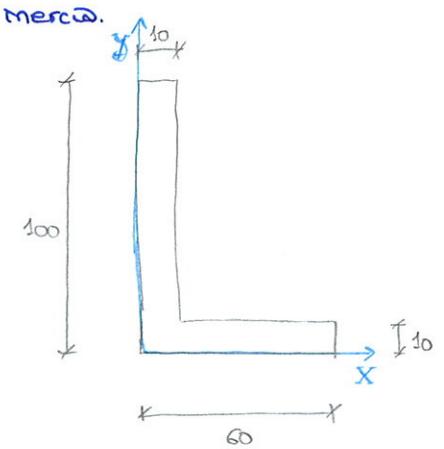
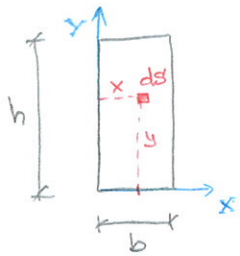
PROBLEMA 1.

Obtener los ejes principales centroidales de la sección dada y sus momentos de inercia.



CENTROIDE: $\vec{r}_G = \dfrac{\int_M \vec{r}\, dm}{\int_M dm} = \dfrac{\iiint_V \vec{r}\, dV}{\iiint_V dV}$ $\qquad \rho = \dfrac{dm}{dV}$

Recordamos el cálculo de centroide de una pieza rectangular:



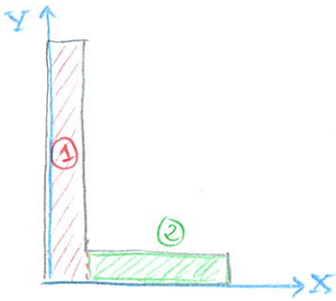
$$x_G = \frac{\iint_S x\, dS}{\iint_S dS} \qquad y_G = \frac{\iint_S y\, dS}{\iint_S dS}$$

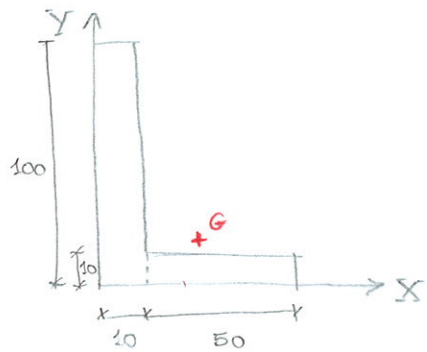
$$dS = dx\,dy \qquad \iint_S dS = \int_{x=0}^{b}\int_{y=0}^{h} dx\,dy = bh$$

$$x_G = \frac{1}{bh}\int_{x=0}^{b}\int_{y=0}^{h} x\,dx\,dy = \frac{1}{b\cancel{h}}\int_{x=0}^{b} \cancel{h}\,x\,dx\,dy = \frac{1}{b}\,\frac{x^2}{2}\Big|_0^b = \frac{b}{2}$$

$$y_G = \frac{1}{bh}\int_{x=0}^{b}\int_{y=0}^{h} y\,dx\,dy = \frac{h}{2}$$

Calculamos el centroide de la pieza:





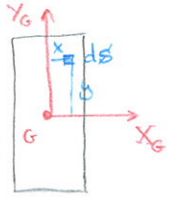
	S	x_{G_i}	y_{G_i}
1	10^3	5	50
2	500	35 (10+25)	5
	1500		

$$x_G = \frac{1000\cdot 5 + 500\cdot 35}{1500} = 15$$

$$y_G = \frac{1000\cdot 50 + 500\cdot 5}{1500} = 35$$

TENSOR DE INERCIA : $I_e = \int_M d_e^2 \, dm \qquad P_{\pi_1 \pi_2} = \int_M d_{\pi_1} d_{\pi_2} \, dm$

Recordamos el cálculo del momento de inercia de una pieza rectangular:



$dS = dx\,dy \qquad dm = \sigma\, dS = \sigma\, dx\,dy$

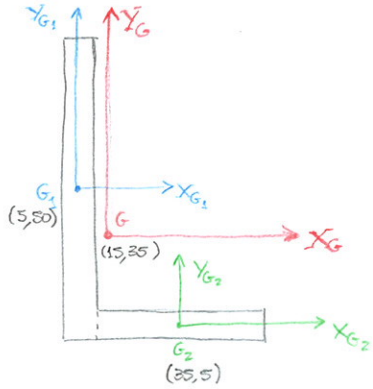
Suponemos densidad unitaria: $\sigma = 1$

$$I_{x_G} = \iint_S y^2 \, dS = \int_{x=-b/2}^{b/2} \int_{y=-h/2}^{h/2} y^2 \, dx\, dy = \frac{1}{12} b h^3$$

$$I_{y_G} = \iint_S x^2 \, dS = \int_{x=-b/2}^{b/2} \int_{y=-h/2}^{h/2} x^2 \, dy\, dx = \frac{1}{12} b^3 h$$

$$P_{x_G y_G} = \iint_S xy \, dS = \int_{x=-b/2}^{b/2} \int_{y=-h/2}^{h/2} xy \, dx\, dy = 0$$

