Some remarks on convex surfaces in simply connected homogeneous three manifolds

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Abstract

Here we extend Hadamards' Theorem to other homogeneous three manifolds, i.e., we prove that a compact orientable immersed surface Σ in general position whose principal curvatures κ_i, $i = 1, 2$, satisfy $\kappa_i \geq \tau$, is an embedded sphere.

2010 Mathematical Subject classification: 53C42, 53C24.
Keywords and phrases: Hadamard's Theorem, homogeneous 3-manifolds.

1 Introduction

When is an immersed surface Σ, that is locally convex, an embedded globally convex surface? There are now many answers to this question that depend on the meaning of *locally convex* and the ambient manifold.

In $\mathbb{R}^3$ (and more generally in the space forms $\mathbb{H}^3$ and $\mathbb{S}^3$), there are complete answers. Hadamard [H] proved that a smooth compact surface that is immersed in $\mathbb{R}^3$ with extrinsic curvature $K_e > 0$, is an embedded sphere that bounds a convex body. Stoker then extended this to complete immersed surfaces in $\mathbb{R}^3$ with $K_e > 0$, proving the surface is an embedded sphere

[1]The author is partially supported by Spanish MEC-FEDER Grant MTM2007-65249, and Regional J. Andalucia Grants P06-FQM-01642 and FQM325

[2]The author is partially supported by Spanish MEC-FEDER Grant MTM2007-65249, and Regional J. Andalucia Grants P06-FQM-01642 and FQM325

or plane, and bounds a convex domain. These results were established also in $\mathbb{S}^3$ and $\mathbb{H}^3$ (see [CW] and [C]).

J. Van Heijenhoort [VH] considered topological immersions of complete surfaces in $\mathbb{R}^3$ that are locally convex. This means each point of the surface has a neighborhood that is on the boundary of a convex body. When this neighborhood contains no line segments, then it is called strictly convex. He then proved that such a locally convex surface in $\mathbb{R}^3$, that has at least one point that is strictly locally convex, is an embedded sphere or plane, the boundary of a convex body.

Notice this fails to be true if there is no strictly convex point. Let $\Sigma := \gamma \times \mathbb{R}$ where γ is an immersed, non embedded, closed curve in the $(x, y)-$plane, of strictly positive curvature.

The authors [EGR] have studied this question in other $3-$manifolds. The simplest, most symmetric such manifolds after the space forms, are the Riemannian submersions $\mathbb{E}(\kappa, \tau)$ over the $\mathbb{M}^2(\kappa)$, where $\mathbb{M}^2(\kappa)$ is either $\mathbb{R}^2$ if $\kappa = 0$, $\mathbb{S}^2(\kappa)$ if $\kappa > 0$, or $\mathbb{H}^2(\kappa)$ if $\kappa < 0$. Here τ is the bundle curvature and the $\mathbb{E}(\kappa, \tau)$ are $\mathbb{S}^2(\kappa) \times \mathbb{R}$, $\mathbb{H}^2(\kappa) \times \mathbb{R}$ for $\tau = 0$, and Berger spheres, Heisenberg space and $\widetilde{\mathrm{PSL}(2, \mathbb{R})}$ for $\tau \neq 0$. We take $\tau = 1/2$ and $\kappa = 0$ or $\kappa = \pm 1$ for this paper.

In $\mathbb{H}^2 \times \mathbb{R}$ we proved a Hadamard-Stoker type theorem

> **Theorem:** *Let Σ be a complete immersed surface in $\mathbb{H}^2 \times \mathbb{R}$ whose extrinsic curvature K_e is strictly positive. Then Σ is an embedded sphere or plane.*

Moreover, in [ES], the authors generalized the aforementioned result to general Hadamard-Killing submersions among others.

In this paper we discuss some extensions of the Hadamard Theorem to the other homogeneous spaces, where we allow K_e to be zero. In fact, we will assume the second fundamental form of the immersion is semi-definite, which is stronger than $K_e \geq 0$. We will prove a theorem which says when such an immersed surface is an embedded sphere.

