An overview on norm attaining linear operators

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Contents

1 Preliminaries

1.1 Notation

★ Notation

X, Y real or complex Banach spaces

- $\mathbb{K}$ base field $\mathbb{R}$ or $\mathbb{C}$,
- $B_X = \{x \in X : \|x\| \leqslant 1\}$ closed unit ball of X,
- $S_X = \{x \in X : \|x\| = 1\}$ unit sphere of X,
- $\mathcal{L}(X, Y)$ bounded linear operators from X to Y,
  - $\|T\| = \sup\{\|T(x)\| : x \in S_X\}$ for $T \in \mathcal{L}(X, Y)$,
- $\mathcal{W}(X, Y)$ weakly compact linear operators from X to Y,
- $\mathcal{K}(X, Y)$ compact linear operators from X to Y,
- $\mathcal{FR}(X, Y)$ bounded linear operators from X to Y with finite rank,
- if $Y = \mathbb{K}$, $X^* = \mathcal{L}(X, Y)$ topological dual of X,

Observe that

$$\mathcal{FR}(X, Y) \subset \mathcal{K}(X, Y) \subset \mathcal{W}(X, Y) \subset \mathcal{L}(X, Y).$$

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1.2 Introducing the topic

★ Norm attaining functionals

Norm attaining functionals
$x^* \in X^*$ *attains its norm* when

$$\exists\, x \in S_X \ : \ |x^*(x)| \ = \ \|x^*\|$$

★ $\mathrm{NA}(X,\mathbb{K}) := \{x^* \in X^* \colon x^* \text{ attains its norm}\}$

Examples and comments

- $\dim(X) < \infty \implies \mathrm{NA}(X,\mathbb{K}) = \mathcal{L}(X,\mathbb{K})$ (Heine–Borel).
- X reflexive $\iff \mathrm{NA}(X,\mathbb{K}) = \mathcal{L}(X,\mathbb{K})$ (Hahn–Banach, James).
- $\mathrm{NA}(c_0,\mathbb{K}) = c_{00} \leqslant \ell_1$,
- $\mathrm{NA}(\ell_1,\mathbb{K}) = \big\{x \in \ell_\infty \colon \|x\|_\infty = \max_n\{|x(n)|\}\big\} \subseteq \ell_\infty$, residual, contains c_0,
- $\mathrm{NA}(X,\mathbb{K})$ can be "wild", for instance:
  - it may contain NO two-dimensional subspaces (Read, 2018; Rmoutil, 2017),
  - it can be NOT norm Borel (Kaufman, 1991).
- (Petunin–Plichko 1974; Godefroy 1987): X separable, $Z \leqslant X^*$ closed, separating for X, $Z \subseteq \mathrm{NA}(X,\mathbb{K}) \implies Z$ is an isometric predual of X.

★ Norm attaining operators

Norm attaining operators
$T \in \mathcal{L}(X,Y)$ *attains its norm* when

$$\exists\, x \in S_X \ : \ \|T(x)\| \ = \ \|T\|$$

★ $\mathrm{NA}(X,Y) := \{T \in \mathcal{L}(X,Y) \colon T \text{ attains its norm}\}$

Some examples and comments

- $\dim(X) < \infty \implies \mathrm{NA}(X,Y) = \mathcal{L}(X,Y)$ for every Y (Heine-Borel),
- $\dim(X) = \infty \implies \mathrm{NA}(X,c_0) \neq \mathcal{L}(X,c_0)$ (see M.-Merí-Payá, 2006).
- X reflexive $\iff \mathcal{K}(X,Y) \subseteq \mathrm{NA}(X,Y)$ for every Y (James).
- $\mathcal{L}(X,\ell_\infty) = \big[\bigoplus_{n\in\mathbb{N}} \mathcal{L}(X,\mathbb{K})\big]_{\ell_\infty} = \ell_\infty(X^*)$.
  $\mathrm{NA}(X,\ell_\infty) = \big\{(x_n^*) \in \ell_\infty(X^*) \colon \exists k \in \mathbb{N},\ \|x_k^*\| = \|(x_n^*)\|_\infty,\ x_k^* \in \mathrm{NA}(X,\mathbb{K})\big\}$.
- $\mathcal{L}(\ell_1,Y) = \big[\bigoplus_{n\in\mathbb{N}} \mathcal{L}(\mathbb{K},Y)\big]_{\ell_\infty} = \ell_\infty(Y)$.
  $\mathrm{NA}(\ell_1,Y) = \big\{(y_n) \in \ell_\infty(Y) \colon \exists k \in \mathbb{N},\ \|y_k\| = \|(y_n)\|_\infty\big\}$.
- $\mathrm{NA}(L_1[0,1], L_\infty[0,1])$???