Calculamos el momento de inercia para la pieza:



Teorema de Steiner:

$$I_e = I_{e_G} + M d_e^2$$

$$I_{x_{G_1}} = \frac{1}{12} b h^3 = \frac{1}{12} \cdot 10 \cdot 100^3 = \frac{2'5 \cdot 10^6}{3}$$

$$I_{x_{G_2}} = \frac{1}{12} \cdot 50 \cdot 10^3 = \frac{12'5 \cdot 10^3}{3}$$

$$I_{x_G}^{①} = \frac{2'5 \cdot 10^6}{3} + 10^3 \cdot (50-35)^2 = \frac{3'175 \cdot 10^6}{3}$$

$$I_{x_G}^{②} = \frac{12'5 \cdot 10^3}{3} + 500\,(35-5)^2 = \frac{1'3625 \cdot 10^6}{3}$$

$$I_{x_G} = I_{x_G}^{①} + I_{x_G}^{②} = 1'5125 \cdot 10^6$$

$$I_{y_{G_1}} = \frac{1}{12} b^3 h = \frac{1}{12} \cdot 10^3 \cdot 100 = \frac{25 \cdot 10^3}{3}$$

$$I_{y_{G_2}} = \frac{1}{12} 50^3 \cdot 10 = \frac{312'5 \cdot 10^3}{3}$$

$$I_{y_G}^{①} = \frac{25 \cdot 10^3}{3} + 10^3 (15-5)^2 = \frac{325 \cdot 10^3}{3}$$

$$I_{y_G}^{②} = \frac{312'5 \cdot 10^3}{3} + 500\,(35-15)^2 = \frac{912'50 \cdot 10^3}{3}$$

$$I_{y_G} = I_{y_G}^{①} + I_{y_G}^{②} = 412'5 \cdot 10^3$$

$$P_{x_{G_1} y_{G_1}} = 0$$

$$P_{x_{G_2} y_{G_2}} = 0$$

$$P_{x_G y_G}^{①} = 0 + 10^3 \cdot (15-5)(35-50) = -150000$$

$$P_{x_G y_G}^{②} = 0 + 500 \cdot (15-35)(35-5) = -300000$$

$$P_{x_G y_G} = P_{x_G y_G}^{①} + P_{x_G y_G}^{②} = -450 \cdot 10^3$$

TENSOR DE INERCIA RESPECTO AL CENTROIDE

$$\underline{I}_G = \begin{pmatrix} 1512500 & 450000 \\ 450000 & 412500 \end{pmatrix}$$

Calculamos los valores principales y direcciones principales:

$$|\underline{I}_G - \lambda \underline{I}| = 0$$

$$\begin{vmatrix} 1512500-\lambda & 450000 \\ 450000 & 412500-\lambda \end{vmatrix} = 0 \qquad (1512500-\lambda)(412500-\lambda) - 450000^2 = 0$$

$$\lambda^2 - 1925000\lambda + 623905'8 \cdot 10^6 = 0$$

$$\lambda = \frac{1925 \cdot 10^3 \pm \sqrt{(1925 \cdot 10^3)^2 - 4 \cdot 623905'8 \cdot 10^6}}{2}$$

$$I_G^{I} = 1673133'52 = I_{x_G^{(P)}}$$

$$I_G^{II} = 251866'48 = I_{y_G^{(P)}}$$

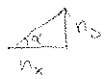
$$\begin{pmatrix} 1512500 - 1673133'52 & 450000 \\ 450000 & 412500 - 1673133'52 \end{pmatrix} \begin{pmatrix} n_x \\ n_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$n_x^2 + n_y^2 = 1$$

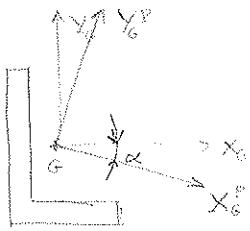
$$(1512500 - 167313'52)\, n_x + 450000\, n_y = 0 \; ; \qquad -160633'52\, n_x + 450000\, n_y = 0$$

$$n_x = 1 \; ; \qquad n_y = \frac{-160633'52}{450000} = -0'357 \qquad \sqrt{1 + (-0'357)^2} = 1'0618$$

$$n_x^{I} = \frac{1}{1'0618} = 0'9418 \qquad n_y^{I} = \frac{-0'357}{1'0618} = -0'336$$



$$\operatorname{tang} \alpha = \frac{-0'336}{0'9418} \Rightarrow \alpha = -0'342 \text{ rd} = -19'645^\circ$$

