CLASSIFICATION OF SEMIGRAPHICAL TRANSLATORS

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ABSTRACT. We complete the classification of semigraphical translators for mean curvature flow in $\mathbf{R}^3$ that was initiated by Hoffman-Martín-White. Specifically, we show that there is no solution to the translator equation on the upper half-plane with alternating positive and negative infinite boundary values, and we prove the uniqueness of pitchfork and helicoid translators. The proofs use Morse-Radó theory for translators and an angular maximum principle.

1. INTRODUCTION

A surface M in $\mathbf{R}^3$ is called a **translator** (for mean curvature flow) with velocity $\mathbf{v}$ if

$$M \mapsto M_t := M + t\mathbf{v}$$

is a mean curvature flow, i.e., if the normal velocity at each point is equal to the mean curvature vector: $\vec{H} = \mathbf{v}^\perp$. By rotating and scaling, it suffices (for nonzero velocities) to consider the case $\mathbf{v} = -\mathbf{e}_3$. In that case, we refer to M simply as a **translator**. Ilmanen [Ilm94] observed that M is a translator if and only if it is minimal with respect to the translator metric $g_{ij}(x, y, z) = e^{-z}\delta_{ij}$.

If the surface M is the graph of a function u over a region of the (x, y)-plane, then the translator equation $\vec{H} = (-\mathbf{e}_3)^\perp$ is equivalent to

$$
(1) \qquad \mathrm{Div}\left(\frac{Du}{\sqrt{1 + |Du|^2}}\right) = -\frac{1}{\sqrt{1 + |Du|^2}},
$$

which is a quasilinear elliptic PDE.

A smooth, connected, complete properly embedded translator M is called **semigraphical** if it contains a non-empty, discrete collection $\{L_n\}_n$ of vertical lines such that $M \setminus \bigcup_n L_n$ is a graph $z = z(x, y)$. This notion was introduced by Hoffman-Martín-White in [HMW22b], which included a partial classification. Building on their work, we complete the classification of semigraphical translators and prove

Theorem 1. *A semigraphical translator M in $\mathbf{R}^3$ is one of the following:*

(1) a (doubly periodic) Scherk translator;
(2) a (singly periodic) Scherkenoid;

(3) a (singly periodic) translating helicoid;

(4) a pitchfork;

(5) a (singly periodic) trident.

Up to isometries of $\mathbf{R}^3$,

 (i) *translating helicoids and pitchforks are uniquely determined by a single parameter, the width of a fundamental region,*

 (ii) *tridents are uniquely determined by a single parameter, the period, and*

 (iii) *Scherk translators and Scherkenoids are uniquely determined by two parameters, the width and an angle.*

See [HMW22b] for pictures of the surfaces in Theorem 1.

Theorem 1, minus Assertions (i), (ii), and (iii), is the same as Theorem 34 of [HMW22b], except that the list in Theorem 34 included a sixth item, sometimes called a "yeti". In [HMW22b], yetis were conjectured not to exist. Asssertions (i), (ii), and (iii) assert that, for the allowed values of the parameters, the indicated surfaces exist and are unique. Existence and uniqueness for the tridents was proved in [HMW22b], and existence and uniqueness for the other surfaces was proved in [HMW22a], except that uniqueness of pitchforks and uniqueness of translating helicoids were left as conjectures.

In this paper, we complete the proof of Theorem 1 by proving those three conjectures: uniqueness for pitchforks (§3) and translating helicoids (§4), and nonexistence of the yeti (§5).

It is natural to study and classify semigraphical translators following the classification of graphical solitons in $\mathbf{R}^3$, due to the work of Wang, Spruck-Xiao, and Hoffman-Ilmanen-Martín-White [Wan11, SX20, HIMW19a]. Another starting point for the study of semigraphical translators is the construction of translating analogs to Scherk's doubly periodic minimal surfaces. These were obtained in [HMW22a] along with more examples, some resembling well-known minimal surfaces and some without such analogs. For example, the pitchfork (4) contains a single vertical line of symmetry (see Figure 1). The uniqueness of pitchforks can therefore be viewed as the translator analog of the characterization of the half Enneper's surface as the unique properly embedded non-flat stable minimal surface with straight line boundary and quadratic area growth, cf. [Pér07].

The doubly periodic ("Scherk") and singly periodic ("Scherkenoid") translating surfaces (2) and (3) of Theorem 1 are constructed in [HMW22a] via repeated Schwarz reflections around vertical axes of the graph of a function solving the translator equation (1) on:

 (i) for Scherk translators, a parallelogram $\mathcal{P}(\alpha, w, L)$ of base angle $\alpha \in (0, \pi)$, width w, and unique length $L(\alpha, w)$, with boundary values $\pm\infty$ on the pairs of opposite sides;

 (ii) for Scherkenoids, the half-strip of boundary data that results from extending the horizontal sides of $\mathcal{P}(\alpha, w, L)$ above into rays in the $(+1, 0)$-direction.

Using a sequential compactness theorem [HMW22a, Theorem 10.1], one can extract subsequential limits from sequences of these Scherk or Scherkenoid surfaces as the base angle α tends to π. The limit surface is called a **translating helicoid** if $w < \pi$ and a **pitchfork** if $w \geq \pi$. The uniqueness results in the current paper show that the limit is unique and thus does not depend on the choice of subsequence.

Pitchforks are simply connected and have entropy 3. They might therefore arise as blowups of mean curvature flows of surfaces that are initially closed and smoothly immersed. (By recent work [BK23] of Bamler and Kleiner, they cannot arise as blowups if the initial surface is embedded.) All semigraphical translators other than pitchforks and tridents have cubic area growth, and therefore infinite entropy, and hence cannot arise as such blowups. Tridents have finite entropy but infinite genus, and thus they cannot arise as blowups at the first singular time.