★ The problems of denseness of norm attaining functionals and operators

Problem
Is $\mathrm{NA}(X,\mathbb{K})$ always dense in X^***?**

Theorem (Bishop–Phelps, 1961)
The set of norm attaining functionals is dense in X^* (for the norm topology).

Problem
Is $\mathrm{NA}(X,Y)$ always dense in $\mathcal{L}(X,Y)$**?**

The answer is **No**, and this is the origin of the study of norm attaining operators.

Modified problem
When is $\mathrm{NA}(X,Y)$ dense in $\mathcal{L}(X,Y)$**?**

The study of this problem was initiated by J. Lindenstrauss in 1963, who provided the first negative and positive examples.

2 An overview on "classical" results

2.1 First results: Lindenstrauss

★ Lindenstrauss' seminal paper of 1963

Negative answer

$$\mathrm{NA}(X,Y) \text{ is NOT always dense}$$

Lemma
Y LUR, $T\colon X \longrightarrow Y$ bounded from below (monomorphism).
If T attains its norm, then it does at a strongly exposed point.

Example
X separable without strongly exposed points (e.g. c_0, $C[0,1]$, $L_1[0,1]$), Y LUR renorming of X. Then, $\mathrm{NA}(X,Y)$ is not dense in $\mathcal{L}(X,Y)$.

Lemma
If Y is strictly convex, then $\mathrm{NA}(c_0,Y)\subseteq \mathcal{FR}(c_0,Y)$.

Example
Y strictly convex, $Y\supset c_0$. Then, $\mathrm{NA}(X,Y)$ is not dense in $\mathcal{L}(X,Y)$.

★ Lindenstrauss properties A and B

Observation

- The question then is for which X and Y the density holds.
- As this problem is too general, Lindenstrauss introduced two properties.

Definition
X, Y Banach spaces,

- X has (Lindenstrauss) *property A* iff $\overline{\mathrm{NA}(X,Z)} = \mathcal{L}(X,Z) \quad \forall Z$
- Y has (Lindenstrauss) *property B* iff $\overline{\mathrm{NA}(Z,Y)} = \mathcal{L}(Z,Y) \quad \forall Z$

First examples

- If X is finite-dimensional, then X has property A,
- $\mathbb{K}$ has property B (Bishop–Phelps theorem),
- c_0, $C[0,1]$, $L_1[0,1]$ fail property A,
- if Y is strictly convex, $Y \supset c_0$, then Y fails property B.

★ Positive results I

Theorem (Lindenstrauss, 1963)
X, Y Banach spaces. Then

$$\{T \in \mathcal{L}(X,Y) : \ T^{**} : X^{**} \longrightarrow Y^{**} \text{ attains its norm}\}$$

is dense in $\mathcal{L}(X,Y)$.

Observation
Given $T \in \mathcal{L}(X,Y)$, there is $S \in \mathcal{K}(X,Y)$ such that $[T+S]^{**} \in \mathrm{NA}(X^{**},Y^{**})$.

Consequence
If X is reflexive, then X has property A.

An improvement (Zizler, 1973)
X, Y Banach spaces. Then

$$\{T \in \mathcal{L}(X,Y) : \ T^{*} : Y^{*} \longrightarrow X^{*} \text{ attains its norm}\}$$

is dense in $\mathcal{L}(X,Y)$.

★ Positive results II

Some examples (using Lindenstrauss' ideas):

- The following spaces have property A: ℓ_1 and **all** finite-dimensional spaces.
- The following spaces have property B: every Y such that $c_0 \subset Y \subset \ell_\infty$, finite-dimensional spaces such that the dual unit ball has finitely many extreme points (up to rotation).
- Every finite-dimensional space has property A, but the only known (in the 1960's) finite-dimensional real spaces with property B were the polyhedral ones. Only a little bit more is known nowadays...

