima, maxima and R.S. sensitivity factors, taking into account the same prior assumptions as in the above examples, i.e. that the prior parameters are $a=2$, $b=7$, $c=5$, $d=3$ and then applying Theorem 4. Finally, Table ref: tab4 shows the results for Theorem 4 and Figure ref: Fig4 analyzes the ranges of variation comparing Theorem 3 and 4 for $x = 1$.

7. Conclusions and further Lines of Research

In this study we have carried out a Bayesian analysis of the Compound Collective Model, taking into consideration a Poisson distribution for the number of claims, and an Exponential distribution for the severity of each accident. The analysis was performed under the assumption that the non-negative random variable *X* represents the total claim size of a portfolio at the end of a fixed time period.

We analyzed the robustness of the posterior premium, or Bayes premium, using the Net Premium Principle, with respect to the prior distribution of the parameter "number of claims" and with respect to the prior distribution of the parameter "distribution of the severity of accidents"; this was done in an independent way for each, maintaining the hypothesis of independence between the parameters. For each of the two robustness analyses, we used classes of contamination, considering two contaminant classes, that of all the distributions, and that of all the unimodal distributions with the same mode. In every case considered, the optimization problem was transformed into one of the maxima and minima of a real function of a real variable. The results are illustrated numerically.

We consider the following problems to be of interest, and these are proposed as lines for future research:

- To study the same problem for other principles of premium calculation, such as the Principle of Variance, the Exponential Principle or the Esscher Principle.
- To study the same problem, with the Net Premium Principle, but analyzing the robustness of the posterior premium with respect to modifications of the joint two-dimensional prior distribution of the two parameters underlying the problem.
- A similar study but assuming *t* periods of observations for the number of claims and the claims size.

8. References

Berger, J. (1994). "An Overview of Robust Bayesian Analysis". Test, 3–1; 5–124.

Dropkin, L.B. (1959). "Some Considerations on Automobile Rating Systems Utilizing Individual Driving Records". PCAS XLVI.

Eichenauer, J.; Lehn, J. and Rettig, S. (1988). "A Gamma-Minimax Result in Credibility Theory". Insurance: Mathematics and Economics, 2; 49–57.

Ellis, R.; Gallup, C. and McGuire, T. (1990). "Should Medical Professional Liability Insurance Be Experience Rated?" The Journal of Risk and Insurance, 57,1.

Feller, W. (1971). *An Introduction to Probability Theory and its applications*. Vol. 2, 2ª edition. John Wiley.

Frangos, N.E. and Vrontos, S.D. (2001). "Design of Optimal Bonus-Malus Systems with a frequency and a severity component on an individual basis in automobile insurance". Astin Bulletin, 31–1; 1–22.

Freifelder, F. (1974). "Statistical Decision Theory and Credibility Theory and Procedures". In *Credibility Theory and Applications*, Edited by P.M. Kahn, Academic Press, 71–88.

Gómez, E. (1996). "Estadística Bayesiana en Credibilidad con aplicación a la fijación de primas de seguros". Tesis doctoral. Universidad de Las Palmas. Spain.

Gómez, E.; Hernández, A. and Vázquez, F.J. (1999 a). "The Variance Premium Principle: A Bayesian Robustness Analysis". Actuarial Research Clearing House, 1; 1–12.

Gómez, E.; Hernández, A. and Vázquez, F.J. (1999 b). "The Esscher Premium Principle in Risk Theory: A Bayesian Sensitivity Study". Insurance: Mathematics and Economics, 25; 387–395.

Gómez, E.; Hernández, A. and Vázquez, F.J. (2002 a). "Bounds for Ratios of Posterior Expectations: Applications in the Collective Risk Model". Scandinavian Actuarial Journal, 1; 37–42.

Gómez, E.; Hernández, A. ; Pérez, J.M. and Vázquez, F.J. (2002 b). "Measuring Sensitivity in a Bonus-Malus System". Insurance: Mathematics and Economics, 31; 105–113.