In [GMM22], pitchforks are characterized as the only complete embedded simply connected translating solitons that are contained in a slab, have entropy 3, and contain a vertical line.

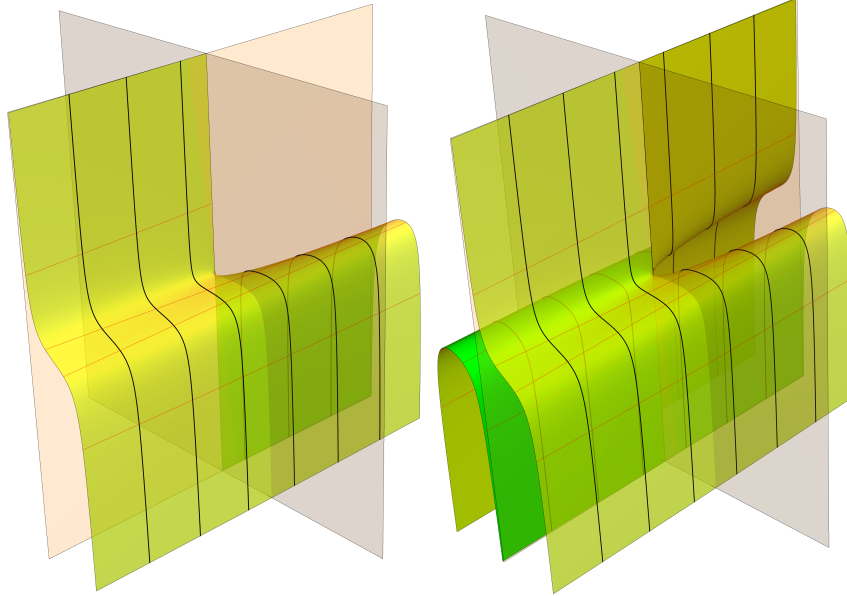See Figure 1 and Figure 2 for example illustrations of these translators and their fundamental pieces.

FIGURE 1. Left: A fundamental piece of the pitchfork of width π. Right: The whole surface, obtained from the fundamental piece by a 180° rotation around the z-axis.)

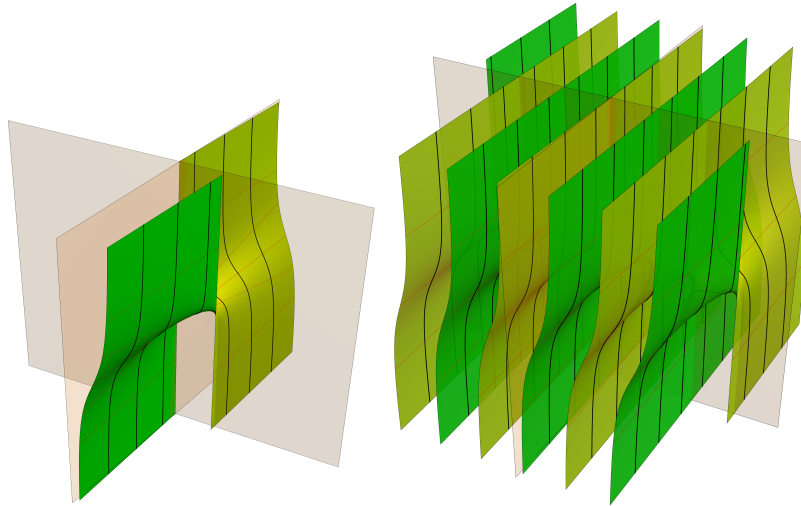FIGURE 2. Left: A fundamental piece of the helicoid of width $\pi/2$. Right: Part of the surface, obtained by successive reflections along the vertical boundary lines.

Tridents, the last item (5) in the list in Theorem 1, were originally constructed for small neck sizes by Nguyen using gluing techniques [Ngu09]. Later, Hoffman-Martín-White constructed tridents as semigraphical translators of arbitrary neck

size $a > 0$ [HMW22b]. They also proved that, as $a \to \infty$, the tridents converge subsequentially to a pitchfork of width π. They observed that if there is a unique pitchfork of width π, then one would get convergence, not merely subsequential convergence, and the limit would coincide with the pitchfork obtained as a limit of Scherkenoids of width π. In this paper, we prove the uniqueness, and thus those consequences of uniqueness follow.

There are interesting translators that are neither graphical nor semigraphical. Examples include: helicoid-like translators that contain a vertical line and are invariant under a screw motion about that line [Hal13], surfaces obtained by gluing [DdPN17, Ngu13, Ngu15, Smi24], rotationally-invariant annuli [CSS07], and a large family [HMW24] of annuli contained in slabs (and therefore not rotationally invariant).

The proofs in this paper use Morse-Radó theory [HMW23a], which studies critical points and level sets for a class of functions arising in minimal surface theory. (Recall that translators are minimal surfaces with respect to the translator metric.) The uniqueness proofs use an angular maximum principle that may be of independent interest. The proof of non-existence of the yeti involves comparison with an appropriately placed grim reaper surface, using a technique first employed by Chini [Chi20].

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2. An Angular Maximum Principle

In this section, we prove a version of the maximum principle that will be used to prove the uniqueness of pitchforks and helicoidal translators.