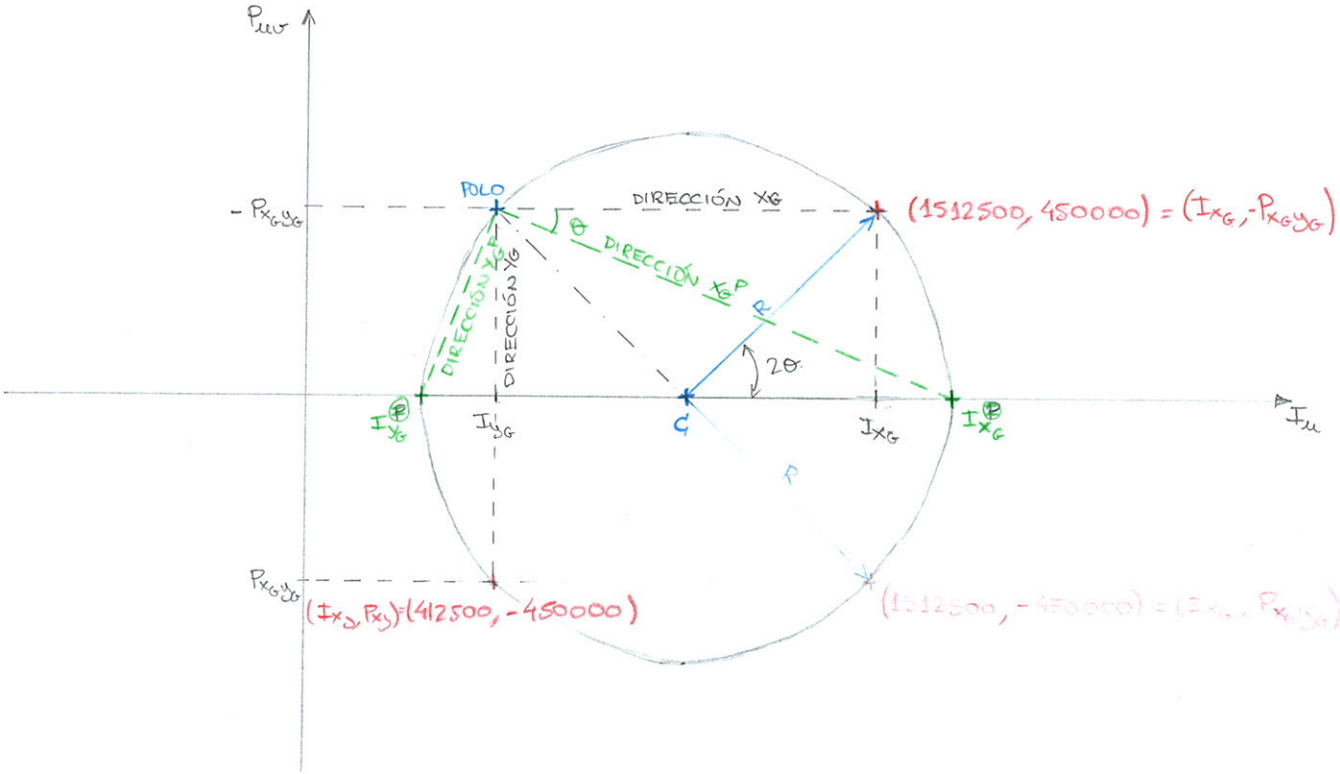


Empleamos el círculo de Mdur:

$I_{x_G} = 1.512.500$

$I_{y_G} = 412.500$

$P_{x_G y_G} = -450.000$



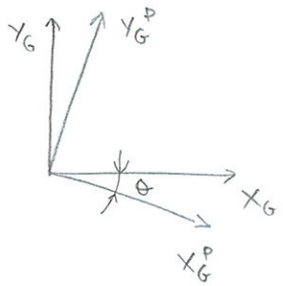
Centro: $C = \left(\frac{I_{x_G} + I_{y_G}}{2}, 0 \right) = (962500, 0)$

Radio = $R = \sqrt{(1512500 - 962500)^2 + 450000^2} = 710633'52$

$$I_{x_G}^P = \frac{I_x + I_y}{2} + \frac{1}{2}\sqrt{(I_x - I_y)^2 + 4P_{xy}^2} = 1673133'52 = C_I + R$$

$$I_{y_G}^P = \frac{I_x + I_y}{2} - \frac{1}{2}\sqrt{(I_x - I_y)^2 + 4P_{xy}^2} = 251866'48 = C_I - R$$

$$tg\,\theta = \frac{450000}{1673133'52 - 412500} = 0'357 \quad ; \quad \theta = 0'343\,rd = 19'645°$$



