# Complete Spacelike Surfaces with a Constant Principal Curvature in the 3-dimensional de Sitter Space

## to the meeting on Lorentzian Geometry

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**Abstract**

In this work we study complete spacelike surfaces with a constant principal curvature $R$ in the 3-dimensional de Sitter space $\mathbf{S}^3_1$, proving that if $R^2 < 1$, such a surface is either totally umbilical or umbilically free. Moreover, in the second case we prove that the surface can be described in terms of a complete regular curve in $\mathbf{S}^3_1$. We also give examples which show that the result is not true when $R^2 \geq 1$

## 1 Introduction and Statement of the Main Result

Spacelike surfaces in the de Sitter space $\mathbf{S}^3_1$ have been of increasing interest in the recent years from different points of view. That interest is motivated, in part, by the fact that they exhibit nice Bernstein-type properties. For instance, Ramanathan [7] proved that every compact spacelike surface in $\mathbf{S}^3_1$ with constant mean curvature is totally umbilical. This result was generalized to hypersurfaces of any dimension by Montiel [6]. On the other hand, Li [5] obtained the same conclusion when the compact spacelike surface has constant Gaussian curvature. More recently, the first author jointly Romero [3] have proved that the totally umbilical round spheres are the only compact spacelike surfaces in the de Sitter space such that the Gaussian curvature of the second fundamental form is constant.

As a natural generalization of Ramanathan and Li results, the authors [1] have recently proved that the only compact linear Weingarten spacelike surfaces in $\mathbf{S}_1^3$, (that is, surfaces satisfying that a linear combination of their mean and Gaussian curvatures is constant) are the totally umbilical round spheres. In the quoted paper, we also study compact spacelike surfaces with a constant principal curvature, proving that such surfaces are totally umbilical round spheres.

In this work we extend the last result to complete spacelike surfaces in the following terms:

**Theorem 1** *Let $\psi : M \longrightarrow \mathbf{S}_1^3$ be a complete spacelike surface with a constant principal curvature $R$ such that $R^2 < 1$. Then $\psi(M)$ is either totally umbilical or umbilically free. In the second case $R > 0$ and the surface is not compact and can be described as*

$$\psi(x, y) = \frac{1}{\sqrt{1-R^2}} \ (R\alpha(y) + \cos(x)v_1(y) + \sin(x)v_2(y)) \qquad (1)$$

*where $\alpha$ is a $C^\infty$ complete regular curve in $\mathbf{S}_1^3$ and $\{v_1(y), v_2(y)\}$ is an orthonormal frame of the normal plane along $\alpha$.*

*Conversely, given a regular curve $\alpha$ in $\mathbf{S}_1^3$, (1) defines an umbilically free spacelike immersion in $\mathbf{S}_1^3$ with a constant principal curvature $R$ such that $0 < R^2 < 1$.*

To finish, in Section 4, we construct some examples which show that the result is false when $R^2 \geq 1$

## 2 Preliminaries

Let $\mathbf{L}^4$ be the 4-dimensional *Lorentz-Minkowski space*, that is, the real vector space $\mathbf{R}^4$ endowed with the Lorentzian metric tensor $\langle , \rangle$ given by

$$\langle , \rangle = dx_1^2 + dx_2^2 + dx_3^2 - dx_4^2,$$

where $(x_1, x_2, x_3, x_4)$ are the canonical coordinates of $\mathbf{R}^4$. The 3-dimensional unitary *de Sitter space* is given as the following hyperquadric of $\mathbf{L}^4$,

$$\mathbf{S}_1^3 = \{x \in \mathbf{L}^4 : \langle x, x \rangle = 1\}.$$

As is well known, $\mathbf{S}_1^3$ inherits from $\mathbf{L}^4$ a time-orientable Lorentzian metric which makes it the standard model of a Lorentzian space of constant sectional curvature one. A smooth immersion $\psi : M^2 \longrightarrow \mathbf{S}_1^3 \subset \mathbf{L}^4$ of a 2-dimensional connected manifold $M$ is said to be a *spacelike surface* if the induced metric via $\psi$ is a Riemannian metric on $M$, which, as usual, is also denoted by $\langle , \rangle$. The time-orientation of $\mathbf{S}_1^3$ allows us to choose a timelike unit normal field $N$ globally defined on $M$, tangent to $\mathbf{S}_1^3$, and hence we may assume that $M$ is oriented by $N$. Finally, we will denote by $\lambda_1, \lambda_2$ the principal curvatures of $M$ associated to $N$.