Theorem 2. *Suppose that Ω is a domain contained in a half-strip*

$$S := [a, \infty) \times (b, c).$$

Suppose that $u_1, u_2 : \Omega \to \mathbf{R}$ are solutions of the translator equation that extend continuously to $\partial\Omega$, and suppose that

$$(u_2 - u_1)|\partial\Omega = d$$

for some constant d. Suppose also that

$$\inf_{\substack{(x,y)\in\Omega \\ x \geq x_0}} \nu_i(x,y) \cdot \mathbf{e}_2 > \epsilon > 0$$

for some $x_0 < \infty$ and ϵ. Then $u_2 - u_1 \equiv d$.

Proof. By translating, we can assume that $b = 0$.

Suppose, contrary to the theorem, that $u_2 - u_1$ is not constant. By relabelling, we may assume that there are points in Ω where $u_2 - u_1 > d$.

By the strong maximum principle,

$$\mu := \max_{\Omega \cap \{x \leq x_0\}} (u_2 - u_1) < \sup(u_2 - u_1).$$

By replacing Ω by $\{(x, y) \in \Omega : u_2 - u_1 \geq \mu\}$ and by replacing u_2 by $u_2 - \mu - d$, we can assume that Ω lies in

$$[x_0, \infty) \times (0, c),$$

that $u_2 = u_1$ on $\partial\Omega$, and that $u_2 \geq u_1$ on Ω. By the strong maximum principle, $u_2 > u_1$ on Ω.

By translating, we can assume that x_0 is as large as we like. In particular, we can assume that

$$x_0 > \frac{c}{\epsilon}.$$

Since $x_0 > 0$, we have well-defined polar coordinates (with $0 \leq \theta < \pi/2$) on $\overline{\Omega}$.

Let $s_i = (1 + |Du_i|^2)^{1/2}$. Then

$$
\begin{aligned}
\frac{\partial u_i}{\partial \theta} &= \frac{\partial u_i}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u_i}{\partial y}\frac{\partial y}{\partial \theta} \\
&= \frac{\partial u_i}{\partial x}(-y) + \frac{\partial u_i}{\partial y}x \\
&= -s_i\,\nu_i \cdot (-y, x, 0) \\
&= s_i(y\,\nu_i \cdot \mathbf{e}_1 - x\,\nu_i \cdot \mathbf{e}_2) \\
&\leq s_i(c - x_0\epsilon) \\
&< 0.
\end{aligned}
$$

Now we define a function $\omega : \Omega \to \mathbf{R}$ as follows. If $p \in \Omega$, let $J(p)$ be the connected component of

$$
\Omega \cap \partial B(0, |p|)
$$

containing p.

Since $u_1 = u_2$ at the endpoints of $J(p)$ and since $\partial u_i/\partial \theta < 0$ at all points of $J(p)$, we see that u_1 and u_2 map $J(p)$ diffeomorphically to the same interval. Thus there is a unique $q \in J(p)$ such that $u_2(q) = u_1(p)$. We define

$$
\omega(p) = \theta(q) - \theta(p).
$$

Since $u_2 > u_1$, $\omega(p) > 0$.

Now ω is a smooth function with the following geometric interpretation. If $p \in \Omega$, then $\omega(p)$ is the smallest positive angle such that

$$
\mathcal{R}_{\omega(p)}(p, u_1(p)) \in \operatorname{graph}(u_2),
$$

where $\mathcal{R}_\theta : \mathbf{R}^3 \to \mathbf{R}^3$ denotes counterclockwise rotation by θ about the z-axis.

Note that ω extends continuously to $\overline{\Omega}$ by setting $\omega = 0$ on $\partial \Omega$. By the strong maximum principle, $\omega(\cdot)$ cannot have an interior local maximum.

Note that if $x(p) > 0$ and $0 \leq y(p) \leq c$, then

$$
\theta(p) \geq 0
$$

and

$$
r(p)\sin\theta(p) \leq c,
$$

so $\theta(p) \leq \arcsin(r(p)^{-1}c)$.

It follows that for every $p \in \Omega$,

$$
0 < \omega(p) < \arcsin(r(p)^{-1}c).
$$

Thus $\omega(p) \to 0$ as $r(p) \to \infty$, so $\omega(\cdot)$ attains its maximum. But that contradicts the strong maximum principle. $\qquad\square$

3. Uniqueness for Pitchforks

Theorem 3. *Let $w \geq \pi$. Up to an additive constant, there exists a unique translator*

$$
f : \mathbf{R} \times (0, w) \to \mathbf{R}
$$

such that f has boundary values

$$
f(x, w) = -\infty,
$$

$$
f(x, 0) = \begin{cases} -\infty & \text{if } x < 0, \\ \infty & \text{if } x > 0, \end{cases}
$$

and such that $M := \operatorname{graph}(f) \cup Z$ is a smooth manifold-with-boundary, where Z is the z-axis.

Here, M is half of a pitchfork, as in Fig. 1 (viewed from a point in the first octant). The other half, also shown in Fig. 1, is obtained by rotating M by π about Z.

Remark 4. The hypothesis that M is a smooth manifold-with-boundary is not necessary; it is implied by the other hypotheses. See Proposition 11.

The hypothesis $w \geq \pi$ is necessary: if $w < \pi$, then no such f exists [HMW22a, Theorem 12.1].

Proof. Existence was proved in [HMW22a, §12], so it suffices to prove uniqueness. To prove uniqueness, it will be convenient to add two points $+\widehat{\infty}$ and $-\widehat{\infty}$ at infinity to $\mathbf{R}^2$ such that, for $p_i \in \mathbf{R}^2$,

(2)
$$p_i \to +\widehat{\infty} \text{ if and only if } x(p_i) \to \infty, \text{ and}$$
$$p_i \to -\widehat{\infty} \text{ if and only if } x(p_i) \to -\infty.$$

The proof of uniqueness uses the following three facts (see [HMW22a, Theorem 12.1]) about solutions f:

(F1). The Gauss map image of $\operatorname{graph}(f)$ depends only on the width w.