$$tg\,2\theta = \frac{450000}{1512500 - 962500} = 0'818$$

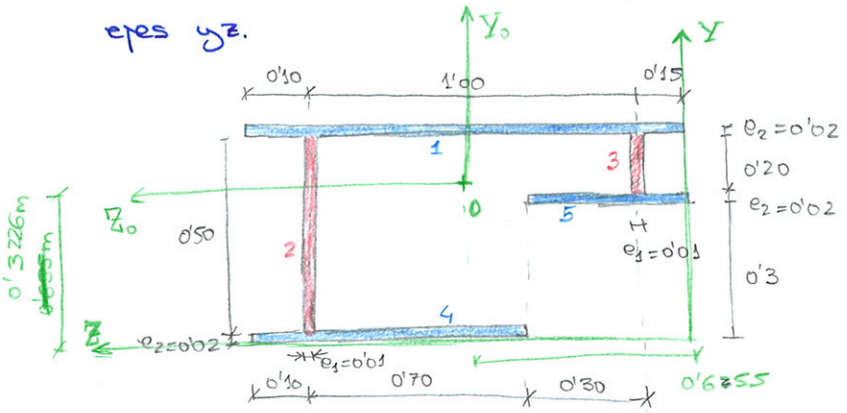
$2\theta = 0'685\,rd = 39'289°$

$\theta = 19'645°$

PROBLEMA 2.

Calcular el momento de inercia (I_y, I_z) y el producto de inercia (P_{yz}) del área indicada, sabiendo que el centroide de la misma (O) es el origen de los ejes y (vertical) y z (horizontal). Suponer que los espesores e_1 y e_2 son muy pequeños en comparación con el resto de longitudes. (Perfil de pared delgada)

Obtener los momentos principales de inercia y el ángulo que forman con los ejes yz.



(Cotas en m)

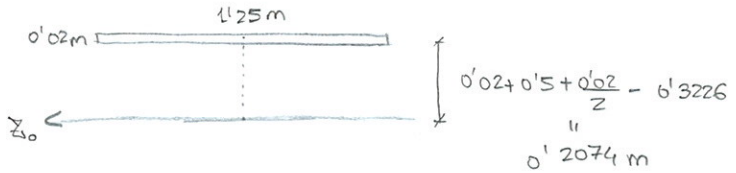
CALCULAMOS LA POSICIÓN DEL CENTROIDE

	$S(m^2)$	z_{0i}	y_{0i}
1	$(0'1+1+0'15)\,0'02 = 0'025$	$(0'1+1+0'15)/2 = 0'625$	$\left(0'02+0'5+\frac{0'02}{2}\right) = 0'53$
2	$0'5 \cdot 0'01 = 0'005$	$\left(0'15+1+\frac{0'01}{2}\right) = 1'155$	$0'02+\frac{0'5}{2} = 0'27$
3	$0'2 \cdot 0'01 = 0'002$	$0'15+\frac{0'01}{2} = 0'155$	$0'02+0'5-\frac{0'2}{2} = 0'42$
4	$(0'1+0'7)\,0'02 = 0'016$	$\left(0'15+0'3+\frac{0'8}{2}\right) = 0'85$	$\frac{0'02}{2} = 0'01$
5	$(0'3+0'15)\cdot 0'02 = 0'009$	$\frac{0'3+0'15}{2} = 0'225$	$0'02+0'5-0'2-\frac{0'02}{2} = 0'31$
	$0'057$		

$$z_0 = \frac{1}{0'057}\left(0'625 \cdot 0'025 + 1'155 \cdot 0'005 + 0'155 \cdot 0'002 + 0'85 \cdot 0'016 + 0'225 \cdot 0'009\right) = 0'655\,m$$

$$y_0 = \frac{1}{0'057}\left(0'53 \cdot 0'025 + 0'27 \cdot 0'005 + 0'42 \cdot 0'002 + 0'01 \cdot 0'016 + 0'31 \cdot 0'009\right) = 0'3226\,m$$

Como nos indican que e_1 y e_2 son muy pequeños, podemos calcular los momentos de inercia como si se tratase de alambres: Vamos a comprobarlo con $I_{z_0}^{①}$



$$I_{z_G}^{①} = \frac{1}{12} \cdot bh^3 = \frac{1}{12} \cdot 1'25 \cdot 0'02^3 = 8'333 \cdot 10^{-7}$$

$$I_{z_0}^{①} = I_{z_G}^{①} + S_1 \cdot d_1^2 = 8'333 \cdot 10^{-7} + 0'025 \cdot 0'2074^2 = 1'075 \cdot 10^{-3}$$