## 3 Proof of the Theorem

Let $\psi : M^2 \longrightarrow \mathbf{S}_1^3 \subset \mathbf{L}^4$ be a complete spacelike surface in $\mathbf{S}_1^3$ with a constant principal curvature $0 \leq \lambda_1 = R < 1$ (up to a change of orientation). If there exists a non umbilical point $p \in M$, then we can consider local parameters $(u, v)$ in a neighborhood $U$ of $p$ without umbilical points, such that

$$\begin{aligned} \langle d\psi, d\psi \rangle &= E\ du^2 + G\ dv^2 \\ \langle d\psi, -dN \rangle &= RE\ du^2 + \lambda_2 G\ dv^2 \end{aligned}$$

where the principal curvature $\lambda_2 \neq R$. Then, the structure equations are given by

$$\begin{aligned} \psi_{uu} &= \frac{E_u}{2E}\ \psi_u - \frac{E_v}{2G}\ \psi_v - REN - E\psi \\ \psi_{uv} &= \frac{E_v}{2E}\ \psi_u + \frac{G_u}{2G}\ \psi_v \\ \psi_{vv} &= -\frac{G_u}{2E}\ \psi_u + \frac{G_v}{2G}\ \psi_v - \lambda_2 GN - G\psi \\ N_u &= -R\ \psi_u \\ N_v &= -\lambda_2\ \psi_v \end{aligned}$$

and the Mainardi-Codazzi equations for the immersion $\psi$ are

$$\begin{aligned} &(R - \lambda_2)E_v = 0 \\ &(R - \lambda_2)\frac{G_u}{2G} + (R - \lambda_2)_u = 0. \end{aligned}$$

Since $\lambda_2 \neq R$, the coefficient $E$ does not depend on $v$, that is, $E = E(u)$. If we consider the new parameters

$$x = \int \sqrt{E(u)}\ du, \qquad y = v$$

the structure equations become

$$\begin{aligned}
\psi_{xx} &= -RN - \psi \\
\psi_{xy} &= \frac{G_x}{2G}\ \psi_y \\
\psi_{yy} &= -\frac{G_x}{2}\ \psi_x + \frac{G_y}{2G}\ \psi_y - \lambda_2 GN - G\psi \\
N_x &= -R\ \psi_x \\
N_y &= -\lambda_2\ \psi_y
\end{aligned} \tag{2}$$

and the Mainardi-Codazzi equation is

$$(R-\lambda_2)\frac{G_x}{2G} + (R-\lambda_2)_x = 0. \tag{3}$$

On the other hand, the Gauss equation is given by

$$\left(\frac{G_x}{2G}\right)_x + \left(\frac{G_x}{2G}\right)^2 = R\lambda_2 - 1 = R(\lambda_2 - R) + R^2 - 1. \tag{4}$$

Thus, if we take

$$\varphi = \frac{1}{R-\lambda_2}$$

we obtain from (3) and (4) that

$$\begin{aligned}
\varphi_x &= \frac{G_x}{2G}\ \varphi \\
\varphi_{xx} &= \left(\left(\frac{G_x}{2G}\right)_x + \left(\frac{G_x}{2G}\right)^2\right)\ \varphi = -R - (1-R^2)\varphi
\end{aligned} \tag{5}$$

Let $\gamma_q$ be the maximal line of curvature passing through a point $q = \psi(x_o, y_o) \in U$ for the principal curvature $R$. Then, from (2) it follows that $\gamma_q(t) = \psi(x_o + t, y_o)$ satisfies

$$\begin{aligned}
(\gamma_q)_{tt} &= -R(N\circ\gamma_q) - \gamma_q \\
(N\circ\gamma_q)_t &= -R(\gamma_q)_t
\end{aligned}$$

so that $\gamma_q$ is a geodesic curve, which is a solution of the differential equation

$$(\gamma_q)_{tt} + (1-R^2)\gamma_q = Rw_o$$

for a constant vector $w_o \in \mathbf{L}^4$. Therefore, taking into account that $0 \le R < 1$, $\gamma_q$ is given by

$$\gamma_q = \cos\left(\sqrt{1-R^2}\ t\right) w_1 + \sin\left(\sqrt{1-R^2}\ t\right) w_2 + \frac{R}{1-R^2}\ w_o \tag{6}$$

for suitable vectors $w_1, w_2 \in \mathbf{L}^4$.

From (5), the principal curvature $\lambda_2$ can be calculated on $\gamma_q$ as

$$R - \lambda_2 = \left( a\cos\left(\sqrt{1-R^2}\; t\right) + b\sin\left(\sqrt{1-R^2}\; t\right) - \frac{R}{1-R^2} \right)^{-1} \tag{7}$$

for real constants $a, b$.