2 Preliminaries

Let $\mathbb{E}$ denote one of the manifolds $\mathbb{H}^2 \times \mathbb{R}$, $\widetilde{\mathrm{PSL}(2, \mathbb{R})}$ or $\mathrm{Nil}(3)$. The first two are Riemannian submersions over $\mathbb{H}^2$ ($\tau = 0$ and $\tau = 1/2$ respectively) and $\mathrm{Nil}(3)$ fibers over $\mathbb{R}^2$ (this is the Heisenberg space: topologically $\mathbb{R}^3$ and $\tau = 1/2$). We denote by $\pi : \mathbb{E} \to \mathbb{H}^2$ or $\mathbb{R}^2$ the bundle submersion.

Let Σ be an orientable compact immersed surface in $\mathbb{E}$. Henceforth we will always assume its second fundamental form II is semi-definite, and we orient Σ by the unit normal N so that $II \geq 0$ on Σ, i.e., the principal curvatures κ_1 and κ_2 of Σ are non-negative on Σ.

Let Ω be a domain of $\mathbb{M}^2(\kappa)$ such that $\Sigma \subset \pi^{-1}(\Omega)$. Let $\mathcal{F}(\Omega)$ be a foliation of Ω whose leaves are geodesics of $\mathbb{M}^2(\kappa)$. For each geodesic $\gamma \in \mathcal{F}(\Omega)$, $P(\gamma) := \pi^{-1}(\gamma)$ is a vertical *plane* in $\mathbb{E}$. $P(\gamma)$ is locally isometric to $\mathbb{R}^2$. In $\mathbb{H}^2 \times \mathbb{R}$, $P(\gamma)$ is totally geodesic, and in $\mathrm{Nil}(3)$ and $\widetilde{\mathrm{PSL}}(2, \mathbb{R})$, $P(\gamma)$ is a minimal surface with principal curvatures $\pm\tau = \pm 1/2$.

The pullback of $\mathcal{F}(\Omega)$ by π induces a foliation $\mathcal{F}$ of $\pi^{-1}(\Omega)$ by the vertical planes $P(\gamma)$, $\gamma \in \mathcal{F}(\Omega)$.

We remark that when Ω is relatively compact in $\mathbb{M}^2(\kappa)$, the space of geodesic foliations of Ω is infinite dimensional. This also holds when $\mathbb{M}^2(\kappa) = \mathbb{H}^2 = \Omega$.

Definition 2.1. *We say Σ is in general position with respect to a foliation $\mathcal{F} := \pi^{-1}(\mathcal{F}(\Omega))$ if each leaf $P(\gamma)$ of $\mathcal{F}$ satisfies one of the following conditions:*

- *$P(\gamma)$ meets Σ transversally,*
- *$P(\gamma) \cap \Sigma$ has a finite number of points where they are tangent and such a point p is the intersection of some k smooth curves of $\Sigma \cap P(\gamma)$ passing through p, and meeting transversally at p, $k \geq 2$ (see Figure 1)*
- *$P(\gamma) \cap \Sigma$ contains vertical segments along which they are tangent,*
- *$P(\gamma) \cap \Sigma$ contains isolated points or is empty.*

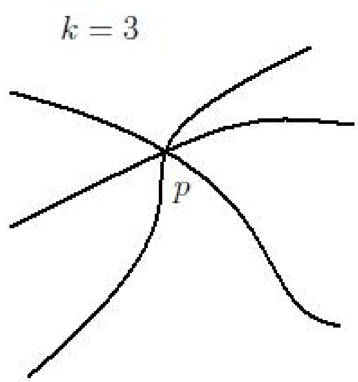

Figure 1

3 The main result

In this section we shall establish:

Theorem 3.1. *Let Σ be a compact orientable surface immersed in $\mathbb{E}$. Assume the principal curvatures κ_i, $i = 1, 2$, of Σ satisfy $\kappa_i \geq \tau$, $\tau = 0$ in $\mathbb{H}^2 \times \mathbb{R}$ and $\tau = 1/2$ in $\mathrm{Nil}(3)$ or $\widetilde{\mathrm{PSL}}(2, \mathbb{R})$.*

If Σ is in general position with respect to a foliation $\mathcal{F} := \pi^{-1}(\mathcal{F}(\Omega))$, then Σ is an embedded sphere.

