

ThE Papers 05/07



Departamento de Teoría e Historia Económica

Universidad de Granada

Mathematical needs and economic interpretations

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¹ My thanks to Victoria Chick and Sheila Dow for valuable comments on an earlier draft of this paper and helpful discussions of these issues.

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Abstract

This paper aims at showing that the disconnection between formal structures and theoretical content in economic model building could lead to undesirable consequences. In this sense, linking formal and verbal contents by means of coherent and relevant interpretations is worth the effort. For it is a relatively simple way of helping to improve the usefulness of mathematical economic theorization. By way of illustration of these ideas, the paper discusses the interpretation which Arrow and Debreu themselves put on the inclusion of free goods in their proof of existence of a general equilibrium.

Key words: Economic Methodology, Existence, General Equilibrium Theory, Interpretation, Mathematical Economics.

1. Introduction

The application of mathematical techniques to the analysis of economic problems is a methodological possibility. During the last sixty years, this methodological option has been linked to the history of a significant part of economic theory. Particularly, in theoretical model building, mathematical techniques are a basic tool used in order to link theoretical models' assumptions to their conclusions. In accordance with this situation, the term *mathematization* will be used in this paper to make reference to “a methodology which requires that all arguments be expressed, or at least expressible, in mathematics” (Chick and Dow, 2001, p. 705)².

This process of mathematization has brought about important benefits (see, e. g., Backhouse (1997, ch. 10), and Katzner (2003, pp. 565-567)). Particularly, it has given economic analysis “greater logical solidity” (Debreu, 1991, p. 3). Nevertheless, the implementation of this methodological option has produced important modifications in the nature of economic analysis itself. In this regard the way in which the mathematization of economics has often been put into

² For a deeper discussion of the meaning of the term *mathematization* -and its relation to the concepts of *formalization* and *axiomatization*- see Backhouse (1998, pp. 1848-1850), and Chick (1998, pp. 1859-1861).

practice implies a new order of priorities. According to this order, the use of mathematics becomes, from a methodological view, an end in itself. This does not mean that mathematical economics bears no relation to economic reality. However, it implies that economists have accepted that the construction of economic analysis -even when its relation to economic reality or economic intuition is not too solid- is methodologically correct if formal or mathematical criteria of deductive reasoning are satisfied. That is, standard practice in mathematical economics suggests that, in case of conflict, there is no problem in considering the formal rigor³ and clarity which the use of mathematics gives to economic arguments as more important aims than other methodological goals such as describing or explaining economic life's real workings or linking theorization to empirical evidence⁴.

Leaving aside this conflict between mathematical economics and the realism of its constructions, one of the consequences which stems from the status of mathematization as an end in itself is that the economic content of a formal model or its possible interpretations are secondary to the building of a mathematically correct formal structure⁵ (Chick, 1998, p. 1865).

³ For the different ways in which the notion of *rigor* can be defined, see Davidson (2003), Dow (2003), and Weintraub (1998, and 2002).

⁴ In this sense, Rosenberg (1994, p. 230) states that "the mystery surrounding the actual development of economic theory -its shifts in formalism, its insulation from empirical assessment, its interest in proving purely formal, abstract possibilities ... - can be comprehended and properly appreciated if we give up on the notion that economics any longer has the aims or makes the claims of an empirical science of human behaviour. Rather we should view it as a branch of mathematics". The list of authors who have pointed out methodological problems associated with the mathematization of economics is very long. Some relevant works are Blaug (1999), Chick (1998), Chick and Dow (2001), Clower (1994 and 1995), Coates (1996), Coddington (1975), Comim (2000, and 2002), Davidson (2003), Davis (1999), Dow (2003), Hausman (1981, pp. 28ff.), Hutchison (1994, pp. 289ff., and 1999, pp. 250ff.), or Katzner (2002, and 2003).

⁵ Following Gibbard and Varian (1978, p. 666), the term *formal structure* of an economic model -or *theory in the strict sense*- makes reference to the set formed by that model's formal assumptions, the theorems which follow from such assumptions, and the mathematical and logical developments which

In this sense, Debreu -taking into consideration that he “appears to equate mathematisation and axiomatisation” (Backhouse, 1998, p. 1849n)- establishes in the preface of *Theory of Value* (p. VIII) that the axiomatization of economic theory is based on the principle that *theory in the strict sense* and interpretations are logically disconnected. Moreover, the former is autonomous. Indeed, in “Theoretic Models”, Debreu (1986, p. 1265) explicitly states that “an axiomatized theory has a mathematical form that is completely separated from its economic content. If one removes the economic interpretation of the primitive concepts, of the assumptions, and of the conclusions of the model, its bare mathematical structure must still stand”.

The usefulness of disassociating formal theory from its interpretations, however, has not been underlined only by Debreu. As Vilks (1999, p. 158) points out, “this distinction can be found, more or less explicitly, in many if not most contributions to mathematical economics after *Theory of Value*”.

To illustrate the principle established in the *Theory of Value*, Debreu mentions how Arrow modified the interpretation of the concept of *commodity* when he added to the traditional characteristics which define a commodity the state of the world in which it will be available. However, the postulated separation between formal structure and interpretation could be sidestepping, on occasion, the difficulties in giving economic content to formal contributions. In this sense, note that the above quoted principle, instead of stating that economic content must precede and be the basis of the formal structure which expresses it, proposes the reverse order. That is, this principle legitimizes a situation in which the first and fundamental step in the construction of a model is given by the elaboration of its formal structure, regardless of the fact that the interpretation later provided could lack coherence or relevance -i.e., “immediate practical impact” (Leontief, 1971, p. 1)- and without even having any guarantee that the formal structure can be interpreted at all.

connect deductively those theorems with the assumptions. According to this paper’s point of view, the concept of *model* includes both the model’s formal structure and its interpretation(s). The notion of *interpretation* will be defined in the next section.