(F2). If $\nu(x, y)$ is the upward unit normal to the graph of f at $(x, y, f(x, y))$, then

$$\nu(p) \cdot \mathbf{e}_2 \to 1$$

as $p \to +\widehat{\infty}$.

(F3). There is a strictly decreasing function $\psi : [0, w] \to [\infty, -\infty]$, depending only on w, such that if $x_i \to -\infty$ and $y_i \to y \in [0, w]$, then

$$\frac{\partial f}{\partial y}(x_i, y_i) \to \psi(y).$$

Fact (F3) is based on the fact that the function

$$(x, y) \mapsto f(\bar{x} + x, y) - f(\bar{x}, 0)$$

converges to a tilted grim reaper over $\mathbf{R} \times (0, w)$ as $\bar{x} \to -\infty$. In particular,

$$\psi(y) = \frac{\partial}{\partial y}\left(\left(\frac{w}{\pi}\right)^2 \log(\sin((\pi/w)y) - x\sqrt{\frac{w^2}{\pi^2} - 1}\right) = \frac{w}{\pi}\frac{\cos((\pi/w)y)}{\sin((\pi/w)y)}.$$

Suppose, contrary to the theorem, that there exist two solutions f_1 and f_2 such that $f_1 - f_2$ is not constant. Let $M_i = \operatorname{graph}(f_i) \cup Z$.

Claim 1. *There exists a $v \in \mathbf{R} \times (-w, w)$ such that the function*

$$g(p) := f_1(p) - f_2(p + v).$$

has a critical point. Moreover, we can choose v so that $y(v) \neq 0$.

Of course, the domain of g is the intersection of the domains of f_1 and of f_2.

Proof. The Gauss map images of the graphs of f_1 and f_2 are the same. Thus if $q \in \mathbf{R} \times (0, w)$, then there is a $q' \in \mathbf{R} \times (0, w)$ such that $Df_2(q') = Df_1(q)$. Hence, if $v = q' - q$, then the function

(*) $f_1(p) - f_2(p + v)$

has a critical point at q. Note that a critical point of (*) is equivalent to a point of tangency between the graph of $f_2(p + v)$ and the foliation whose leaves are vertical translates of $\operatorname{graph}(f_1)$. It follows, by [HMW23a, Corollary 40], that the function (*) has a critical point for all v sufficiently close to $q' - q$. (Indeed, the number of critical points, counting multiplicity, of (*) is a lower-semicontinuous function of v.) In particular, there exists such a v with $y(v) \neq 0$.   □

By relabeling f_1 and f_2, if necessary, we can assume that $y(v) > 0$ in Claim 1. Thus we have a translator $u_1 = f_1$ on $\mathbf{R} \times (0, w)$, and a translator

$$u_2 : \mathbf{R} \times (-y(v), w - y(v)) \to \mathbf{R}$$

given by $u_2(p) = f_2(p + v)$. Note that $g := u_1 - u_2$ is defined over the strip

$$\mathbf{R} \times (0, b),$$

where $b = w - y(v)$. Note that $0 < b < w$, and that g has boundary values

$$g(x, 0) = -\infty \quad \text{for } x < 0,$$
$$g(x, 0) = \infty \quad \text{for } x > 0, \text{ and}$$
$$g(x, b) = \infty \quad \text{for } x \in \mathbf{R}.$$

Also, g has a critical point p. By adding a constant to f_1, we can assume that $g(p) = 0$.

For $i = 1, 2$, note that if ν_i is the upward unit normal to graph of u_i, then

(3) $$\nu_i(p) \to \mathbf{e}_2 \text{ as } p \to +\widehat{\infty},$$

by Fact (F2) above.

Let C be a component of the zero set of g that contains the critical point p. Then C has the structure of a network in which p is a node, and in which the valence of each node is an even number ≥ 4. (See, for example, [CM11, Theorem 7.3].) By the maximum principle, C contains no closed loops, so it is a tree with at least 4 ends. From the boundary values of g, we see that each end tends to $-\widehat{\infty}$, to $+\widehat{\infty}$, or to $(0, 0)$.

Since $M_1 := \text{graph}(u_1) \cup Z$ is a smooth manifold-with-boundary, we see that there is a small disk $D \subset \mathbf{R}^2$ centered at $(0, 0)$ such that $D \cap g^{-1}(0)$ consists of a single curve γ joining $(0, 0)$ to a point on ∂D. Thus at most one end of C tends to $(0, 0)$. (There might be no such end since γ might lie a component of $g^{-1}(0)$ other than C.)

We claim that there is a $\tilde{x} < 0$ such that

(4) $$\frac{\partial g}{\partial y}(x, y) > 0 \quad \text{on } (-\infty, \tilde{x}] \times (0, b).$$

For if not, there would be (x_i, y_i) with $x_i \to -\infty$ such that

$$\frac{\partial g}{\partial y}(x_i, y_i) \leq 0.$$

Passing to a subsequence, we can assume that $y_i \to y \in [0, b]$, and therefore

$$0 \geq \frac{\partial g}{\partial y}(x_i, y_i)$$
$$= \frac{\partial u_1}{\partial y}(x_i, y_i) - \frac{\partial u_2}{\partial y}(x_i, y_i)$$
$$\to \psi(y) - \psi(y + y(v))$$
$$> 0,$$

since ψ is strictly decreasing, a contradiction. Thus, there is an $\tilde{x}$ for which (4) holds. Hence, for $x \leq \tilde{x}$, there is at most one y such that $(x, y) \in C$. Thus, at most one end of C tends to $-\widehat{\infty}$.

