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DIAGRAMAS DE PENROSE DE MÉTRICAS COSMOLÓGICAS TIPO FRW

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Resumen

El siguiente trabajo de fin de grado estudia el modelo de universo homogéneo e isótropo, conocido como el modelo de Friedmann-Lemaître-Robertson-Walker (FLRW), y calcula los diagramas de Penrose de diferentes soluciones cosmológicas descritas por este. El alcance de este trabajo está limitado a los universos en los cuales está presente un único tipo de fluido perfecto. Para describir este modelo, se utilizarán las denominadas *coordenadas conformes* y se hallará en este sistema de coordenadas una expresión general para cualquier tipo de materia y para cualquier curvatura de las secciones espaciales. A partir de estas soluciones, se realizarán transformaciones conformes sobre las métricas encontradas para describir ciertas propiedades de estos modelos cosmológicos en los diagramas de Penrose.

Abstract

The following final degree project studies the model of an isotropic and homogeneous universe, known as the Friedmann-Lemaître-Robertson-Walker model (FLRW model for short), and calculates the Penrose diagrams for different cosmological solutions of Einstein's field equation. The scope of this thesis is limited by the solutions of universes filled with only one type of perfect fluid. To analyse this model, we will describe the problem in *conformal coordinates*, and we will find a general solution, valid for every kind of matter and every possible curvature of spatial hypersurfaces, in this coordinate system. We will describe different cases of matter and curvature, making use of Penrose diagrams, which can illustrate properties of the whole universe on a finite piece of paper. To achieve this, conformal transformations will be needed.

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1 Introduction

1.1 General Relativity

General Relativity began as an extension of Special Relativity to include gravity, thus its name, but shortly began to answer questions that no other theory had been capable of formulating. The main strength of General Relativity, and the core complexity within this formalism, comes from the mathematical analysis via tensor calculus. In General Relativity, tensors are needed to explain the nature of spacetime. By the use of tensors, Riemann was capable of describing the curvature of a manifold. Einstein adopted Riemann's ideas to describe gravitation not as a force, but as a consequence of the curvature of spacetime. Einstein proposed a strong version of the **Equivalence Principle**, he stated that locally, the effects of a uniform gravitational field are indistinguishable from the effects of acceleration (d'Inverno, 1992). Following this line of thought Einstein was capable of formulating the key idea of Relativity, the Einstein field equation. This tensorial equation describes the correlation between the dynamics of matter and the curvature of spacetime. Paraphrasing Charles W. Misner: "Spacetime tells matter how to move, matter tells spacetime how to curve" (Misner et al., 1973).

General Relativity brought about a paradigm shift in physics, resulting in the rise of new branches of research. Particularly, we are interested in that of **Relativistic Cosmology**, which studies the dynamics of the whole universe using Einstein's framework. The cornerstone of Relativistic Cosmology is the **Cosmological Principle**. It states that at large scales, the cosmos is the same at every point in space; hence, it provides homogeneity and isotropy to the universe. Following this principle, Alexander Friedman and Georges Lemaître, independently, proposed a model for an isotropic and homogeneous universe. This model was revised by Howard P. Robertson and Arthur Walker. Therefore, it is often called the FLRW model of the cosmos, by the initials of its founders. The FLRW model describes the possible structure and geometry of the universe in different **cosmological solutions**, that is to say, different manifolds that satisfy Einstein equations for the whole cosmos. The key idea that FLRW introduced is the expansion of the universe, which resulted in the theory of the Big Bang by Lemaître in 1931. Current cosmological models (e.g. the Λ CMD model) are based on the FLRW model.

1.2 Penrose Diagrams

The relation of causality between events of a manifold tells us that an event x causally precedes an event y if an occurrence on x can influence what happens at y (Kronheimer and Penrose, 1967). Using the relation of causality, we can structure a manifold by examining the causal precedence of all the events in the manifold. In General Relativity, this causal structure is governed by the invariant speed of light, which defines a local set of light cones at every point in spacetime. Influence between two points x and y can only occur if the vector that starts from x and goes to y is not spacelike. This states that light cones are the boundaries of the

causality relation of events. For any event x , only those events located inside the past light cone whose vertex is x can causally precede x . x only precedes those located inside the future light cone whose vertex is x . The preservation of this network is paramount, as it dictates the limits of information exchange. However, charting the causal structure of an infinite manifold into a finite domain requires a mathematical intervention.

This is achieved through the application of **conformal transformations**, a mathematical operation that rescales the metric in a position-dependent factor. This operation has a critical property: it maintains null geodesics as null geodesics. What this means in practise is that the causal structure is preserved under conformal transformations. Crucially, by selecting an appropriate scale factor that decays at large distances, we can "compactify" the manifold. Through this conformal rescaling, the infinite boundaries of spacetime are compressed down to a finite distance.