This paper's aim is to discuss how and why taking the disconnection between formal structures and interpretations as a methodological starting point for model building could sometimes lead to undesirable and unexpected consequences. In addition, its objective is to show that linking formal and verbal contents by means of coherent and relevant interpretations is worth the effort. For it is a relatively simple way of helping to improve the methodological legitimacy of mathematical economic theorization. To illustrate such aims, part of the argumentation will be focused on a particular problem concerning the proof of existence of general competitive equilibrium (PEGE) provided by Arrow and Debreu (1954) (AD); namely, the way in which these authors interpreted the introduction of free goods in general equilibrium analysis.

Three reasons justify the choice of AD's (1954) model as a case-study. First, AD provided an interpretation of the formal aspects of their model. Second, their interpretation is problematic in a sense which makes the discussion of this model particularly illustrative of this paper's aims. Third, this model has an undeniable relevance in the history of formal economic analysis.

Nevertheless, before we proceed, it might be advisable to emphasize that this paper concerns itself with interpretations and aim at suggesting a simple condition which could contribute to the improvement of mathematical economics. Accordingly, the concrete model which AD (1954) built is taken into consideration, on the one hand, for the purpose of showing that this paper's conclusions are not just *a priori* statements, but are based on the way in which the development of economic analysis takes place. On the other hand, the free-goods problem in AD's (1954) model is discussed for illustrative purposes; that is, for the aim of illustrating how a particular formal structure and its interpretations interacts. This means that this paper is not interested, for example, in the realistic/unrealistic nature of AD's (1954) model. But in the consequences which might stem from the problematic nature of its interpretations. In addition, it means that this paper will not discuss analytical possibilities which were not considered by AD (1954) -for instance, how free goods would be affected if monotonic preferences were assumed

or if negative prices were possible in order to analyze the specific problems posed by the interpretations of this model's formal structure⁶.

In this sense, every single formal structure has its own interpretative solutions and problems. There is no sense, therefore, in asserting that the interpretative problems of a specific model are not worthy of consideration because there exists a previous, contemporary or later model which adopts a different analytical approach to the same subject. Hence, inquiring, for instance, how the particular formal structure of a model like AD's (1954) model can be interpreted is itself a relevant question. There would be no point in the attempt to dispute its relevance by arguing, for example, that AD (1954) could have assumed a different approach. In short, previous formal structures will be modified, improved and even superseded by other formal structures. Nevertheless, using this argument to discredit the methodological conclusions which stem from a discussion about the difficulty in interpreting a formal structure which is a landmark in the history of economic analysis implies, implicitly, that we have to be ready to accept that discussing either the interpretative or analytical problems of any contemporary formal structure becomes superfluous. For, in the future, there will always be a different formal structure which will propose an alternative analytical view.

For a suitable discussion of these aims, this paper has been divided into three sections. In Section 1, the concept of *interpretation* is defined. In Section 2, the problematic relation between interpretations and formal structures in model building is illustrated by analyzing the inclusion of free goods in AD's (1954) proof of existence. The concluding section draws the paper to a close by stressing the main implications which can be drawn from the fact that a certain part of the mathematization of theoretical economics is carried out, implicitly or explicitly, on the basis that mathematical reasoning is not subject to economic content.

⁶ A contemporary paper of AD (1954) which assumes monotonic preferences is, for example, Nikaidô (1956). As regards negative equilibrium prices, after the publication of AD (1954), subsequent theoretical developments introduced this possibility. One of the ways in which such a possibility has been put forward assumes that the set of prices, instead of being the unit simplex, is either the closed unit ball or the unit sphere. See, for example, Bergstrom (1976), Debreu (1956), and Hart and Kuhn (1975).

2. The meaning of *interpretation*

Following Gibbard and Varian's concept of *story*, the notion of *interpretations of a model's formal structure* will be defined as the set of verbal argumentations and definitions which try to give economic verbal content to that model's formal variables, assumptions, deductions, and theorems. To illustrate it with an extremely simple example: by means of its interpretation, we can go from the formal expression $y_j = (y_{j1}, -y_{j2}, \dots, -y_{jl})$, where $y_{j1} = F(y_{j2}, \dots, y_{jl})$, to deal with a firm which would be using certain quantities of production factors to produce efficiently a certain amount of output.

Rubinstein (1991, p. 909) has defined *interpretation* as "a mapping which links a formal theory with everyday language". However, this definition is, in a certain sense, too restrictive. For, besides interpretations which are based on everyday language, there are interpretations which express the verbal content of a formal structure by means of the technical language provided by economic theory. In fact, the main aspect concerning interpretations is not the kind of language in which they are expressed; but the fact that they give verbal content to formal structures by making use of economic theory *in a non-strict sense* (following Debreu's terminology), arguments which are supposed to be intuitively evident, or common sense. Indeed, we should not lose sight of the fact that the reverse is often true too; i.e., many contributions to formal economic theory have their roots in economic intuition, common sense, and ordinary logic (see, e.g., Chick and Dow (2001); Coates (1996); Comim (2000, and 2002); and Davis (1999)).

Accordingly, in our view, interpretations involve an attempt to justify, by making use of non-formal economic theory, economic intuition or common sense, why a certain formal element is introduced in a model. This does not imply that, once interpreted, it becomes *true* in some sense, empirically correct, or necessarily acceptable. At best, what it means is that, without the justification provided by the corresponding interpretation, that formal element is flatly unacceptable -or that its only justification is a mathematical need lacking economic content. By way of illustration, consider the formal assumption that consumer's preferences

are strictly convex. This formal assumption can be viewed, in everyday language, as the formal expression of a basic inclination of agents for diversity; that is, the expression of people's preference for balanced consumption plans, as opposed to plans which are made up of large amounts of a few goods and very small quantities of the rest of them. But it can also be interpreted technically, i.e., in terms of economic theory. Particularly, the strict convexity assumption can be given economic verbal content by means of the concept of *diminishing marginal rate of substitution*.