Alambre:

$$I_{z_0}^{1} = 0'025 \cdot 0'2074^2 = 1'075 \cdot 10^{-3}$$

	$S(m^2)$	$I_{z_{0i}}$
1	0'025	$0'075\ 0'2074^2 = 1'075 \cdot 10^{-3}$
2	0'005	$\frac{1}{12}\ 0'04 \cdot 0'5^3 + 0'005\ \left(0'02 + \frac{0'5}{2} - 0'3226\right)^2 = 1'18 \cdot 10^{-4}$
3	0'002	$\frac{1}{12} \cdot 0'01 \cdot 0'2^3 + 0'002\ \left(0'3 + 0'02 + \frac{0'2}{2} - 0'3226\right)^2 = 2'564 \cdot 10^{-5}$
4	0'016	$0'016 \cdot (0'3226 - 0'01)^2 = 1'5636\ 10^{-3}$
5	0'009	$0'009\ \left(0'3226 - 0'3 - \frac{0'02}{1}\right)^2 = 1'428 \cdot 10^{-6}$

$I_{z_0} = 27'839 \cdot 10^{-9}\ m^4$

	$S(m^2)$	$I_{y_{0i}}$
1	0'025	$\frac{1}{12} \cdot 1'25^3 \cdot 0'02 + 0'025 \cdot \left(\frac{1'25}{2} - 0'655\right)^2 = 3'278 \cdot 10^{-3}$
2	0'005	$0'005 \cdot (1'15 - 0'655)^2 = 1'225 \cdot 10^{-3}$
3	0'002	$0'002\ (0'655 - 0'15)^2 = 5'1 \cdot 10^{-4}$
4	0'016	$\frac{1}{12}\ 0'8^3 \cdot 0'02 + 0'016\ \left(0'15 + 0'3 + 0'4 - 0'655\right)^2 = 1'461 \cdot 10^{-3}$
5	0'009	$\frac{1}{12} \cdot 0'45^3 \cdot 0'02 + 0'009\ \left(0'655 - \frac{0'45}{2}\right)^2 = 1'816 \cdot 10^{-3}$

$I_{y_0} = 82'906 \cdot 10^{-4}\ m^4$

Sabemos que el producto de inercia respecto a $x_{0i}\ y_{0i}$ es nulo.

	$S(m^2)$	z_i	y_i	$P_{y_{0i} z_{0i}}$
1	0'025	$\frac{1'25}{2} - 0'655 = -0'03$	0'2074	$-1'555 \cdot 10^{-4}$
2	0'005	$1'15 - 0'655 = 0'495$	$0'02 + \frac{0'5}{2} - 0'3226 = -0'0526$	$-1'302 \cdot 10^{-4}$
3	0'002	$0'655 - 0'15 = 0'505$	$0'3 + 0'02 + \frac{0'2}{2} - 0'3226 = 0'0974$	$9'838 \cdot 10^{-5}$
4	0'016	$0'15 + 0'3 + 0'4 - 0'655 = 0'195$	$-0'3226 + 0'01 = -0'3126$	$-9'75 \cdot 10^{-4}$
5	0'009	$0'655 - \frac{0'45}{2} = 0'43$	$-0'3226 + 0'3 + 0'01 = -0'0126$	$-4'876 \cdot 10^{-5}$

$P_{y_0 z_0} = -12'114 \cdot 10^{-4}\ m^4$

CALCULAMOS LOS MOMENTOS DE INERCIA PRINCIPALES Y SU DIRECCIÓN

$$\underline{I}_0 = \begin{pmatrix} I_{y_0} & -P_{y_0z_0} \\ -P_{y_0z_0} & I_{z_0} \end{pmatrix} = \begin{pmatrix} 82'906\cdot 10^{-4} & 12'114\cdot 10^{-4} \\ 12'114\cdot 10^{-4} & 27'839\cdot 10^{-4} \end{pmatrix}$$

$$\begin{vmatrix} 82'906\cdot 10^{-4}-\lambda & 12'114\cdot 10^{-4} \\ 12'114\cdot 10^{-4} & 27'839\cdot 10^{-4}-\lambda \end{vmatrix} = 0 \qquad (82'906\,10^{-4}-\lambda)(27'839\cdot 10^{-4}-\lambda)-(12'114\cdot 10^{-4})^2 = 0$$

$$I^P_{y_0} = 85'4531\cdot 10^{-4}$$

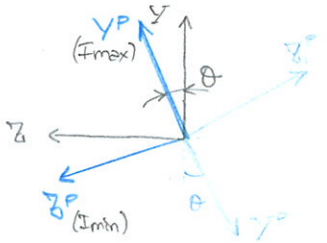
$$I^P_{z_0} = 25'292\cdot 10^{-4}$$

$$\begin{pmatrix} 82'906\cdot 10^{-4}-85'4531\cdot 10^{-4} & 12'114\cdot 10^{-4} \\ 12'114\cdot 10^{-4} & 27'839\cdot 10^{-4}-85'4531\cdot 10^{-4} \end{pmatrix}\begin{pmatrix} n_y \\ n_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$n_x^2+n_y^2 = 1$$