Hence, if $\gamma_q(t_1)$ is the first umbilical point on $\gamma_q$, we obtain from (7) and the continuity of $\lambda_2$ that

$$0 = R - \lambda_2(\gamma_q(t_1)) = \lim_{t\to t_1} R - \lambda_2(\gamma_q(t)) \neq 0$$

which is a contradiction. Therefore, there is no umbilical point on $\gamma_q$.

Observe that, since $M$ is complete, it follows that the geodesic $\gamma_q$ is defined for all $t \in \mathbf{R}$. Moreover $R \neq 0$, because in that case

$$a\cos\left(\sqrt{1-R^2}\; t\right) + b\sin\left(\sqrt{1-R^2}\; t\right) = 0$$

for some $t \in \mathbf{R}$, which contradicts the continuity of $\lambda_2$.

Let $\widetilde{U}$ be the connected component of non umbilical points containing $p$. Note that $\widetilde{U}$ is an open set, and from the above reasoning, can be parametrized by $(x, y) \in (-\infty, \infty) \times (\beta_1, \beta_2)$ for certain $\beta_1, \beta_2$, where $-\infty \leq \beta_1 < \beta_2 \leq \infty$, so that the immersion can be expressed from (6) as

$$\begin{aligned} \psi(x,y) &= \cos\left(\sqrt{1-R^2}\; x\right) w_1(y) \\ &\quad + \sin\left(\sqrt{1-R^2}\; x\right) w_2(y) + \frac{R}{1-R^2}\; w_o(y) \end{aligned} \tag{8}$$

Let us suppose now that there exists an umbilical point $\widetilde{q} \in \partial\psi(\widetilde{U})$. Then there exists a sequence of points $q_n = \psi(x_n, y_n)$ tending to $\widetilde{q}$, being $(x_n, y_n) \in [0, 2\pi/\sqrt{1-R^2}] \times (\beta_1, \beta_2)$. Therefore the sequence of compact geodesics $\gamma_n$ of length $2\pi/\sqrt{1-R^2}$ passing through $q_n$ associated to the principal curvature $R$, converges to a compact geodesic $\gamma_{\widetilde{q}}$ passing through $\widetilde{q}$ which is also a line of curvature for the principal curvature $R$.

Now, from the above argument, it is sufficient to prove that there exists a non umbilical point on $\gamma_{\widetilde{q}}$. In fact, from (7) we are able to choose a point $p_n \in \gamma_n$ such that $\lambda_2(p_n) = 1/R \neq R$. Finally, from an argument of compactness, there exists a subsequence $\{p_k\}$ of $\{p_n\}$ converging to a non umbilical point $\widetilde{p} \in \gamma_{\widetilde{q}}$.

Consequently $M$ is either umbilically free or totally umbilical.

Observe that (8) can be rewritten as

$$\psi(x,y) = \frac{1}{\sqrt{1-R^2}} \left( R\alpha(y) + \cos\left(\sqrt{1-R^2}\, x\right) v_1(y) + \sin\left(\sqrt{1-R^2}\, x\right) v_2(y) \right)$$

where

$$\alpha = \frac{1}{\sqrt{1-R^2}}\, w_o, \qquad v_1 = \sqrt{1-R^2}\, w_1, \qquad v_2 = \sqrt{1-R^2}\, w_2.$$

From the expressions of $\psi$ and $\psi_{xx}$, the Gauss map $N$ can be calculated using the first equation in (2). Thus, since $\langle\psi,\psi\rangle = 1$, $\langle\psi_x,\psi_x\rangle = 1$, $\langle N,N\rangle = -1$ and they are mutually orthogonal, it follows that $\alpha, v_1, v_2$ are orthogonal, and $\langle\alpha,\alpha\rangle = -1$, $\langle v_1,v_1\rangle = 1$ and $\langle v_2,v_2\rangle = 1$.

On the other hand, since $\psi_y$ is orthogonal to $\psi_x$ and $N$, we get using also that $\langle\psi_{xy},N\rangle = 0$

$$\alpha' = \mu_o P, \qquad v_1' = \mu_1 P, \qquad v_2' = \mu_2 P,$$

where $P$ is the wedge product of $\alpha, v_1$ and $v_2$ in $\mathbf{L}^4$, and $\mu_o$, $\mu_1$, $\mu_2$ are $C^\infty$ functions. Moreover, since

$$\langle\psi_y,\psi_y\rangle = \frac{1}{1-R^2}\left(R\mu_o + \cos\left(\sqrt{1-R^2}\,x\right)\mu_1 + \sin\left(\sqrt{1-R^2}\,x\right)\mu_2\right)^2$$

is positive, it follows that $\mu_o \neq 0$ and therefore $\alpha$ is a regular curve with tangent vector $P$. In particular, $\{v_1(y), v_2(y)\}$ is an orthonormal frame of the normal plane along $\alpha$. Finally, the completeness of $\alpha$ follows from the completeness of $M$.