However, even if this assumption is interpretable in both ways, it is still, as Cornwall (1984, p. 35), for instance, has suggested, a "somewhat restrictive" condition placed on agents' preferences. Therefore, mathematical or analytical needs apart, there is no room for stating that it is either true or necessarily acceptable in order to understand agents' behavior. Nevertheless, at the same time, given that it is interpretable, we have arguments -non-mathematical arguments- which help to justify why strict convexity can be assumed in the theory of consumer behavior. At least, to say it more prosaically, transmitting the economic knowledge implicit in mathematical models to, for instance, students of economic theory would be easier if formal structures were coherently interpretable. That is, non-mathematical explanations do not make assumptions true or realistic, but help to justify why we assume what we assume.

But why is interpretation important as far as economic modelization is concerned? Other reasons apart, the answer to this question, once again, has its roots in the fact that empirical criteria do not normally play a definitive role in relation to the problem of theory selection in actual economics (see, for instance, Hands (1984, p. 59)). Given this situation, an ideal solution to improve the methodological status of economic theorization would be to propose that any economic model which is not empirically testable or does not satisfy the empirical criteria which have been agreed should be unacceptable. Thus, most of the interpretative problems concerning the mathematization of economics would probably vanish. Nevertheless, this kind of solution is sort of unrealistic. For it is based on the belief that things can be improved much more than what seems possible.

Probably, these are the reasons why different authors have tried to establish the usefulness of economic theoretical models on a different basis. In this sense, according to Hausman (1992), economic models would be useful for “conceptual exploration” (see, also, Backhouse (1999, p. 297)). Other authors have not resigned themselves to the fact that the usefulness of economic models is only conceptual exploration, and have proposed a way of understanding them which attempts to bridge the gap between modelization and the aims of explaining and describing economic reality. In this spirit, assuming that many models are theoretical constructions which would not stand rigorous empirical testing, Gibbard and Varian (1978), for instance, have stated that they could be considered “caricatures of economic reality”. More recently, Sugden (2000) have described them as “credible worlds”.

But the point is that the statement that economic modelization is useful in any of these senses would be underpinned if formal structures were coherently interpretable. Otherwise, we would run the risk of building mathematical constructions which lack relevant, coherent or just articulated verbal content. In other words, we would have accepted a flaw which can be avoided in order to improve economic models’ usefulness, whether for conceptual exploration, for drawing caricatures of economic reality, for the creation of credible worlds, or just for the process of thinking economically.

In order to illustrate these ideas, the significance of interpreting formal structures will be analyzed below by means of interpretations provided by economic theory itself. These interpretations, as far as AD’s (1954) formal structure is concerned, follow the interpretative criterion which Davis mentions in relation to the *Theory of Value*; i.e., regarding this book “interpretation can never be more than an illustration of the logic of his [of Debreu’s] assumptions on production and consumption sets and maximization behavior” (1999, p. 506). The problem, nevertheless, is that following this interpretative criterion -as the interpretations considered do- does not provide a coherent interpretation of AD’s (1954) formal structure. That is, as a result of the disconnection between formal and economic content, this formal structure cannot be coherently interpreted even in terms of AD’s own logic of maximizing agents.

3. Mathematical needs and economic interpretations: by way of example.

PEGEs are, if not “the crowning achievement of mathematical economics” (Rosenberg 1994, p. 229), one of its most outstanding achievements. Far from the scarcely rigorous procedure of counting equations and unknowns -which had already been questioned by Neisser and Stackelberg at the beginning of the 1930s- PEGEs establish the set of conditions which guarantee, in mathematical terms, that the interaction of a great number of agents who decide independently and autonomously gives rise to a state of equilibrium in which every agent, given the available resources, maximizes his/her aims.

Although Wald had already provided a PEGE at the beginning of the 1930s (see, e.g., Arrow and Debreu (1954, pp. 60-61), Arrow and Hahn (1971, pp. 8-9), Wald, (1951 [1936]), and Weintraub (1983, pp. 3-12)), AD’s paper “Existence of an Equilibrium for a Competitive Economy” is usually considered the beginning of “the contemporary phase of development [as regards PEGE]” (Debreu, 1986, p. 1268). In a similar sense, Weintraub (2002, p. 183) asserts that “economists with even scant knowledge of the history of economics can identify Kenneth Arrow and Gerard Debreu’s (1954) *Econometrica* paper as having provided the proof that settled the issue [of PEGE]”.

In this paper, AD model a competitive economy in which there are a finite number of commodities, l , of production units, n , and of consumers, m , which are denoted by the letters h, j and i , respectively. On the basis provided by the assumptions which AD (1954, pp. 267-70) make in relation to producers and consumers, they establish theorem I, which asserts the existence of general equilibrium. According to their definition, and taking into consideration that x_i and y_j are, respectively, a possible consumption plan of the i^{th} consumption unit and a possible production plan of the j^{th} production unit, a general equilibrium is a set of vectors $(x^*_1, \dots, x^*_m, y^*_1, \dots, y^*_n, p^*)$ which satisfy the following four conditions:

“1. y^*_j maximizes $p^* \cdot y_j$ over the set Y_j , for each j

“2. x^*_i maximizes $u_i(x_i)$ over the set $\{x_i/ x_i \in X_i, p^* \cdot x_i \leq p^* \cdot \zeta_i + \sum_{j=1}^n \alpha_{ij} \cdot p^* \cdot y^*_j\}$