$$n_y^I = +0'9786$$

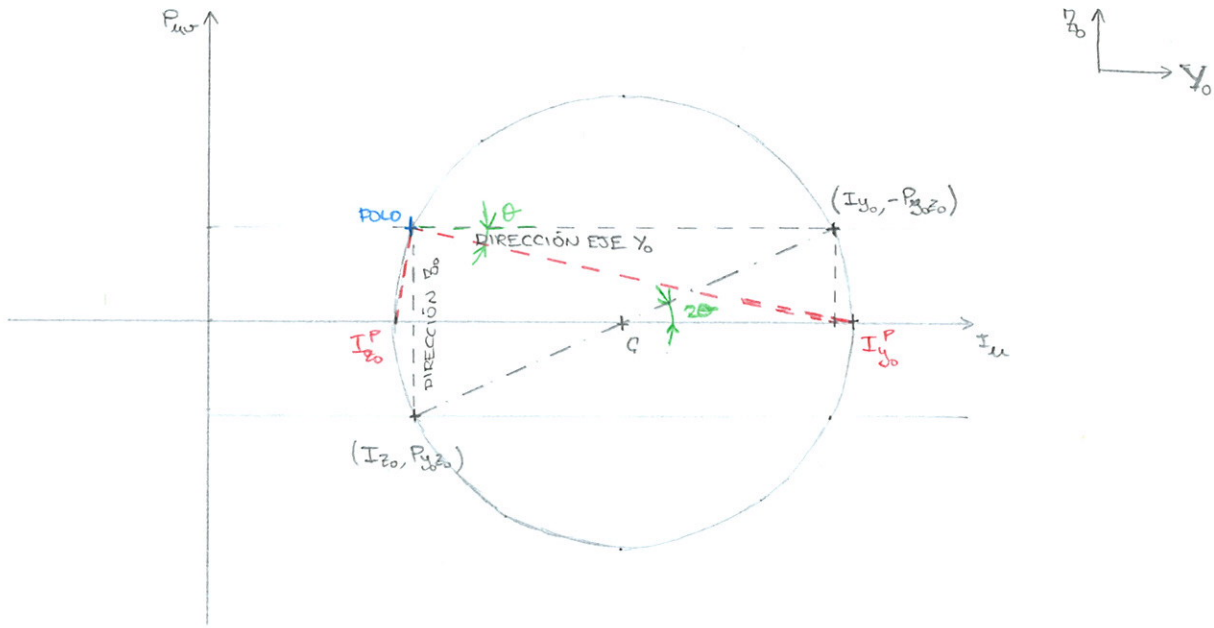
$$n_z^I = +0'2057$$



$$Tg\,\theta = \frac{0'2057}{0'9786} = 0'2103$$

$$\theta = 0'2072\ rad = 11'874^\circ$$

CÍRCULO DE MOHR



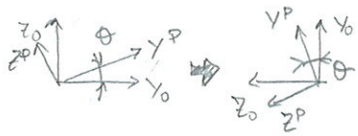
$$C = \left(\frac{I_{y_0}+I_{z_0}}{2}, 0\right) = (65'3725\cdot 10^{-4}, 0)$$

$$R = \sqrt{(I_{y_0}-C_{I_u})^2 + P_{y_0z_0}^2} = 30'081\cdot 10^{-4}$$

$$I^P_{y_0} = C_{I_u} + R = 65'3725\cdot 10^{-4} + 30'081\cdot 10^{-4} = 85'453\cdot 10^{-4}$$

$$I^P_{z_0} = C_{I_u} - R = 65'3725\cdot 10^{-4} - 30'081\cdot 10^{-4} = 25'292\cdot 10^{-4}$$

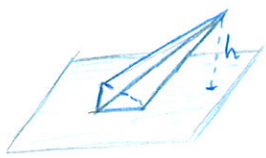
$$tg\,\theta = \frac{12'114\cdot 10^{-4}}{85'453\cdot 10^{-4} - 27'839\cdot 10^{-4}} = 0'2103 \longrightarrow \theta = 11'874^\circ$$

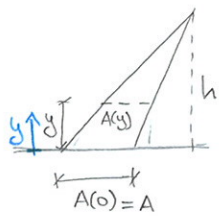


$$tg\,2\theta = \frac{12'114\cdot 10^{-4}}{82'906\cdot 10^{-4} - 65'3725\cdot 10^{-4}} = 0'6909 \rightarrow 2\theta = 0'6045\ rad = 34'64^\circ\ ;\ \theta = 17'32^\circ \quad ¿?$$

PROBLEMA 3.

Hallar la altura del centroide de una pirámide oblícua de base irregular.