The converse is a straightforward computation. Anyway, it is worth pointing out that $R \neq 0$ because in other case from (7) and the completeness of the immersion, there would exist a point on $\gamma_q$ where

$$a\cos\left(\sqrt{1-R^2}\,t\right) + b\sin\left(\sqrt{1-R^2}\,t\right) = 0$$

for some $t \in \mathbf{R}$, which contradicts the continuity of $\lambda_2$. Therefore the surface must be a totally umbilical round sphere. ∎

**Remark 2** Observe that we have not assumed that the principal curvatures $\lambda_1, \lambda_2$ *are necessarily ordered*, but

$$(\lambda_1 - R)(\lambda_2 - R) = 0.$$

∎

## 4 Examples

We use the following well-known result (see, for instance, [4]) *Given a Riemannian metric $I$ and a symmetric (2,0)-tensor $II$ on a simply-connected 2-dimensional manifold $M$, if $I$ and $II$ satisfy the Gauss and Mainardi-Codazzi equations of the de Sitter space $\mathbf{S}_1^3$, then there exists an only immersion (up to an isometry) $\psi : M \longrightarrow \mathbf{S}_1^3$ such that $I$ and $II$ are its first and second fundamental forms, respectively.*

First, we construct a family of complete orientable surfaces in $\mathbf{S}_1^3$ with umbilical and non umbilical points, and a constant principal curvature $R = 1$.

**Example 3** Let us consider $M = \mathbf{R}^2$ and $\psi_h$ the only immersion (up to an isometry) with first and second fundamental forms given by

$$I_h = dx^2 + \frac{1}{4}\Big(2 + (x^2 - 2)h(y)\Big)^2 dy^2$$

and

$$II_h = dx^2 + \frac{1}{4}\Big(2 + x^2 h(y)\Big)\Big(2 + (x^2 - 2)h(y)\Big) dy^2$$

respectively, where $h : \mathbf{R} \longrightarrow \mathbf{R}$ is a $C^\infty$ function which vanishes at some point such that $0 \leq h(y) \leq c < 1$ for a constant $c$. Since

$$I = dx^2 + \frac{1}{4}\Big(x^2 h(y) + 2(1 - h(y))\Big)^2 dy^2 \geq dx^2 + (1-c)^2\, dy^2$$

and $dx^2 + (1-c)^2\, dy^2$ is a complete metric, the immersion $\psi_h$ is complete with principal curvatures

$$1 \qquad \text{and} \qquad \frac{2 + x^2 h(y)}{2 + (x^2 - 2)h(y)}.$$

Thus, the set of umbilical points is $\Omega = \{(x, y) : h(y) = 0\}$.

It is worth pointing out that the interior of $\Omega$ may be non empty. ■

Now we are going to construct a family of complete orientable surfaces in $\mathbf{S}_1^3$ with umbilical and non umbilical points, and a constant principal curvature $R > 1$.

**Example 4** Let us consider again $M = \mathbf{R}^2$ and $\psi_{R,h}$ the only immersion (up to an isometry) with first and second fundamental forms

$$I_{R,h} = \frac{1}{R^2}\, dx^2 + \frac{\left(R^3 h(y) + (R^2 - 1)e^{\sqrt{1 - \frac{1}{R^2}}\; x}\right)^2}{R^2 (R^2 - 1)^2}\, dy^2$$

and

$$\begin{aligned} II_{R,h} &= \frac{1}{R}\,dx^2 + \left(\frac{1}{R}\,e^{\sqrt{1-\frac{1}{R^2}}\;2x}\right. \\ &\quad \left. +\frac{h(y)}{(R^2-1)^2}\left(R^3h(y)+(R^4-1)e^{\sqrt{1-\frac{1}{R^2}}\;x}\right)\right)\,dy^2 \end{aligned}$$

respectively, where $h:\mathbf{R}\longrightarrow\mathbf{R}$ is a non negative $C^\infty$ function which vanishes at some point and $R>1$. Since

$$I_{R,h} \geq \frac{1}{R^2}\left(dx^2 + e^{\sqrt{1-\frac{1}{R^2}}\;2x}dy^2\right),$$

the immersion $\psi_h$ is complete, with principal curvatures

$$R \qquad \text{and} \qquad R\,\frac{Rh(y)+(R^2-1)e^{\sqrt{1-\frac{1}{R^2}}\;x}}{R^3h(y)+(R^2-1)e^{\sqrt{1-\frac{1}{R^2}}\;x}}$$

Again, the set of umbilical points is $\Omega=\{(x,y):h(y)=0\}$.

As above, note that this surface can meet a totally umbilical surface in an open set. ∎

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