“3. $p^* \in P = \{p/ p \in R^l, p \geq 0, \sum_{h=1}^l p_h = 1\}$

“4. $z^* \leq 0$, $p^* \cdot z^* = 0$ ” (1954, pp. 268-71),

where $P \subset R^l$ is the set of prices; $p \in P$ is a l -dimensional price vector; $Y_j \subset R^l$ is the set of possible production plans; $p \cdot y_j$ is the profit made by the j^{th} production unit if, at prices p , it chooses the production plan y_j ; $u_i(x_i)$ is the utility function of the i^{th} consumer; $\zeta_i \in R^l$ is the vector of initial holdings of the i^{th} consumer; α_{ij} is the share of the i^{th} consumer in the profit of the j^{th} production unit; $p \cdot x_i \leq p \cdot \zeta_i + \sum_{j=1}^n \alpha_{ij} \cdot p \cdot y_j$ is the set of vectors which satisfy the budget restraint at prices p ; $z \in R^l$ is a vector whose components denote the excess of demand over supply on each of the l markets; 0 is a l -dimensional vector whose components are zero; asterisks denote equilibrium quantities; and $x \geq x'$ means that, for every component h of such vectors, $x_h \geq x'_h$.

The fourth of the above conditions (C4) implies that, “for some commodity h ..., either $z^*_h = 0$ or $z^*_h < 0$ and $p^*_h = 0$ ” (AD, 1954, pp. 271-272); that is, at equilibrium, supply equals demand on all markets, with the exceptions of markets for free goods⁷, in which supply, at a zero equilibrium price, exceeds demand.

As regards our aims, the question lies in determining why AD’s (1954) proof of existence needs a definition of general equilibrium which -following the line opened by Wald- takes free goods into consideration. In other words, why the condition C4 does not restrict the definition of general equilibrium -according to what can be considered the usual convention in standard economic theory- to those situations in which demand for all commodities equals supply. In this regard, note that later theoretical developments have been focused on the construction of PEGEs which either minimize the effects of free goods on the definition of equilibrium by assuming free disposal or do not take them into consideration at all (see, e.g., Bergstrom (1976), or Debreu (1959)).

⁷ A free good is a useful but a not scarce good, such that, at equilibrium, the supply of it exceeds its demand, and its price -in the sense of social opportunity cost- is zero.

Mathematical needs.

In order to proof Theorem I, AD (1954) make use of an economy whose particular form is the following one: “ $E = [X_1, \dots, X_m, Y_1, \dots, Y_n, P, u_1(x_1), \dots, u_m(x_m), p \cdot y_1, \dots, p \cdot y_n, p \cdot z, A_1(x'_1), \dots, A_m(x'_m), Y_1, \dots, Y_n, P]$ ” (1954, p. 274), where $A_i(x'_i)$ is the budget constraint of the i^{th} consumer, and $p \cdot z = \sum_{h=1}^l p_h \cdot z_h$ is the pay-off function of an agent named *the market participant* -to which we will return below.

The conditions required in order to guarantee the existence of general equilibrium in this economy are based on a lemma (see AD (1954, p. 274)) which extends to an abstract economy the theorem of existence of an equilibrium point in n -person games put forward by Nash (1950). Both this theorem and the lemma are founded on the generalization of Brower’s fixed point theorem carried out by Kakutani (see Debreu (1982, pp. 698-701), and Weintraub (1983, pp. 24-5)). One of these conditions⁸ is that the set of possible price vectors, P , must be closed, bounded, convex, and non-empty. To satisfy this requirement, AD (1954) chose the following mathematical possibility (for other possibilities, see n. 5): on the basis of the homogeneity of degree zero in prices of the excess demand function, P is considered to be given by the $l-1$ -dimensional closed unit simplex; that is, possible price vectors are those whose l components add to one and are non-negative. Thus, P is defined in the following way: $P = \{p/p \in R^l, p \geq 0, \sum_{h=1}^l p_h = 1\}$, which is the third formal condition (C3) that characterizes AD’s (1954) definition of general equilibrium

The condition C3 guarantees that the set of possible price vectors satisfies, in particular, the requirement of being a closed set. However, it does so at the cost of accepting the possibility that, at equilibrium, some prices can be zero. Moreover, a possible equilibrium price vector

⁸ The conditions required for the existence of a general equilibrium are the following ones -with $i = 1, \dots, m$, and $j = 1, \dots, n$: (1) X_i, Y_j and P must be compact and convex sets; (2) $u_i(x_i), p \cdot y_j$ and $p \cdot z$ must be continuous and quasi-concave functions; (3) Y_j and P must be continuous and non-empty; (4) $A_i(x'_i)$ must be a continuous function whose graph is a closed set, and for every x'_i , the set $A_i(x'_i)$ must be convex and non-empty.

could be one which involves that all prices but one are zero. This possibility, however, would be problematic unless the presence of free goods is accepted. For scarce commodities cannot have, by definition, zero equilibrium prices. Accordingly, once AD (1954) open the possibility that some prices can be zero, they add that, if a commodity's equilibrium price were zero, this commodity would be a free good. As regards this kind of commodities, taking into consideration that AD (1954) do not accept negative prices, the law of supply and demand would not clear their markets: at a zero equilibrium price, supply exceeds demand on these markets. Expressed in terms of the model's formal structure: from " $z^* \leq 0, p^* \cdot z^* = 0$ " the consequence is drawn that " $z^*_h < 0$ for some commodity h is possible, but only if $p^*_h = 0$ " (AD, 1954, p. 271).

Thus, it could be stated, with Blaug (1999, p. 275), that "formalism always solves the problems it itself creates": in order to satisfy the formal condition which requires the set of possible prices to be closed, AD (1954) assumed that prices could be zero. This, also, provoked a new problem, whose solution was achieved just by qualifying the idea that economics deals with scarce goods; i.e., to solve this new problem, AD resorted to free goods. They even accepted that, according to C3 and C4, a possible general equilibrium is given by a situation which implies that all commodities of the economy but one are free goods⁹.

Economic interpretations.