Since at most one end of C tends to $-\widehat{\infty}$ and at most one end tends to $(0, 0)$, at least two ends must tend to $+\widehat{\infty}$. Hence, C contains two embedded curves α and β that join p to $+\widehat{\infty}$ and that are disjoint except at p. Let Ω be the open region bounded by $\alpha \cup \beta$. Then $g = 0$ on $\partial \Omega$ and g does not vanish anywhere on Ω. But, by (3) and the angular maximum principle (Theorem 2), $g \equiv 0$ on Ω, a contradiction. $\qquad\square$

4. Uniqueness for Translating Helicoids

Theorem 5. *Let $w < \pi$. Up to an additive constant, there exists a unique translator*

$$f : \mathbf{R} \times (0, w) \to \mathbf{R}$$

with the following property. There exists an $a \in \mathbf{R}$ such that f has boundary values

$$f(x, 0) = \begin{cases} \infty & \text{if } x < 0, \\ -\infty & \text{if } x > 0, \end{cases}$$

$$f(x, w) = \begin{cases} -\infty & \text{if } x < a, \\ \infty & \text{if } x > a, \end{cases}$$

and such that $M := \operatorname{graph}(f) \cup Z \cup L$ is a smooth manifold-with-boundary, where L is the vertical line through (a, w).

Here, M is a fundamental piece of a translating helicoid, the rest of which is obtained from M by iterated Schwarz reflections, as in Fig. 2. The hypothesis $w < \pi$ is necessary: if $w \geq \pi$, then no such f exists [HMW22a, Theorem 11.1].

Remark 6. The hypothesis that M is a smooth manifold-with-boundary is not necessary; it is implied by the other hypotheses. See Proposition 11.

Proof. Existence was proved in [HMW22a, §11], so it suffices to prove uniqueness. As proved there, solutions f have the following properties:

(P1). The Gauss map image of the graph of f depends only on w.
(P2). $\nu(p)$ converges to $\mathbf{e}_2$ as $p \to -\widehat{\infty}$ and to $-\mathbf{e}_2$ as $p \to +\widehat{\infty}$.
Here $\pm\widehat{\infty}$ are the points at $x = \pm\infty$ given by (2).

Suppose, contrary to the theorem, that there exist two solutions f_1 and f_2. Claim 1 in the proof of Theorem 3 also holds here (with exactly the same proof). That is, there exists a $v \in \mathbf{R} \times (0, w)$ with $y(v) \neq 0$ such that the function

$$g(q) = f_1(q) - f_2(q + v)$$

has a critical point p. By relabeling, if necessary, we can assume that $y(v) > 0$. Note that g is defined over a strip $\mathbf{R} \times (0, b)$, where $0 < b < w$. We may assume (by adding a constant to f_1) that $g(p) = 0$.

Note that g has boundary values

$$g(x, 0) = \begin{cases} \infty & \text{if } x < 0, \\ -\infty & \text{if } x > 0 \end{cases}$$

and

$$g(x, b) = \begin{cases} \infty & \text{if } x < a, \\ -\infty & \text{if } x > a. \end{cases}$$

Let C be the connected component of $g^{-1}(0)$ containing p. Exactly as in the proof of Theorem 3, C is a tree with at least four ends, and each end tends to $(0, 0)$, (a, b), $-\widehat{\infty}$, or $+\widehat{\infty}$. Also, as in that proof, at most one end tends to each of those points. (In the cases of $\pm\widehat{\infty}$, this is by Property (P2) and the angular maximum principle, Theorem 2.) Thus, exactly one end tends to each of those points. Consequently, C contains a curve C' that joins $(0, 0)$ to (a, b).

Let U be the component of $(\mathbf{R} \times (0, b)) \setminus C'$ on which x is bounded below:

$$\inf_U x(\cdot) > -\infty.$$

Since C has an end tending to $+\widehat{\infty}$, there are points in U at which $g = 0$. By the strong maximum principle, there are also points at which $g > 0$. Thus the set

$$\Omega := U \cap \{g > 0\}$$

is nonempty. Now $\partial\Omega$ contains no points on the rays $\{(x,0) : x > 0\}$ and $\{(x,b) : x > a\}$, since $g \equiv -\infty$ on those rays. Hence $g \equiv 0$ on $\partial\Omega$. But by Property (P2) and the angular maximum principle (Theorem 2), $g \equiv 0$ on Ω, a contradiction. $\qquad\square$

5. Nonexistence of The Yeti

We will use the following facts about tilted grim reaper translators:

Lemma 7. *Given a vector $v \in \mathbf{R}^2$, there is an $a \geq \pi/2$, a tilted grim reaper*

$$f : \mathbf{R} \times (-a, a) \to \mathbf{R},$$

and a $y \in (-a, a)$ such that

$$Df(x, y) = v$$

for all x. If $|y| \leq \pi/4$, then $|f_y(x,y)| \leq 1$.

Proof. Up to an additive constant, a tilted grim reaper over the indicated strip is given by

$$f(x, y) = \kappa^2 \log(\cos(y/\kappa)) \pm x\sqrt{\kappa^2 - 1},$$

where $\kappa = 2a/\pi \geq 1$. Thus $f_x = \pm\sqrt{\kappa^2 - 1}$, so we can choose κ and the sign of $\pm$ so that $f_x \equiv v_1$. Note that

$$f_y(x, y) = -\kappa \tan(y/\kappa),$$

which ranges from $-\infty$ to $+\infty$, so (given κ) we can choose y so that $f_y(x, y) = v_2$.