Positive results, up to renorming:
(Partington, 1982; Schachermayer, 1983; Godun-Troyanski, 1993)

- Every Banach space can be renormed with property B.
- Every Banach space admitting a long biorthogonal system (in particular, separable) can be renormed with property A.

2.2 The relation with the RNP: Bourgain

★ The Radon-Nikodým property

Definitions

X Banach space.

- X has the Radon-Nikodým property (RNP) if the Radon-Nikodým theorem is valid for X-valued vector measures (with respect to every finite positive measure).
- $C \subset X$ is dentable if it contains slices of arbitrarily small diameter.
- $C \subset X$ is subset-dentable (or RNP set) if every subset of C is dentable.

Theorem (Rieffel, Maynard, Huff, David, Phelps, 1970's)

$$X \text{ RNP} \iff \text{every bounded } C \subset X \text{ is dentable} \iff B_X \text{ RNP set.}$$

Remark

In the book

J. Diestel and J. J. Uhl, *Vector Measures*, Math. Surveys **15**, AMS, Providence, 1977.

there are more than 30 different reformulations of the RNP.

★ The RNP and property A: positive results

Theorem (Bourgain, 1977)

X Banach space, $C \subset X$ absolutely convex closed bounded subset-dentable, Y Banach space. Then

$$\{T \in \mathcal{L}(X,Y)\colon \text{the norm of } T \text{ attains its supremum on } C\}$$

is dense in $\mathcal{L}(X,Y)$.

★ In particular, RNP $\Longrightarrow$ property A.

Remark

It is actually shown that for every bounded linear operator there are arbitrary closed **compact** perturbations of it attaining the norm.

Non-linear Bourgain–Stegall variational principle (Stegall, 1978)

X, Y Banach spaces, $C \subset X$ bounded subset-dentable, $\varphi : C \longrightarrow Y$ uniformly bounded such that $x \longmapsto \|\varphi(x)\|$ is upper semicontinuous. Then for every $\delta > 0$, there exists $x_0^* \in X^*$ with $\|x_0^*\| < \delta$ and $y_0 \in S_Y$ such that the function $x \longmapsto \|\varphi(x) + x^*(x)y_0\|$ attains its supremum on C.

★ The RNP and property A: negative results

Theorem (Bourgain, 1977)

$C \subset X$ separable, bounded, closed and convex,

$\{T \in \mathcal{L}(X,Y)\colon \text{the norm of } T \text{ attains its supremum on } C\}$ dense in $\mathcal{L}(X,Y)$.

$\Longrightarrow$ C is dentable.

★ In particular, if X is separable and has property A $\Longrightarrow$ B_X is dentable.

Remark

Lindenstrauss actually showed that if X is separable and has property A

$\Longrightarrow$ B_X is the closed convex hull of its strongly exposed points.

A refinement (Huff, 1980)

X Banach space failing the RNP. Then there exist X_1 and X_2 equivalent renorming of X such that $\mathrm{Id} \notin \overline{\mathrm{NA}(X_1,X_2)}$, hence $\mathrm{NA}(X_1,X_2)$ is NOT dense in $\mathcal{L}(X_1,X_2)$.

★ The RNP and property A: isomorphic characterization

Main consequence
Every renorming of X has property A $\iff$ X has the RNP.

Example
ℓ_1 has property A in every equivalent norm.

Another consequence
Every renorming of X has property B $\implies$ X has the RNP.

Observations

1. The converse of the implication above is NOT TRUE (Gowers, 1990)
2. To get an equivalence, a weaker property is needed, quasi norm attainment (to be seen latter).

2.3 Counterexamples for property B

★ Counterexamples for property B

Observation
It was an open question in the 1980's whether RNP $\implies$ property B

Counterexamples: spaces which fail property B

- ℓ_p, $1 < p < \infty$ (Gowers, 1990).
- every infinite-dimensional strictly convex space (Acosta, 1999).
- ℓ_1 (Acosta, 1999).
- ℓ_p-sums of finite-dimensional spaces, $1 < p < \infty$ (Fovelle, 2024).

Consequence
Y separable, every renorming of Y has property B $\implies$ Y is finite-dimensional

On the converse...

- We do not know if finite-dimensional spaces have property B.
- (Acosta-Aguirre-Payá, 1996): there are *some* non polyhedral finite-dimensional spaces with property B.