In the preceding section, continuous references have been made to concepts such as *price*, *commodity* or *demand*. Without these references, we could say that analytical economics is just a branch of applied mathematics (see Blaug (1999, p. 276), Clower (1995, pp. 311-316), Hands (1984, p. 65), or Rosenberg (1994, p. 230)). Besides these references, the preceding section has shown how the need to solve a mathematical problem -namely, satisfying the

⁹ As regards the argument which states that the inclusion of free goods in PEGEs can be justified by stating that whether the scarce or free nature of goods cannot be assumed, but is discovered by market forces, see below.

requirements of Kakutani's theorem- gave rise to an incompatibility with the economic concept of *scarcity*. This incompatibility becomes apparent in the fact that AD (1954) included free goods in their model. In this regard, the inclusion of free goods has been referred to as a theoretical improvement (see, e. g., Arrow and Hahn (1971, p. 22)). But, as we have already mentioned, later models have constructed proofs of existence which either exclude free goods or minimize their effects on the definition of equilibrium -even at the cost of assuming free disposal.

If this were the whole story, we could conclude that defining general equilibrium in a way in which supply can exceed demand at equilibrium would be an *ad hoc* strategy -in the sense which Hands (1988, p. 132) mentions: "for economist theorists, the sin of ad hocness seems to be infidelity to the metaphysical presuppositions of the neoclassical program rather than face-saving adjustments in response to recalcitrant data". That is, we would have an *ad hoc* manoeuvre if the way in which concepts are normally used in economic theory is modified in order to guarantee that, mathematically, everything is correct.

Nevertheless, AD (1954) provided an economic interpretation which tried to justify why their definition of general equilibrium includes free goods. The question is therefore to appraise this interpretation.

This interpretation is based on adding to consumers and producers a third type of agent, named the *market participant*. In particular, showing the compatibility between the condition C4 and the market participant's behavior AD (1954) provide an interpretation of the inclusion of free goods in their PEGE based on this agent's behavior and the law of supply and demand¹⁰.

¹⁰ It is advisable to make a brief comment regarding the way in which AD (1954) use the market participant in order to interpret the inclusion of free goods in their proof of existence. When the condition C4 is introduced, in Section 1.4.1. of AD (1954), it is directly interpreted in terms of the "dynamic picture of the classical 'law of supply and demand'" (1954, p. 271). The market participant makes his/her appearance in Section 3. In particular, in Section 3.1.1., the form of the pay-off function of the market participant is justified, too, in terms of "the classical 'law of supply and demand'" (AD, p. 275). As these quotations show, the connection between the arguments in Section 1.4.1 and Section 3.1.1. is very close.

This agent's aim is to maximize his/her pay-off function, $p \cdot z^{11}$. To achieve this goal, AD (1954, pp. 274-275) assume that "the market participant does not maximize instantaneously [the function $p \cdot z$] but, taking other participants' choices as given, adjusts his choice of prices so as to increase his pay-off". That is, taking z as given, the market participant would maximize his/her pay-off function by modifying $p \in P$. Therefore, his/her behavior would be given by a norm which, according to AD (1954, p. 275), would fit "the classical 'law of supply and demand'": "For given z , $p \cdot z$ is a linear function of p ; it can be increased by increasing p_h for those commodities for which $z_h > 0$, decreasing p_h if $z_h < 0$ (provided p_h is not already 0)".

As a result, for a particular value of the excess demands, z^0 , the market participant's behavior would consist in solving the following optimization problem:

$$\text{Max. } p \cdot z^0, \text{ with } p \in P.$$

If excess demands -"taking other participants' choices as given"- are at their equilibrium level -i. e., $z^0 = z^*$ - then the solution of the problem above would be the equilibrium price vector, p^* :

$$\text{Max. } p \cdot z^* = p^* \cdot z^*, \text{ with } z^0 = z^* \text{ and } p \in P.$$

Expressed in a different way:

$$p^* \cdot z^* \geq p \cdot z^*, \text{ with } p \in P.$$

If this expression is combined with Walras' Law -namely, $p \cdot z = 0$ (see AD (1954, pp. 272 and 275)- the following inequality could be stated:

So much so that AD, just after their reference to the law of supply and demand in the latter Section, recommend to consider again what they had stated as regards this law in the former Section. Nevertheless, besides the "intuitive comments" which AD make in relation to the pay-off function of the market participant in Section 3.1.1., they assert that the justification of the form of the pay-off function of the market participant is to be found in Section 3.2. In this Section, by showing the compatibility between the condition C4 and the market participant's behavior, AD provide an interpretation of the inclusion of free goods in their PEGE based on this agent's behavior. This interpretation is discussed below.

¹¹ The conception of the market as a maximizer agent had already been put forward, *mutatis mutandis*, by Debreu (1951, p. 288). Nikaidô (1956), too, based his PEGE on a very similar idea.

$$“0 = p^* \cdot z^* \geq \delta^h \cdot z^* = z^*_h” \text{ (AD, 1954, p. 276),}$$

where $\delta^h \in P$ is a price vector “in which every component is 0, except the h^{th} , which is 1” (1954, p. 276). If the value of h is modified from one to l , the following series of inequalities would be obtained:

$$0 = p^* \cdot z^* \geq \delta^1 \cdot z^* = z^*_1$$

$$0 = p^* \cdot z^* \geq \delta^2 \cdot z^* = z^*_2$$

...

$$0 = p^* \cdot z^* \geq \delta^l \cdot z^* = z^*_l.$$

From these inequalities, the following conclusion is drawn:

$$“z^* \leq \mathbf{0}” \text{ (AD, 1954, p. 276).}$$

That is, at equilibrium, the value of the excess demand can be zero or negative. This conclusion, besides Walras’ Law, involves that, if $p^*_h = 0$, then, $z^*_h < 0$.