We need a preliminary result

Lemma 3.1. *Let P and Σ be orientable immersed submanifolds of $\mathbb{E}$ that meet transversally along an immersed curve β. Let κ_i, $i = 1, 2$, be the principal curvatures of Σ and τ_i, $i = 1, 2$, those of P. If $\kappa_1 \geq \kappa_2$ along β and $\kappa_2 \geq \max\{|\tau_i| : i = 1, 2\}$, then the curvature vector of β in P points to one side of β in P (it may vanish at points of β).*

Proof. We use the notation $\nabla^{\mathbb{E}}$, ∇^{Σ}, ∇^{P}, II_Σ and II_P to denote the connections and second fundamental forms of the spaces involved. Let N_P and N_Σ denote the unit normals to P and Σ, respectively, defining the κ_i and τ_i.

Write

$$N_\Sigma(\beta(t)) = e_1(t) + e_2(t)$$

along β, where e_1 is tangent to P and e_2 orthogonal to P. Since P and Σ meet transversally along β we have $e_1(t) \neq \vec{0}$ for all t.

Also, $\langle e_1(t), \beta'(t)\rangle = 0$ for all t, so $e_1(t)$ is a non-zero section of the normal bundle of β in P. To prove the lemma it suffices to show $\langle \nabla^P_{\beta'}\beta', e_1\rangle(t) \geq 0$ for all t.

By the Gauss equation for Σ and P, we have

$$\begin{aligned}\nabla^{\mathbb{E}}_{\beta'}\beta' &= \nabla^{\Sigma}_{\beta'}\beta' + II_\Sigma(\beta', \beta')N_\Sigma \\ &= \nabla^{P}_{\beta'}\beta' + II_P(\beta', \beta')N_P,\end{aligned}$$

so

$$\nabla^{P}_{\beta'}\beta' = \nabla^{\Sigma}_{\beta'}\beta' + II_\Sigma(\beta', \beta')N_\Sigma - II_P(\beta', \beta')N_P.$$

Taking the scalar product with N_Σ, we obtain

$$\langle \nabla^{P}_{\beta'}\beta', N_\Sigma\rangle = II_\Sigma(\beta', \beta') - II_P(\beta', \beta')\langle N_P, N_\Sigma\rangle,$$

the first term $II_\Sigma(\beta', \beta')$ is between κ_1 and κ_2 and the second term, in absolute value, is at most $\max\{|\tau_i| : i = 1, 2\}$. Hence, $\langle \nabla^{P}_{\beta'}\beta', N_\Sigma\rangle \geq 0$ along β.

Since $\nabla^{P}_{\beta'}\beta'$ is tangent to P and $N_\Sigma = e_1 + e_2$ with e_2 normal to P, we conclude $\langle \nabla^{P}_{\beta'}\beta', e_1\rangle \geq 0$, as desired. □

Proof of Theorem 3.1. We can assume the foliation by vertical planes is parametrized by t and $P(0)$ is a vertical plane transverse to Σ. Let $C(0) \subset P(0) \cap \Sigma$ be an immersed closed curve. Denote by $\vec{k}(0)$ the curvature vector of $C(0)$ in $P(0)$. By Lemma 3.1 and our hypothesis, we know that $\vec{k}(0)$ points to one side of $C(0)$ in $P(0)$; we call this the convex side of $C(0)$. For t near 0, $P(t) \cap \Sigma$ contains an immersed closed curve $C(t)$ along which $P(t)$ is transverse to Σ and $C(t)$ is a smooth variation of $C(0)$. Thus, the convex side of $C(0)$ varies continuously for t near 0. As t increases, there will be a first $t = T$ where the curves $C(t)$ fail to be a transverse intersection.