Now, given $|y| \leq \pi/4$, we know that $\kappa \mapsto |\kappa \tan(y/\kappa)|$ is a decreasing function of $\kappa \in [1, \infty)$, so

$$\begin{aligned} |f_y(x, y)| &= |\kappa \tan(y/\kappa)| \\ &\leq |\tan y| \\ &\leq 1. \end{aligned}$$

$\qquad\square$

By definition, a yeti is (up to translation and rotation) a complete translator M such that $Z \subset M$ and such that $M \setminus Z$ is the graph of a function over $\mathbf{R}^2 \setminus \{y = 0\}$. If u is the restriction of that function to the halfplane $\{y > 0\}$, then (by completeness) u has boundary value $+\infty$ on one component of $\{y = 0\} \setminus \{(0, 0)\}$ and $-\infty$ on the other component. The following theorem proves that such translators do not exist.

Theorem 8. *There is no solution*

$$u : \mathbf{R} \times (0, \infty) \to \mathbf{R}$$

of the translator equation such that

$$u(x, 0) = \begin{cases} \infty & \text{for } x < 0, \text{ and} \\ -\infty & \text{for } x > 0, \end{cases}$$

and such that $\mathrm{graph}(u) \cup Z$ is a smooth manifold-with-boundary.

Remark 9. The hypothesis that $\mathrm{graph}(u) \cup Z$ is a smooth manifold-with-boundary is not necessary. See Proposition 11.

The idea of the proof is as follows. We suppose that there is such a function u, and we take a subsequential limit of

$$u(x + s, y) - u(s, 1)$$

as $s \to \infty$. (Here 1 is arbitrary; we could use any positive number.) We show that the subsequential limit is a complete, translating graph over all of $\mathbf{R} \times (0, \infty)$, which is impossible since, by the classification [HIMW19a] of complete graphical translators, no such translator exists.

A priori, the subsequential limit might not exist, or it might be a graph over a proper subset of $\mathbf{R} \times (0, \infty)$. To exclude those possibilities, we need suitable gradient bounds on u. By [HIMW19a, Theorem 6.7], to get such bounds, it suffices to show that $|Du|$ is bounded on some connected, unbounded set contained in a halfstrip of the form $[x_1, \infty) \times I$, where I is compact interval in $(0, \infty)$. Fortunately, we are able to produce the necessary set, namely the graph of a certain smooth function $Y : [x_1, \infty) \to I$ such that, on that graph, $u_y = 0$ and $|u_x|$ is bounded.

Proof of Theorem 8. Suppose, contrary to the theorem, that there is such a function u.

Claim 1. *Let $a \geq \pi/2$, and let*

$$f : \mathbf{R} \times (-a, a) \to \mathbf{R}$$

be a function whose graph is a complete translator. (Thus $f = -\infty$ on the boundary.) Then the infimum of $u - f$ on the half-strip

$$S := (-\infty, -1] \times (0, a)$$

is attained at a point in the right edge $\{-1\} \times (0, a)$ of S.

(We only need the claim when f is a tilted grim reaper.)

Proof. Let $\mathcal{R}_\theta : \mathbf{R}^2 \to \mathbf{R}^2$ be counterclockwise rotation through θ about the point $(-1, 0)$. For $\theta \in (0, \pi/2)$, let

$$S(\theta) := (\mathcal{R}_\theta S) \cap \{y > 0\}.$$

Thus $S(\theta)$ is a triangular region with $(-1, 0)$ as one vertex. Let $E(\theta)$ be the interior of the edge joining $(-1, 0)$ to the vertex in $\{y > 0\}$, and let $E(0)$ be its limit as θ tends 0:

$$E(0) = \{-1\} \times (0, a).$$

Define

$$f_\theta : S(\theta) \to \mathbf{R}$$

by

$$f_\theta(p) = f(\mathcal{R}_{-\theta}(p)).$$

Note that

$$u - f_\theta : S(\theta) \to \mathbf{R}$$

has boundary value $+\infty$ at all boundary points except those in the edge $E(\theta)$. By the maximum principle, the minimum of $u - f_\theta$ is attained at a point in $E(\theta)$:

$$\tag{5} \min_{S(\theta)} (u - f_\theta) = \min_{E(\theta)} (u - f_\theta).$$

The result follows by letting $\theta \to 0$. (Note that the right-hand side of (5) depends continuously on θ.) $\qquad\square$

Claim 2. *There is a $c \in (0, \infty)$ with the following property. For every $x > 0$, there is a $y \in (0, c)$ such that $u_y(x, y) < 0$.*

Proof. Since $u(-1, 0) = \infty$,

$$\liminf_{y \to 0} u_y(-1, y) = -\infty.$$

Thus we can choose a point p with $x(p) = -1$ such that

$$\tag{6} \begin{aligned} 0 &< y(p) < \pi/4, \text{ and} \\ u_y(p) &< -1. \end{aligned}$$