Gómez, E.; Vázquez, F.J and Hernández, A. (2000). "Robust Bayesian Premium Principles in Actuarial Science". Journal of The Royal Statistical Society- Series D- The

Statistician, 49-2; 1–12.

Heilman, W. and Schröter, K.J. (1987). “On the Robustness of Premium Principles". Insurance: Mathematics and Economics, 6; 145–149.

Hürlimann, W. (1990). “On Maximum Likelihood Estimation for Count Data Models". Insurance: Mathematics and Economics, 9, 1.

Insua, S.R.; Martin, J.; Insua, D.R. and Ruggeri, F. (1999). “Bayesian Forecasting for Accident Proneness Evaluation". Scandinavian Actuarial Journal, 2; 134–156.

Keffer, R. (1929). “An Experience Rating Formula". Transactions of the Actuarial Society of America, 30.

Klugman, S.; Panjer, H. and Willmot, G. (1998). *Loss Models From Data to Decisions*. John Wiley and Sons.

Kozubowski, T.J. and Panoska, A.K. (2005). “A mixed bivariate distribution with exponential and geometric marginals". Journal of Statistical Planning and Inference, 134, 501–520.

Miller, R. and Hickman, J. (1974). “Insurance Credibility Theory and Bayesian Estimation". In *Credibility Theory and Applications*, Edited by P.M. Kahn, Academic Press, 249–270.

Nadarajah, S. and Kotz, S. (2006 a). “Compound mixed Poisson distributions I". Scandinavian Actuarial Journal, 3; 141–162.

Nadarajah, S. and Kotz, S. (2006 b). “Compound mixed Poisson distributions II". Scandinavian Actuarial Journal, 3; 163–181.

Nye, B. and Hofflander, A. (1988). “Experience Rating in Medical Professional Liability Insurance". The Journal of Risk and Insurance, 60, 1.

Panjer, H.H. (1980). “The Aggregate Claims Distribution and Stop-Loss Reinsurance". Trans of the Society of Actuaries, XXXII.

Panjer, H.H. (1981). “Recursive evaluation of a family of Compound Distributions". Astin Bulletin, 12; 22–26.

Panjer, H.H. and Willmot, G.E. (1983). “Compound Poisson Models in Actuarial Risk Theory". Journal of Econometrics, 23; 63–76.

Ruohonen, M. (1988). “A Model for the Claim Number Process". Astin Bulletin, 18, 1.

Schmidt, K.D. (1998). “Bayesian models in Actuarial Mathematics". Mathematical Methods of Operational Research, 48; 117–146.

Sivaganesan, S. (1988). “Range of Posterior Measures for Priors with Arbitrary Contaminations". Comm. Stat. Theory and Methods, 17–5; 1591–1612.

Sivaganesan, S. (1989). “Sensitivity of Posterior Mean to Unimodality Preserving Contaminations". Statistics and Decisions, 7; 77–93.

Sivaganesan, S. and Berger, J. (1987). “Ranges of Posterior Measures for Priors with Unimodal Contaminations". Technical Report, Dpt. Statistics, Univ. Purdue.

Sivaganesan, S. and Berger, J, (1989). “Ranges of Posterior Measures for Priors with Unimodal Contaminations". The Annals of Statistics, 17–2; 868–889.

Sivaganesan, S. (1991). “Sensitivity of Some Posterior Summaries when the Prior is Unimodal with Specified Quantiles". The Canadian Journal of Statistics, 19–1; 57–65.

Sundt, B. and Jewel, W.S. (1981). “Further Results on Recursive Evaluation of Compound Distributions". Astin Bulletin, 12; 27–39.

Tremblay, L. (1992). “Using the Poisson Inverse Gaussian in Bonus-Malus Systems". Astin Bulletin, 22–1; 97–106.

Venter, G.G. (1991). “Effects of Variations from Gamma-Poisson Assumptions". Proceedings of the Casualty Actuarial Society, LXXVIII; 41–56.