Para un corte transversal se puede demostrar:

$$A(y) = A\,\frac{(h-y)^2}{h^2}$$

$$y_G = \frac{\int_M y\,dm}{\int_M dm} = \frac{\iiint_V y\rho\,dV}{\iiint_V \rho\,dV} = \frac{\iiint_V y\,dV}{\iiint_V dV}$$

$$dV = A(y)\,dy = A\,\frac{(h-y)^2}{h^2}\,dy$$

$$\iiint_V dV = \int_{y=0}^{h} \frac{A(h-y)^2}{h^2}\,dy = \frac{A}{h^2}\int_0^h (h-y)^2\,dy = \frac{A}{h^2}\,\frac{-(h-y)^3}{3}\Big|_0^h = \frac{A}{\cancel{h^2}}\,\frac{+h^3}{3} = \frac{Ah}{3}$$

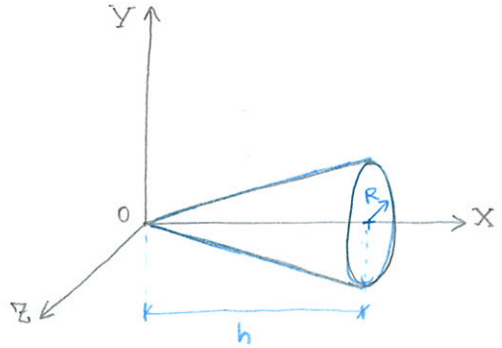
$$\iiint_V y\,dV = \int_{y=0}^{h} y\,\frac{A(h-y)^2}{h^2}\,dy = \frac{A}{h^2}\int_{y=0}^{h}\left(h^2y + y^3 - 2hy^2\right)dy = \frac{A}{h^2}\left(\frac{h^2y^2}{2} + \frac{y^4}{4} - \frac{2hy^3}{3}\right)\Big|_0^h =$$

$$= \frac{A}{h^2}\left(\frac{h^4}{2} + \frac{h^4}{4} - \frac{2h^4}{3}\right) = Ah^2\left(\frac{1}{2} + \frac{1}{4} - \frac{2}{3}\right) = \frac{1}{12}Ah^2$$

$$y_G = \frac{1}{12}\cancel{A}h^{\cancel{2}} \cdot \frac{3}{\cancel{A}\cancel{h}} = \frac{h}{4}$$

PROBLEMA 4.

Calcular: (i) el tensor de inercia para los ejes cartesianos. (ii) el momento de inercia según la generatriz del cono en el plano xy. (iii) ejes ~~principales~~ paralelo por el centroidales.

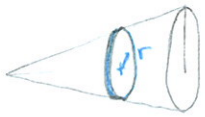


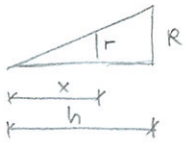
$$I_0 = \begin{pmatrix} I_x & -P_{xy} & -P_{yz} \\ -P_{xy} & I_y & -P_{yz} \\ -P_{xz} & -P_{yz} & I_z \end{pmatrix}$$

$$I_{xy} = \int_M z^2\,dm \qquad I_{xz} = \int_M y^2\,dm \qquad I_{yz} = \int_M x^2\,dm \qquad P_{xy} = \int_M xy\,dm$$

$$I_x = \int_M (y^2+z^2)\,dm \qquad I_y = \int_M (x^2+z^2)\,dm \qquad I_z = \int_M (x^2+y^2)\,dm \qquad P_{xz} = \int_M xz\,dm$$

$$I_x = I_{xy} + I_{xz} \qquad I_y = I_{xy} + I_{yz} \qquad I_z = I_{xz} + I_{yz} \qquad P_{yz} = \int_M yz\,dm$$

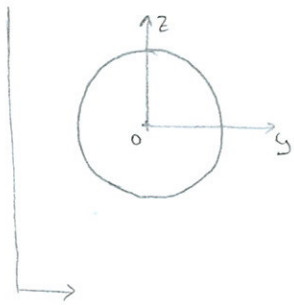




$$\frac{x}{h} = \frac{r}{R} \; ; \quad r = \frac{R}{h}x$$

$$dm = \rho\,dV = \rho\pi r^2 dx = \rho\pi \frac{R^2}{h^2} x^2 dx$$

Para el disco sabemos:



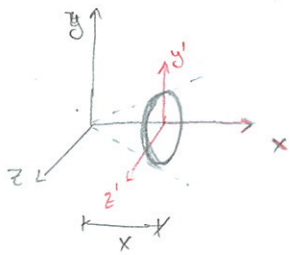
$$I_y = I_z = \frac{1}{4}\pi r^4 \sigma = \frac{1}{4} m r^2$$

$$I_x = I_0 = I_y + I_z = \frac{1}{2}\sigma\pi r^4 \qquad \sigma = \frac{m}{S} = \frac{m}{\pi r^2} \; ; \quad m = \sigma\pi r^2$$

$$I_x = \frac{1}{2} m r^2 \longrightarrow dI_x = \frac{1}{2} r^2 dm = \frac{1}{2}\left(\frac{R}{h}x\right)^2 dm = \frac{1}{2}\frac{R^2}{h^2}x^2\,dm$$

$$I_x = \int dI_x = \int_{x=0}^{h} \frac{1}{2}\frac{R^2}{h^2}x^2\,\rho\pi\frac{R^2}{h^2}x^2\,dx = \frac{1}{2}\rho\pi\frac{R^4}{h^4}\int_0^h x^4\,dx = \frac{1}{2}\rho\pi\frac{R^4}{h^4}\cdot\frac{x^5}{5}\Big|_0^h$$

$$I_x = \frac{1}{2}\rho\pi\frac{R^4}{h^4}\frac{h^5}{5} = \frac{1}{10}\rho\pi R^4 h$$



$$I_{y'} = \frac{1}{4} m r^2$$

$$I_y = \frac{1}{4} m r^2 + m x^2$$

$$dI_y = \left[\frac{1}{4}\left(\frac{R}{h}x\right)^2 + x^2\right] dm$$

$$I_y = \int_{x=0}^{h} \left(\frac{1}{4}\left(\frac{R}{h}x\right)^2 + x^2\right) \rho \pi \left(\frac{R}{h}x\right)^2 dx = \frac{1}{20}\rho\pi R^2 h\,(4h^2+R^2)$$

$I_z = I_y$

$P_{xy} = \int_M xy\,dm = 0$ porque es simétrico respecto a xy

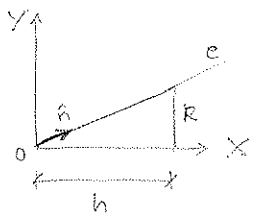
$P_{xz} = \int_M xz\,dm = 0$ " " " " " xz

$P_{yz} = \int_M yz\,dm = 0$ por simetría a 2 planos coordenados.

$$\underline{\underline{I}}_o = \begin{pmatrix} \frac{1}{10}\rho\pi R^4 h & 0 & 0 \\ 0 & \frac{1}{20}\rho\pi R^2 h(4h^2+R^2) & 0 \\ 0 & 0 & \frac{1}{20}\rho\pi R^2 h(4h^2+R^2) \end{pmatrix}$$

Son los ejes principales de inercia.

(ii)



Dirección de la generatriz:

$\vec{e} = (h, R, 0)$

$|\vec{e}| = \sqrt{h^2+R^2}$

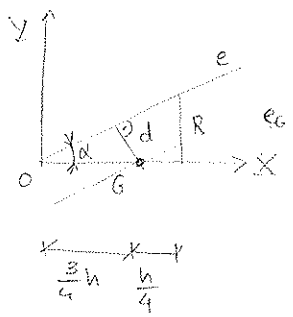
$\vec{n} = \left(\frac{h}{\sqrt{h^2+R^2}}, \frac{R}{\sqrt{h^2+R^2}}, 0\right)$

$$I_e = \frac{1}{\sqrt{h^2+R^2}}\begin{pmatrix} h & R & 0 \end{pmatrix}\begin{pmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{pmatrix}\frac{1}{\sqrt{h^2+R^2}}\begin{pmatrix} h \\ R \\ 0 \end{pmatrix} = \frac{1}{h^2+R^2}\begin{pmatrix} h & R & 0 \end{pmatrix}\begin{pmatrix} hI_x \\ RI_y \\ 0 \end{pmatrix}$$

$$I_e = \frac{1}{h^2+R^2}\left(h^2 I_x + R^2 I_y\right) = \frac{1}{h^2+R^2}\left(h^2\frac{1}{10}\rho\pi R^4 h + R^2\frac{1}{20}\rho\pi R^2 h(4h^2+R^2)\right) =$$

$$I_e = \frac{1}{20}\rho\pi R^4 h\,\frac{6h^2+R^2}{h^2+R^2}$$

(iii)



$I_e = I_{e_G} + M\,d^2$

$M = \rho\,V = \rho\,\frac{1}{3}\pi R^2 h$

$\operatorname{sen}\alpha = \frac{R}{\sqrt{R^2+h^2}} = \frac{d}{\frac{3}{4}h}$; $d = \frac{3}{4}h\frac{R}{\sqrt{h^2+R^2}}$

$$I_{e_G} = I_e - M d^2 = \frac{1}{20}\rho\pi R^4 h\,\frac{6h^2+R^2}{h^2+R^2} - \rho\,\frac{1}{3}\pi R^2 h\left(\frac{3}{4}h\frac{R}{\sqrt{h^2+R^2}}\right)^2 = \frac{1}{80}\rho\pi R^4 h\,\frac{9h^2+4R^2}{h^2+R^2}$$