There are several possibilities for $C(T)$. If $C(T)$ is a point or an interval, then the curves $C(t)$, $t < T$, and t near T, are embedded convex curves in $P(t)$; otherwise Σ would not be an immersed surface.

Now, a continuous variation of embedded convex curves in a Euclidean plane, among compact convex curves (allowing points of zero curvature along the curve) remains embedded. Thus the curves $C(t)$ are embedded for all t, $0 \leq t < T$. So $C(0)$ bounds an embedded disk in Σ, the union of the $C(t)$, $0 \leq t < T$.

In the case (i.e., when $C(T)$ is a point or interval), consider the curves $C(t)$ for $t < 0$. There is then a first accident at some $t = T_1$ where $P(T_1)$ is not transverse to Σ along $C(T_1)$. Again, if $C(T_1)$ is a point or an interval, we conclude the $C(t)$ are embedded for $T_1 < t \leq 0$. Hence Σ is an embedded sphere.

Then to complete the proof, we will show that no other accident can happen at $t = T$. The other possibility is $C(T)$ contains a point p, and there are k branches (smooth) of $C(T)$ passing through p. Σ and $P(T)$ are tangent at p and p is an isolated tangency in a neighborhood. In a disk neighborhood D of p in Σ, we have the induced foliation of D (singular at p) which has $2k$ sectors as in Figure 2.

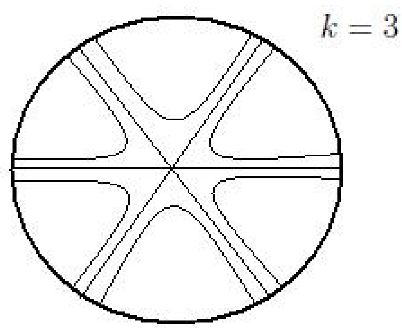

Figure 2

The curves β of this foliation, that do not pass through p, separate D into 2 components. Let $F(\beta)$ denote the component not containing p. For D sufficiently small, there are points of β (near p), where the curvature vector $\vec{k}(\beta)$ is not zero and points into $F(\beta)$.

Now for $t < T$, t near T, $C(t) \cap D$ contains such an arc β. So, the curvature vector of $C(t)$ points into $F(\beta)$ and is non zero at some point of $C(t)$ near p (see Figure 3).

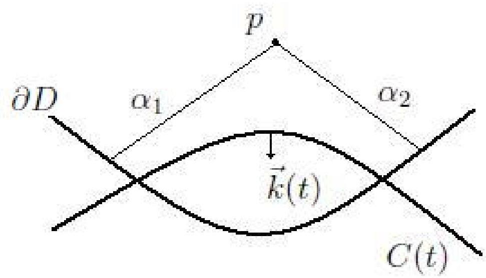

Figure 3

As $t \to T$, these arcs of $C(t) \cap D$ converge to the two arcs α_1 and α_2 meeting at p.

Suppose some point $q \in \alpha_1 \cup \alpha_2$, $q \neq p$, has $\vec{k}_\alpha(q) \neq \vec{0}$, $\alpha = \alpha_1$ or α_2, depending in which q belongs. Then, for t near T, there are points $q(t) \in C(t)$ such that $\vec{k}(q(t)) \neq \vec{0}$. Thus the convex side of α is the sector bounded by $\alpha_1 \cup \alpha_2$; i.e., the vector $\vec{k}_\alpha(q)$ points towards the arc of $C(t)$ containing $q(t)$ (see Figure 4).

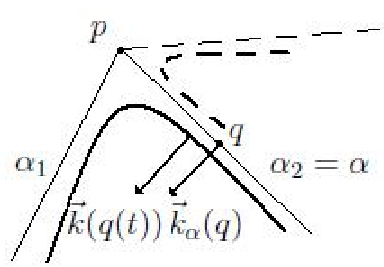

Figure 4

Now consider the arcs of $\Sigma \cap P(s)$, $s > T$, s near T, on the other side of α than $C(t)$; let $C(s)$ denote the arc (see Figure 5).