Willmot, G.E. (1986). “Mixed Compound Poisson Distributions". Astin Bulletin, 16S

Willmot, G. (1988).“Mixed Compound Poisson Distributions". Astin Bulletin, 16; S59–S79.

Young, V.R. (1998). “Robust Bayesian Credibility using Semiparametic Models". Astin Bulletin, 28, 1; 187–203.

Figure 1.

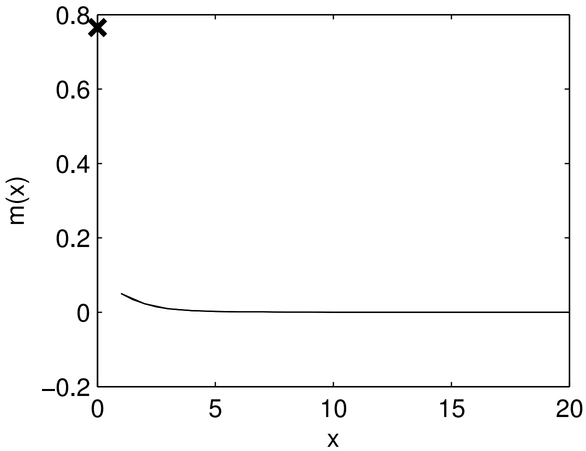

Figure 2.

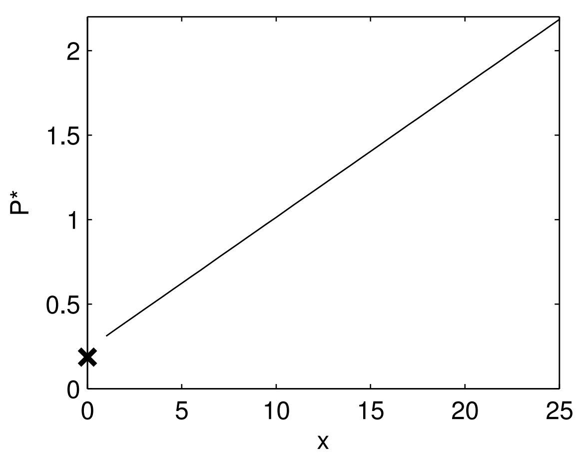

Figure 3.

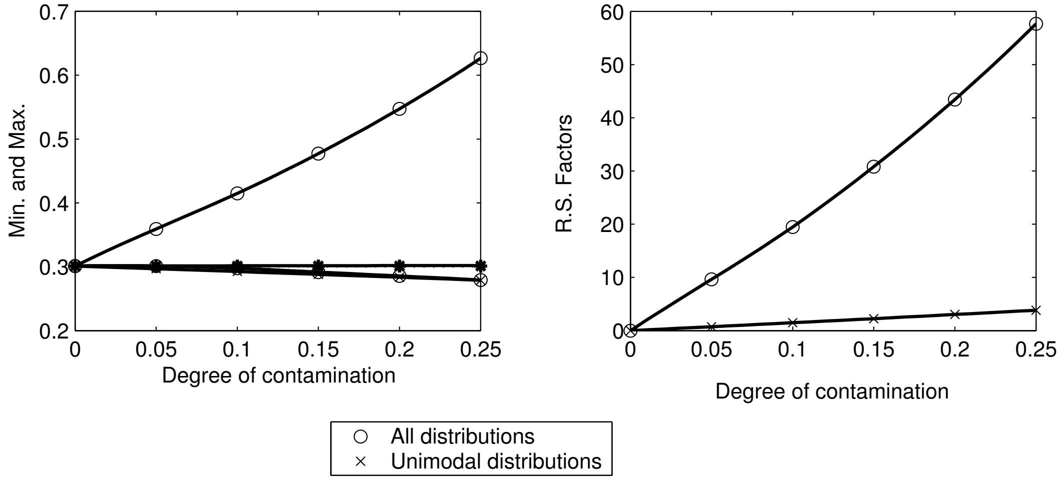

Figure 4.