The visual manifestation of this compactification is the **Penrose diagrams** (or Penrose-Carter diagrams). By mapping the conformally transformed spacetime onto a two-dimensional plane, we depict the causal structure of the manifold in a geometrical shape, where light rays travel at 45° angles. This allows us to intuitively deduce the global causal relationship of the universe, identify horizons and rigorously map causal barriers such as black hole horizons.

2 Definitions and Conventions

To begin with, we remark that **natural units** will be used. The velocity of light, c , will be defined in a particular set of units so that $c = 1$. Therefore, time and longitude will have the same units.

With respect to the signature of the metric, we will use the **mostly minus signature** $(+, -, -, -)$.

The **curvature tensors** are defined the following way:

- The Riemann tensor:

$$\mathcal{R}_{\mu\nu\rho}{}^\lambda = \partial_\mu \Gamma_{\nu\rho}^\lambda - \partial_\nu \Gamma_{\mu\rho}^\lambda + \Gamma_{\mu\sigma}^\lambda \Gamma_{\nu\rho}^\sigma - \Gamma_{\nu\sigma}^\lambda \Gamma_{\mu\rho}^\sigma, \quad (2.1)$$

where $\Gamma_{\nu\rho}^\mu$ is the Levi-Civita connection.

- The Ricci tensor:

$$\mathcal{R}_{\mu\nu} = \mathcal{R}_{\mu\rho\nu}{}^\rho = g^{\rho\lambda} \mathcal{R}_{\mu\rho\nu\lambda}. \quad (2.2)$$

- The Ricci scalar:

$$\mathcal{R} = g^{\mu\nu} \mathcal{R}_{\mu\nu} = \mathcal{R}_\mu{}^\mu. \quad (2.3)$$

- Weyl tensor:

$$C_{\mu\nu\rho\sigma} = \mathcal{R}_{\mu\nu\rho\sigma} - \frac{1}{2} (g_{\mu\rho} \mathcal{R}_{\nu\sigma} - g_{\mu\sigma} \mathcal{R}_{\nu\rho} - g_{\nu\rho} \mathcal{R}_{\mu\sigma} + g_{\nu\sigma} \mathcal{R}_{\mu\rho}) + \frac{1}{6} \mathcal{R} (g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}). \quad (2.4)$$

8 Conclusiones

La primera y más importante conclusión de este trabajo es el hallazgo de una fórmula general para el factor de escala, a , mediante el uso de coordenadas conformes. Esta fórmula, (4.13), es capaz de describir cualquier tipo de universo de un solo componente mediante el modelo FLRW. El precio a pagar por este hallazgo se halla en la dificultad de hallar una relación exacta entre el tiempo comóvil, t , y el tiempo conforme, τ . Pese a esta limitación, se ha demostrado en varias ocasiones que el uso de la fórmula general facilita los cálculos en diversos ámbitos, como por ejemplo el cálculo de singularidades, explicado en 4.3.1, que muestra que toda la materia con $w > -1$ muestra singularidades en los puntos donde se anula el factor de escala; mientras que las soluciones cosmológicas de energía fantasma, $w < -1$, muestran singularidades en los puntos donde el factor de escala diverge a valores infinitos.

Una comprobación para la veracidad de la fórmula general se ha realizado al describir la solución de De Sitter de universo en expansión exponencial. Aquí se ha comprobado que la expresión para tiempo conforme corresponde a las expresiones conocidas en tiempo comóvil. Tras esto, se ha estudiado la curvatura constante de De Sitter hallando que presenta una curvatura $K = -\Lambda/3$.

En la descripción de los modelos de materia fría ($v \ll c$) y materia relativista, el uso de la fórmula general ha sido fundamental para hallar diferentes propiedades. Por ejemplo, el tiempo de vida del universo antes de un Big Crunch.

En la segunda parte del TFG, centrada en las transformaciones conformes y los diagramas de Penrose, la solución general simplificó los cálculos necesarios. En esta sección se hallaron las siguientes propiedades para los horizontes cósmicos:

- En los casos donde $\mathcal{I}^+$ y $\mathcal{I}^-$ son hipersuperficies de tipo espacial, la variedad presenta horizontes de eventos y partículas, respectivamente.
- En los casos donde los infinitos nulos $\mathcal{I}$ son hipersuperficies nulas, los observadores inerciales no apreciarán ningún horizonte cósmico.

Estas propiedades han sido utilizadas posteriormente en el análisis de la estructura causal de las diferentes soluciones cosmológicas FLRW.

En este trabajo de fin de grado se han mostrado **todos los diagramas de Penrose** para soluciones cosmológicas del tipo FRW, se han mostrado específicamente para $k = 0$ y $k = -1$. Y para el caso de $k = -1$ se ha discutido que la altura del diagrama es proporcional a n . El cálculo de todos estos diagramas no habría sido posible sin la fórmula general para el factor de escala.

En conclusión, el presente trabajo de fin de grado ha cumplido sus objetivos iniciales con éxito y ha llegado a superarlos y a hallar una expresión general para el factor de escala en coordenadas conformes.

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