Thus, the behavior of the market participant provides an interpretation of the formal condition C4¹². In particular, given that AD (1954) state that this economic agent’s behavior fits the law of supply and demand in competitive economies, the presence of free goods in general equilibrium could be interpreted -and, therefore, explained and justified- in terms of this law. Consequently, C4 would not only be one of the formal elements on which AD’s (1954) proof of

¹² Note that this interpretation is just the result of combining two ideas. On the one hand, the intuitive statement that “the market participant does not maximize instantaneously [the function $p \cdot z$] but, taking other participants’ choices as given, adjusts his choice of prices so as to increase his pay-off” (AD 1954, pp. 274-5). On the other hand, the fact that, given a certain value of the excess demands, z^0 , the problem $Max. p \cdot z^0$, with $p \in P$, synthesizes the market participant’s behavior. In other words, the interpretation of C4 is based on the idea that, once consumers and producers choose their consumption and production plans respectively, we would have the excess demand vector, z^0 , which gives the market participant the opportunity for maximizing the function $p \cdot z^0$.

existence is based. In addition, it would be interpretable, by means of the market participant, according to one of the most widely accepted theoretical ideas in economics.

Nevertheless, the expounded interpretation is untenable. But the cause is not that it is empirically false, excessively abstract or fictitious. It is untenable in relation to certain fundamental aspects of the formal structure of AD's (1954) model itself. In particular, the problem stems from the fact that there is a contradiction between, on the hand, the way in which AD interprets the formal implications of the first two conditions which define general equilibrium (C1 and C2), and, on the other hand, the interpretation of C4 and its implications.

According to C1 -namely, y^*_j maximizes $p^* \cdot y_j$ over the set Y_j , for each j - a certain producer's behavior would consist in choosing the production plan(s) which, "taking prices as given" (AD, 1954, p. 268), maximize(s) his/her profits. In addition, in accordance with C2 - namely, x^*_i maximizes $u_i(x_i)$ over the set $\{x_i / x_i \in X_i, p^* \cdot x_i \leq p^* \cdot \zeta_i + \sum_{j=1}^n \alpha_{ij} \cdot p^* \cdot y^*_{jj}\}$ - the consumer's problem is to choose the consumption vector(s) which maximize(s) his/her utility function and "whose cost at market prices does not exceed the individual's income" (AD, 1954, p. 270). Therefore, from C1 and C2, AD (1954, p. 271) draw the following conclusion: "the supply and demand for all commodities is determined as a function of p (not necessarily single valued) if we vary p and at the same time instruct each production and consumption unit to behave as if the announced value of p were the equilibrium value".

That is, given that individual demand and supply functions depend on prices, such functions will only have concrete values when we have a particular price vector. However, the problems of how prices are established and how the price system co-ordinates agents' actions in a competitive economy remain unresolved in economic terms. For "the assumption of perfect competition ... left no room for any actual agent to change price" (Hahn, 1982, p. 745). To solve these problems AD (1954) make use of a recurrent interpretation in the neo-Walrasian tradition, i.e., they assume the existence of a being to which we will refer by the well-known concept of

*Walrasian auctioneer*¹³. This auctioneer announces a price vector so that consumers and producers, given their demand and supply functions, calculate their consumption and production plans under such prices. If these plans were not an equilibrium allocation, economic agents would not carry out their plans and the auctioneer would modify prices again in order to influence economic agents' supply and demand decisions.

Note that the previous reasoning, which involves a dynamic process of *tâtonnement*, has value only as a verbal interpretation of the formal structure of AD's (1954) model, which is a static model whose equilibrium is found by solving a set of simultaneous equations. Indeed, right at the beginning of their paper, AD (1954, p. 266) state clearly that they are not concerned with investigating the stability of equilibrium, a study which "would require specification of the dynamics of a competitive market". However, at the same time, in order to provide an interpretation of their static formal structure, AD themselves use dynamic arguments. First, when they refer to the process of *tâtonnement* which has just been mentioned (AD, 1954, p. 271). Second, when, in relation to the market participant's behavior, they state that "the market participant does not maximize instantaneously..." (AD, 1954, p. 275). Third, when they talk about "the dynamic picture of the classical 'law of supply and demand'" (AD, 1954, p. 271). Accordingly, the fact that AD's paper does not investigate the problem of stability of equilibrium bears no relation to the fact that they use dynamic verbal arguments to interpret their static formal structure.

Once the Walrasian auctioneer has been introduced in the interpretation of the formal structure of AD's (1954) model, we can go on to argue in terms of the excess demand function. Given the way in which this function is constructed, C1 and C2 imply that the excess demand function depends on prices too:

$$z(p) = \sum_{i=1}^m x_i(p) - \sum_{j=1}^n y_j(p) - \sum_{i=1}^m \zeta_i.$$

¹³ AD (1954) do not mention explicitly the Walrasian auctioneer. However, as the quotation above shows, they are implicitly referring to him/her when they describe the process of *tâtonnement*: "...if we vary p and at the same time instruct each production and consumption unit to behave as if the announced value of p were the equilibrium value".

Therefore, prices are the exogenous variable of the system of simultaneous equations composed of the excess demand functions of each of the markets. When interpreting this formal consequence, the task of modifying this independent variable is delegated to the Walrasian auctioneer, assuming that, by means of the process of *tâtonnement*, a general equilibrium will be achieved. In this sense, the auctioneer has been widely accepted in the neo-Walrasian tradition as an interpretative keystone of competitive markets because his/her behavior corresponds to the law of supply and demand: if the demand for (supply of) a commodity exceeds supply (demand), the auctioneer would rise (lower) the price of that commodity. But it would behave this way in order to decrease the value of the excess demand (supply). That is, if Giffen goods are excluded, the price of a commodity and the excess demand for it are mutually and inversely related. Thus, if at p_h the excess demand for the commodity h is z_h , and at p'_h it is z'_h , then, in the case that p_h were lower than p'_h , we would have that z_h is higher than z'_h , and so we could not conclude, *a priori*, whether $p_h \cdot z_h$ is higher or lower than $p'_h \cdot z'_h$. As a result, the following optimization problem makes no sense as far as the Walrasian auctioneer is concerned:

$$\text{Max. } p \cdot z^0, \text{ with } p \in P, \text{ and } z^0 \text{ constant.}$$

Neither does the following inequality:

$$p^* \cdot z^* \geq p \cdot z^*, \text{ with } p \in P.$$

However, as we have discussed, this inequality and this problem -which synthesize the market participant's behavior- are precisely the basis of AD's (1954) interpretation of the condition C4, which implies that, at equilibrium, excess demands are non-positive, $z^* \leq 0$.