Let

$$g : \mathbf{R} \times (b, c) \to \mathbf{R}$$

be the tilted grim reaper surface defined over a strip parallel to the x-axis such that $g(p) = u(p)$ and $Dg(p) = Du(p)$. (The function g exists by Lemma 7.) By (6)

and Lemma 7, the distance from p to the center line $\{y = (b+c)/2\}$ of the strip is greater than $\pi/4$, and thus

$$\frac{b+c}{2} < y(p) - \frac{\pi}{4} < 0.$$

Consequently,

(7) $$g_y < 0 \text{ on } \mathbf{R} \times [0, c).$$

Choose a large enough that $[b, c] \subset (-a, a)$, and let

$$f : \mathbf{R} \times (-a, a) \to \mathbf{R}$$

be a tilted grim reaper function such that $f_x < 0$. By Claim 1, and by adding a constant to f, we can assume that

$$u - f \geq 0 \text{ on } (-\infty, -1] \times (0, c).$$

Since $[b, c] \subset (-a, a)$, we see that $f_x - g_x$ is a constant $-k < 0$. Thus

$$f(x, y) - g(x, y) = -kx + (f(0, y) - g(0, y))$$
$$\geq -kx + f(0, c) - g(0, 0)$$

since $f(0, y)$ and $g(0, y)$ are decreasing functions of $y \in [0, c]$. Thus, on $(-\infty, -1] \times [0, c)$,

$$u(x, y) - g(x, y) \geq f(x, y) - g(x, y)$$
$$\geq \delta - kx$$

for a constant δ. Hence

(8) $$\lim_{x \to -\infty} \min_{y \in [0, c)} (u(x, y) - g(x, y)) = \infty.$$

Let $h = u - g$. Thus the domain of h is the strip

$$\mathbf{R} \times (0, c),$$

and h has boundary values

$$h(x, 0) = \begin{cases} \infty & \text{if } x < 0, \\ -\infty & \text{if } x > 0, \end{cases}$$

and

$$h(x, c) = \infty \quad (x \in \mathbf{R}).$$

Let C be the connected component of $h^{-1}(0)$ that contains p.

Since $Dh(p) = 0$, C is a tree with at least 4 ends. From the boundary values of h, we see that each end of C tends to $-\widehat{\infty}$, to $\widehat{\infty}$, or to $(0, 0)$. Here $\pm\widehat{\infty}$ are the points at $x = \pm\infty$ given by (2).

By (8), no end tends to $-\widehat{\infty}$.

As in the proof of Theorem 3, at most one end tends to $(0, 0)$.

Thus, at least three ends of C tend to $+\widehat{\infty}$. Consequently, for each $x > 0$, there are at least 3 values of $y \in [0, c)$ for which $h(x, y) = 0$. Let $0 < y_1 < y_2 < c$ be two of those values. Then

$$u(x, y_1) - g(x, y_1) = u(x, y_2) - g(x, y_2) = 0.$$

so

$$u(x, y_2) = g(x, y_2) < g(x, y_1) = u(x, y_1),$$

since $g_y(x, y) < 0$ for $y \in [0, c)$. Thus there is a $y \in (y_1, y_2) \subset (0, c)$ such that $u_y(x, y) < 0$. $\qquad\square$

Claim 3. *Let c be as in Claim 2. Let Q be the set of $(x, y) \in \mathbf{R} \times (0, c)$ such that $u_y(x, y) = 0$. There is an $x_1 \in (0, \infty)$ such that $Q \cap \{x \geq x_1\}$ is the graph of a smooth function $Y : [x_1, \infty) \to (\pi/4, c)$. Furthermore,*

$$\sup_{Q \cap \{x \geq x_1\}} |Du| < \infty.$$

Proof. Suppose $p_n \in Q$ and $x(p_n) \to \infty$. After passing to subsequence, $y(p_n)$ converges to a limit $\hat{y} \in [0, c]$, and

$$\operatorname{graph}(u) - (x(p_n), 0, u(p_n))$$

converges to a complete translator that lies in the halfspace $\{y \geq 0\}$. Let $\widehat{M}$ be the connected component of the limit that contains $p := (0, \hat{y}, 0)$.

Note that the y-axis is tangent to $\widehat{M}$ at p. Thus, $\widehat{M}$ cannot be a vertical plane (since $\widehat{M}$ lies in $\{y \geq 0\}$). Hence, $\widehat{M}$ is a graph. By the classification of graphical translators [HIMW19a], it is the graph of a function ϕ over a strip in $\{y \geq 0\}$ of width $\geq \pi$. Also, by that classification, $\phi_{yy} < 0$ at all points, and $\phi_y = 0$ only on the midline of the strip. Since $\phi_y(0, \hat{y}) = 0$, we see that the line $\{y = \hat{y}\}$ is the midline of the strip. Since the strip has width $\geq \pi$,

$$\hat{y} \geq \pi/2.$$

Thus, we have shown that every sequence $p_n \in Q$ with $x(p_n) \to \infty$ has a subsequence $p_{n(i)}$ such that

(1) $y(p_{n(i)})$ converges to a limit $\hat{y} \geq \pi/2$.
(2) $u_{yy}(p_{n(i)})$ converges to a limit $\phi_{yy}(0, \hat{y}) < 0$.
(3) $Du(p_{n(i)})$ converges to a limit $D\phi(0, \hat{y})$.

Hence, we can choose $x_1 > 0$ large enough so that, for all $p \in Q$ with $x(p) \geq x_1$,

$$(9) \qquad\qquad\qquad\qquad y(p) > \frac{\pi}{4},$$

$$(10) \qquad\qquad\qquad\qquad u_{yy}(p) < 0,$$

and so that

$$(11) \qquad\qquad\qquad\qquad \sup_{Q \cap \{x \geq x_1\}} |Du| < \infty.$$

Let $x \geq x_1$. We claim there is exactly one $y \in (0, c)$ for which $u_y(x, y) = 0$, i.e., for which $(x, y) \in Q$.

Now c was chosen according to Claim 2. Thus there is a $\tilde{y} \in (0, c)$ for which $u_y(x, \tilde{y}) < 0$. Since $u(x, 0) = -\infty$, there is at least one value of $y \in (0, \tilde{y})$ for which $u_y(x, y) = 0$.