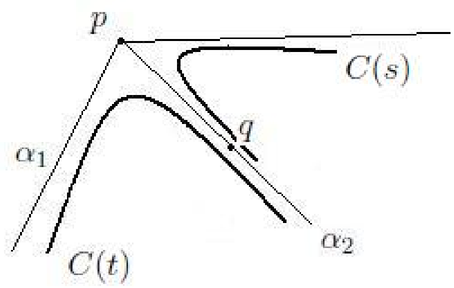

Figure 5

For $z \in C(s)$, z near q, the curvature vector of $C(s)$ at q points towards q by continuity. But at the points of $C(s)$ near p (for s close enough to T) the curvature vector points to the other side of $C(s)$. If we choose s so Σ and $P(s)$ meet transversally, this contradicts the convexity of $C(s)$.

Thus, there are no points of $\alpha_1 \cup \alpha_2$ of non zero curvature. The same reasoning applies to the connected component of $P(T) \cap \Sigma$ containing p. We conclude all the arcs of this component are straight lines. But these straight lines extend through the tangency points of $\Sigma \cap P(s)$. Since this is impossible, this completes the proof. $\square$

Remark 3.1. *Suppose Σ is an immersed compact surface in $\mathbb{R}^3$ with $K_e \geq 0$ and has, at least, one point at which the extrinsic curvature in strictly positive. Then for almost every line in $\mathbb{R}^3$, the projection of Σ onto the line is a Morse function on Σ, so Σ is in general position with respect to the foliation by planes orthogonal to the line. Start with a planar foliation coming from a Morse function so that the strictly convex point is a non degenerate minimum. Then, the level curves start out convex and embedded. Moreover, note that they can not became non compact. So, the first accident occurs at a saddle point (see Figure 3). But this can not happen by arguing as in the proof of Theorem 3.1. Thus, Σ is an embedded sphere and bounds a convex body in $\mathbb{R}^3$. This gives an alternative proof of Van Heijenhoort Theorem [VH] for the closed case.*

4 An example

In this Section we present a complete noncompact surface in $\mathbb{H}^2\times\mathbb{R}$ whose extrinsic curvature is greater than or equal than zero, and it is not embedded. This example shows that the conditions on the previous Theorem 3.1 are sharp.

Let us denote by $\mathbb{R}^4_1$, the real vector space $\mathbb{R}^4$ endowed with linear coordinates (x_1, x_2, x_3, x_4) and the metric $\langle,\rangle$ induced by the quadratic form $-x_1^2+x_2^2+x_3^2+x_4^2$. $\mathbb{H}^2\times\mathbb{R}$ will be considered as the submanifold of the Lorentzian space $\mathbb{R}^4_1$, given by

$$\mathbb{H}^2\times\mathbb{R} = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 :\ -x_1^2 + x_2^2 + x_3^2 = -1,\ x_1 > 0\}.$$

Let γ and β be horizontal geodesics parametrized by arc-length, i.e, both geodesic are contained in $\mathbb{H}^2 \times \{0\}$, so that they are orthogonal at $p = \gamma(0) = \beta(0)$. For simplicity, assume $p = (0, 0, 1)$. Let ψ_t be the the one parameter family of isometries generated by translation along γ, then the trajectories along ψ_t of a point $q \in \beta$ is either the geodesic γ if $q = \beta(0)$, or an equidistant curve to γ if $q \neq \beta(0)$, denote these curves by $\gamma_s(t)$. Set $P = \beta \times \mathbb{R}$, P is isometrically $\mathbb{R}^2$. Let (x, y) cartesian coordinates in P so that x is the arc-length parameter along β and y is the height, the origin $(0, 0)$ is the point $p = \beta(0)$ and $e_1 = (1, 0) = \beta'(0)$. Let $\alpha(s) = (r(s), R(s))$ be a closed strictly convex immersed curve in P so that the critical points of $R(s)$ are not inflection points, i.e., if $R'(s_0) = 0$ then $R''(s_0) \neq 0$, and moreover, if s_0 is a critical point of R, then $\alpha(s_0) \in \{0\} \times \mathbb{R}$, i.e., it is contained in the $y-$axis. Then, one can see that the curvature vector field $\vec{k}_\alpha(s)$ along α (with the orientation that makes α convex) points in the e_1 direction if $r(s) < 0$ and points in the $-e_1$ direction if $r(s) > 0$. Moreover, the curvature vector $\vec{k}_s(t)$ along $\gamma_s(t)$ (with the orientation that makes $\gamma_s(t)$ convex, recall they are either geodesics or equidistant curves) points in the e_1 direction if $r(s) < 0$ and in the $-e_1$ direction if $r(s) > 0$, also, the curvature vanishes when $r(s_0) = 0$ since, in this case, $\gamma_{s_0}(t)$ is a horizontal geodesic. Thus, from these observations, the surface $\Sigma = \psi_t(\alpha(s))$ has principal curvatures greater or equals than zero at any point (w.r.t. the inward orientation) and so, Σ is a locally convex immersed surface in $\mathbb{H}^2 \times \mathbb{R}$. We describe next an explicit example:

Let $r, R : \mathbb{R} \to \mathbb{R}$ be smooth $4\pi-$periodic functions and consider

$$\psi(s, t) := (\cosh(t)\cosh(r(s)), \sinh(r(s)), \sinh(t)\cosh(r(s)), R(s)).$$

Given a point p of the horizontal geodesic $\alpha(t) = (\cosh(t), 0, \sinh(t), 0)$ and the vertical plane P orthogonal to α at p, then, $P \cap \Sigma$, where Σ is the image of ψ, is nothing but the curve $(r(s), R(s))$ in P "centered" at p.

The extrinsic curvature of this immersion is given by

$$K_e = \frac{\tanh(r(s))R'(s)\left(r'(s)R''(s) - r''(s)R'(s)\right)}{\left(r'(s)^2 + R'(s)^2\right)^2}$$

Thus, it is not difficult to choose two functions $r(s), R(s)$ such that the pair $(r(s), R(s))$ is a non embedded curve and the associated extrinsic curvature is non negative. For instance, if we take

$$r(s) := \sin\left(\frac{s}{2}\right)\cos(s), \qquad R(s) := \cos\left(\frac{s}{2}\right) - \frac{1}{3}\cos\left(\frac{3s}{2}\right),$$

the immersion ψ is well defined in the cylinder $\mathbb{R}/(4\pi\mathbb{Z}) \times \mathbb{R}$ and it is not embedded since $\psi(\frac{\pi}{2}, t) = \psi(\frac{7\pi}{2}, t)$ (see Figure 6 for the curve $(r(s), R(s))$). In addition, its extrinsic curvature

$$K_e = 32\frac{\tanh\left(\frac{1}{2}\left(\sin\left(\frac{s}{2}\right) - \sin\left(\frac{3s}{2}\right)\right)\right)\,\frac{1}{2}\,\left(\sin\left(\frac{s}{2}\right) - \sin\left(\frac{3s}{2}\right)\right)(5 - 4\cos(s) + \cos(2s))}{(-17\cos(s) + 2\cos(2s) + 5\cos(3s) + 18)^2}$$

is non negative because $\tanh(a)\, a \geq 0$ for all real number a.

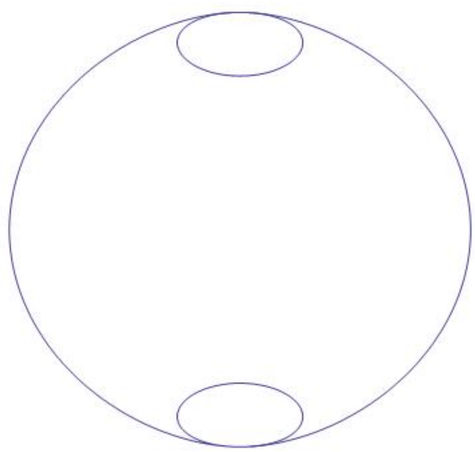

Figure 6

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