In summary, as regards the interpretation of the formal structure of AD (1954), it can be stated the following: the formal implications which stem from C1 and C2 are interpreted by AD in terms of the Walrasian auctioneer, but the formal condition C4 is interpreted by means of the market participant. However, both interpretations -regardless of the fact that AD's (1954) model is impeccable from a mathematical view- are incompatible. Both the Walrasian auctioneer and the market participant, according to AD (1954), stand for market forces. More specifically, they represent the law of supply and demand in competitive economies. Nevertheless, their behaviors

are different. The auctioneer's behavior consists in modifying prices to influence the excess demand for the different commodities, and his/her aim -prices being the exogenous variable- is to find a price vector which gives rise to an equilibrium allocation. As regards the market participant, his/her behavior could be described as follows: "taking other participants' choices as given" -i.e., once the value of excess demands (which are the exogenous variable now) is determined- the market participant would modify prices with the aim of maximizing the function $p \cdot z$; but this modification of prices -in contrast to what has been stated in relation to the auctioneer- would not affect the value of excess demands.

Consequently, following the interpretations which AD (1954) provided, the workings of the economy can be described as a process divided in two phases. In the first stage, market forces -disguised as the Walrasian auctioneer- would announce a price vector, p^0 , which gives rise to a particular excess demand vector, z^0 . In the second stage, those forces -now disguised as the market participant- would take this z^0 as given and would find out the price vector, p^1 , which not only maximizes the function $p \cdot z^0$, but also does not cause any alteration to the value of z^0 . However, with respect to the same agents and commodities, we cannot postulate that prices influence excess demands and, at the same time, that excess demands remain unchanged under variations in prices. Put it simply, if coherence is an unavoidable value as far as economic interpretations are concerned, the AD's (1954) interpretative schema is untenable because of its own internal contradictions.

These contradictions are not (just) the result of using dynamic verbal arguments to interpret a static formal structure. Otherwise, no static model—particularly, no PEGE -could have ever been coherently interpreted or justified by resorting to verbal arguments like, for instance, the *tâtonnement* process or the Walrasian auctioneer. The problem is deeper. Its origin is to be found in the mathematical needs which led AD (1954) to assume a closed set of prices, P , and to include, accordingly, free goods in their proof of existence. In other words, it is not a problem of choosing between stability and existence. It is a problem of disconnection between formal and verbal contents.

The fact that a formal structure has not been interpreted in a coherent way does not imply, however, that the possibility of interpreting it coherently does not exist. In this sense, what we can assert in relation to AD's (1954) model is, on the one hand, that the interpretative schema which they proposed is untenable, and, on the other hand, that there has not been -and we have not been able to conceive of- an alternative way of interpreting that model's formal structure. That is, it could be said that AD's (1954) formal structure is waiting for a coherent interpretation.

Note in this regard that economic analysis has provided different interpretations concerning the inclusion of free goods in PEGE. For instance, Wald stated that "whether factors are free or scarce cannot be considered, a priori, a datum of the economy; it can only be determined on the basis of the production equations" (1951 [1936], p. 371). That is, according to this interpretation, which goods are scarce and how scarce they are become questions which must be solved by the system of equations which synthesizes the decisions of the individuals who participate in the market. But this interpretation is hardly compatible with the full-knowledge assumption on which the decisions of the agents of AD's (1954) model are based.

Moreover, AD's (1954) model, like any other theoretical model which provides a PEGE, aims at establishing the formal conditions under which a certain system of equations has a solution which is named *general competitive equilibrium*. Therefore, if a system of equations which satisfies these conditions is set out, the existence of a solution is guaranteed. However, the resolution of the system of equations has not discovered anything which had not been previously assumed -whether the scarce or free nature of the goods considered. In Katzner (2003, p. 565) words, "the notion that the introduction of mathematical argument would produce knowledge where none previously existed is specious".

Another example of alternative interpretation is Gale's (1955, pp. 159-160) interpretation, which is based on the idea that certain processes of production generate by-products. But this kind of interpretations cannot solve the interpretative problems of AD's (1954) model. For there exists a serious obstacle to the attempt to interpret this model's formal structure without the idea which underlies the market participant's behavior—that is, the idea

that there exists either *something* or *someone* that, given z^0 , maximizes the function $p \cdot z^0$. In this sense, if we ignore this idea and make use of an argument like Gale's one in order to interpret AD's (1954) formal structure, we might be able to elaborate an interpretation which justifies why AD included free goods in their PEGE. However, we could not interpret an essential aspect of that formal structure which was closely related to the problem of free goods by these authors; namely, we could not interpret the role of the function $p \cdot z$ in their proof of existence. In this regard, note, first, that this function is one of the keystones on which the application of Kakutani's theorem to AD's proof was based. Second, if we ignore *the idea which underlies* the market participant's behavior, we will probably have difficulties in interpreting the role of this function, because $p \cdot z$ is the market participant's pay-off function. Third, the fact that the market participant's pay-off function is closely related to the free goods problem has been discussed above.