If there were two such values $y_1 < y_2$, then by (10), the function

$$y \in (y_1, y_2) \mapsto u(x, y)$$

would attain its minimum at some $\hat{y} \in (y_1, y_2)$. Thus $u_y(x, \hat{y}) = 0$ and $u_{yy}(x, \hat{y}) \geq 0$, contrary to (10).

Hence there is exactly one $y \in (0, c)$ – call it $Y(x)$ – such that $(x, y) \in Q$. Note that $Y(x) \in (\pi/4, c)$ (by (9)), and that $|Du|$ is bounded on the graph of Y (by (11)). Finally, $Y(x)$ is a smooth function of $x \in [x_1, \infty)$, by (10) and the implicit function theorem. $\qquad\square$

We are now ready to complete the proof of Theorem 8. We use the following special case of Lemma 6.3 in [HIMW19a]:

Lemma 10. *Suppose $u : [x_1, \infty) \times (0, A) \to \mathbf{R}$ is a translator. Suppose I is a compact interval in $(0, A)$ and that $Y : [x_1, \infty) \to I$ is continuous function such*

that $|Du|$ is bounded on the graph of $Y(\cdot)$. Then for every compact subset K of $(0, A)$,

(12) $$\sup_{[x_1,\infty) \times K} |Du| < \infty.$$

We have shown that, in our situation, the hypotheses of Lemma 10 hold with $A = \infty$ and $I = [\pi/4, c]$. Hence, (12) holds for every compact $K \subset (0, \infty)$. It follows that the functions

$$(x, y) \in \mathbf{R} \times (0, \infty) \mapsto u(x + s, y) - u(s, 1)$$

converge subsequentially as $s \to \infty$ to a function

$$w : \mathbf{R} \times (0, \infty) \to \mathbf{R}$$

whose graph is a complete translator. But there is no such function by [HIMW19a, Theorem 6.7]. $\qquad\square$

The following proposition was not used in this paper, but it justifies the assertions in Remarks 4, 6, and 9.

Proposition 11. *Suppose that*

$$u : \{(x, y) \in B^2(0, R) : y > 0\} \to \mathbf{R}$$

is a smooth solution to the translator equation such that

$$u(x, 0) = \begin{cases} -\infty & \text{if } x < 0, \\ \infty & \text{if } x > 0. \end{cases}$$

Then $\mathrm{graph}(u) \cup Z$ *is a smooth-manifold-with-boundary.*

Proof. Let $C(R)$ be the cylinder $B^2(0, R) \times \mathbf{R}$. Let M be the graph of u. Then M minimizes the translator area in the following sense: if K is a compact portion of M and if K' is a compact surface in $C(R) \cap \{y > 0\}$ with $\partial K' = \partial K$, then

$$\mathrm{area}(K) \leq \mathrm{area}(K').$$

Here and in what follows, we use $\mathrm{area}(\Sigma)$ to refer to the translator area of a surface Σ, computed with respect to Ilmanen's translator metric $g_{ij}(x, y, z) = e^{-z}\delta_{ij}$.

Actually, the assumption that K' lies in $C(R) \cap \{y > 0\}$ is not needed. For suppose that K' is any compact surface with $\partial K' = \partial K$. Let Q be a compact convex subset of $B^2(0, R) \cap \{y > 0\}$ such that $Q \times \mathbf{R}$ contains K and let K'' be the projection of K' to $Q \times \mathbf{R}$. Then

$$\mathrm{area}(K) \leq \mathrm{area}(K'') \leq \mathrm{area}(K').$$

In particular, if K is a convex, compact subset of $C(R) \cap \{y > 0\}$, then $M \cap K$ and $V \cap \partial K$ have the same boundary, where

$$V = \{(x, y, z) \in C(R) : y > 0, \ z < u(x, y)\}$$

is the subgraph of u. Thus

$$\mathrm{area}(M \cap K) \leq \mathrm{area}(V \cap \partial K)$$
$$\leq \mathrm{area}(\partial K)$$

Thus, if K is any compact, convex subset of $C(R)$, then

$$\mathrm{area}(M \cap K \cap \{y \geq \epsilon\}) \leq \mathrm{area}(\partial K \cap \{y \geq \epsilon\})$$
$$\leq \mathrm{area}(\partial K).$$

Thus, letting $\epsilon \to 0$,

$$\mathrm{area}(M \cap K) \leq \mathrm{area}(\partial K).$$

The fact that $M \cap K$ has finite area together with the fact that $M \cap \{y = 0\}$ has area 0 implies that the surfaces

(*) $$M \cap K \cap \{y > \epsilon\}$$

converge (as currents, with the mass norm) to $M \cap K$ as $\epsilon \to 0$. Thus, $M \cap K$ is area-minimizing, since it is the limit of the area-minimizing surfaces (*).

Let S be the strip

$$S = \{(x, 0, z) : x \in (0, R),\, z \in \mathbf{R}\}.$$

Let $[\![M]\!]$, $[\![S]\!]$, and $[\![V]\!]$ be the locally integral currents obtained by orienting M by the upward unit normal, and S by the unit normal $\mathbf{e}_2$, and by giving V the standard orientation.

Then

$$[\![\partial R]\!] \, \llcorner \, C(R) = [\![M]\!] - [\![S]\!],$$

so

$$[\![\partial M]\!] \, \llcorner \, C(R) = [\![\partial S]\!] \, \llcorner \, C(R)$$
$$= [\![Z]\!].$$

We have shown: M is translator-area minimizing and has boundary (in $C(R)$) the line Z with multiplicity 1. Hence by [HS79], $M \cup Z$ is a smooth manifold-with-boundary. $\qquad\square$

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