In other words, in the case of AD's (1954) model, free goods, the condition C4, the market participant's behavior and his/her pay-off function are formal and interpretative aspects which can hardly be separated. Indeed, once AD put forward the market participant's pay-off function and her behavior, their aim is to show explicitly that this fictitious agent's pay-off function is consistent with C3 and C4 (see AD (1954, pp. 274-276; particularly, 3.1.1. and 3.2.)). But, at the same time, and as we have shown, AD's (1954) interpretative schema of their formal structure is untenable.

In conclusion, demanding -as Debreu (1959, p. VIII, and 1986, p. 1265) does- the autonomy of the formal structure of models with respect to their economic content involves the risk that, once the set of variables deductively linked by the mathematical machinery is put forward, we might have a formal structure which is not interpretable in a coherent way. In the particular case of condition C4, the analyzed interpretative problems are an example which shows explicitly this risk.

AD's (1954) model was a pioneering model which has inspired a large body of theoretical work. However, looking at the future, the above discussion of the interpretative problems of this model could help to understand some of the problems concerning the

methodological status of formal economic models and could cast some light on the ways to improve their usefulness -at an acceptable cost. In this sense, linking tightly the verbal and formal contents of economic models is an accessible path which could be followed in order to strengthen the usefulness of theoretical models, whether in Hausman's *conceptual exploration*, or in the creation of Sugden's *credible worlds*.

As a reflection of what has just been stated, when we have mathematically faultless formal structures which have not been interpreted in a coherent or relevant way, the transmission of economic knowledge by means of those formal structures becomes much harder -unless we give up the idea that mathematical economics differs and must differ from mathematics in that the former possesses economic content. In addition, when the development of economic analysis is achieved through this kind of models we are giving up a very simple way of connecting modelization either to economic theory in a non-strict sense or to economic intuition. That is, we are missing an opportunity to strengthen the acceptability of modeled economic knowledge.

4. Conclusion: the relevance of interpretations

Many examples could be mentioned to support the assertion that a typical feature of neoclassical thinking is that it takes into consideration those commodities which are scarce. In this sense, reference can be made to Robbins' (1949, p. 16) well-known definition of economic science, or to Walras' (1954 [1874], p. 65) statement that the object of economics is "the social wealth"—i.e., "all things, material or immaterial ... that are scarce". However, Menger deserves to be specially mentioned; because AD (1954, p. 272n) refer to him when they discuss the inclusion of free goods in their definition of general equilibrium According to Menger (1950 [1871], p. 99), as regards those goods whose available quantity exceeds the amount necessary to satisfy human needs -i.e., non-economic goods- economic activity is excluded.

However, AD (1954) -as Schlesinger and Wald did before- broke with the idea that the goods which deserve to be analyzed are only scarce goods. This break, in itself, is not negative. As a matter of fact, scientific development often happens as a consequence of a shift in a firmly

held (up to that moment) conception. In this regard, encompassing the case of free goods as well as the case of scarce goods could be considered, in principle, a theoretical improvement. Nevertheless, this consideration seems to be contradicted by the fact later analytical developments have been focused on the construction of PEGEs which either exclude free goods or minimize their effects on the definition of equilibrium.

Whether a theoretical improvement or not, free goods were introduced in AD's (1954) model in order to satisfy the mathematical conditions required by Kakutani's theorem. Thus, although Debreu (1986, p. 1262) states that, "in a global historical view, the perfect fit between the mathematical concept of a fixed point and the social science concept of an equilibrium stands out", it can be said that this *perfect fit* was possible only because, previously, anything that could hinder it was modified or removed.

Mentioning this way of proceeding could be considered important in the sense that it shows the alteration in the relation between mathematical means and theoretical ends. But it is not the basis of this paper's argument. For, on the one hand, the appraisal of a model should not be based on the intentions on which it was built, and, on the other hand, interpretative questions are this paper's main concern. Accordingly, the problem lies in the fact that the result of the described way of proceeding was a mathematically correct PEGE which, nevertheless, cannot be interpreted in a coherent way -not even by making use of fictitious agents such as the Walrasian auctioneer and the market participant.

The mathematical analysis of economic problems is a methodological option which makes it possible to satisfy desirable scientific values to a greater extent, such as precision or transparency -in the sense that, for example, all the assumptions leading to a result have to be laid carefully. However, this does not imply that the application of mathematical techniques to economics can be carried out anyhow and independently of its costs. In this sense, the main conclusion which stems from the preceding discussion is that the condition that formal structure and interpretation are closely linked is a minimum requirement which could underpin the usefulness of economic models -whether *usefulness* is understood according to Hausman,

Gibbard and Varian, or Sugden. Moreover, despite Debreu's claim in the preface of *Theory of Value*, formal structure should be a means to express economic content.

Against Samuelson's (1947, p. 6) statement that "the laborious literary working over of essentially simple mathematical concepts ... is ... not only unrewarding from the standpoint of advancing the science, but involves as well mental gymnastics of a peculiar depraved type", the opposite could be asserted. To say it in Rosser's words (2003, p. 586), "equations are here to stay", but increasing "the ratio of words to equations" would be desirable. In this regard, note that one of the tasks, among others, fulfilled by economic formal models is the communication of economic knowledge. However, as regards those models whose formal structure is not coherently interpretable, there is considerable room for improvement in the way they carry out this task.

Considering the significance of AD's (1954) model in the development of mathematical economics, the fact that it violates the minimum methodological requirement of linking verbal and formal content is in itself of some concern. However, the hermeneutic problems attached to this model are just an example of a situation which has certain legitimacy among -or, at least, it is not rejected by- mathematical economists; namely, that theoretical contributions may not only lack relevance, but also, as far as a formal element helps to build a mathematical proof, there is no need for it to be interpretable. Given this situation and its consequences, demanding that formal structures must be interpretable and that the use of mathematics should be conditioned by the economic content which we want to express are requirements which can be easily satisfied, but could help to improve the acceptability of the scientific results which the mathematization of economics produces. For satisfying these conditions would imply that we are using available resources in order to bridge the gap between economic intuition and mathematical economics.

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