2.4 Some results on classical spaces

★ Some classical spaces: positive results

Example (Johnson-Wolfe, 1979)
In the real case, $\mathrm{NA}(C(K_1), C(K_2))$ is dense in $\mathcal{L}(C(K_1), C(K_2))$.

Example (Iwanik, 1979)
$\mathrm{NA}(L_1(\mu), L_1(\nu))$ is dense in $\mathcal{L}(L_1(\mu), L_1(\nu))$.

Example (Schachermayer, 1983)

For every compact space K, for every Banach space Y:

$$\mathcal{W}(C(K),Y) = \overline{\mathrm{NA}(C(K),X) \cap \mathcal{W}(C(K),Y)}$$

Consequence (Schachermayer, 1983)
$\mathrm{NA}(C(K), L_p(\mu))$ is dense in $\mathcal{L}(C(K), L_p(\mu))$ for $1 \leqslant p < \infty$.

Example (Finet-Payá, 1998)
$\mathrm{NA}(L_1[0,1], L_\infty[0,1])$ is dense in $\mathcal{L}(L_1[0,1], L_\infty[0,1])$.

⋆ Some classical spaces: negative results

Example (Schachermayer, 1983)
$\mathrm{NA}(L_1[0,1], C[0,1])$ is NOT dense in $\mathcal{L}(L_1[0,1], C[0,1])$.

Consequence
$C[0,1]$ does not have property B and it was the first "classical" example.

Example (Uhl, 1976)

- If Y has the RNP, then $\mathrm{NA}(L_1[0,1], Y)$ is dense in $\mathcal{L}(L_1[0,1], Y)$.
- If Y is strictly convex and $\mathrm{NA}(L_1[0,1], Y)$ is dense in $\mathcal{L}(L_1[0,1], Y)$, then Y has the RNP.

2.5 Compact operators

⋆ The question of norm attainment for compact operators

Question (open from 1970's till 2014)
Can every compact operator be approximated by norm attaining operators?

Observations

- In all the negative examples of the previous sections, the authors constructed NON COMPACT operators which cannot be approximated by norm attaining ones.
- Actually, the idea of the proofs is to use that the operator which is not going to be approximated is not compact or, even, it is an isomorphism.
- In most examples, it was even known that compact operators attaining the norm are dense.

⋆ Positive results on norm attaining compact operators

Positive results

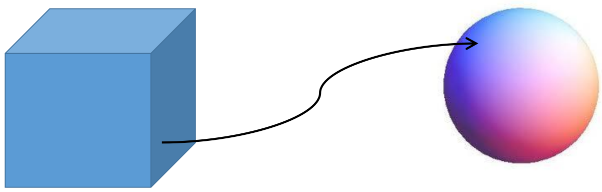
- If X is reflexive, then ALL compact operators from X into Y are norm attaining. (Indeed, compact operators carry weak convergent sequences to norm convergent sequences.)
- Some classical spaces (Johnson–Wolfe, 1979):
  - $X = C_0(L)$ or $X = L_1(\mu)$, Y arbitrary;
  - X arbitrary, $Y = L_1(\mu)$ (only real case) or $Y^* \equiv L_1(\mu)$;
  - X arbitrary, $Y \leqslant c_0$ with AP.
- More recent results:
  - (Cascales–Guirao–Kadets, 2013) X arbitrary, Y uniform algebra;
  - (M. 2014) $X^* = \ell_1$, Y arbitrary.

★ The solution

Negative answer (M. 2014)
There are compact operators which cannot be approximated by norm attaining ones

Ideas

- (Extending Lindenstrauss) $X \leqslant c_0$, Y strictly convex $\implies$ $\mathrm{NA}(X,Y) \subset \mathcal{FR}(X,Y)$.
- (Enflo) There is $X \leqslant c_0$ such that X^* fails the approximation property.
- There is Y sc and $T\colon X \longrightarrow Y$ compact not approx. by finite-rank operators.
- $\mathcal{K}(X,Y)$ is not contained in $\overline{\mathrm{NA}(X,Y)} \subset \overline{\mathcal{FR}(X,Y)}$



Example
There is $X \leqslant c_0$ *with Schauder basis* and Y such that
$\mathcal{K}(X,Y)$ is not contained in $\overline{\mathrm{NA}(X,Y)}$.

★ Some interesting open problems, to be (partially) solved here!

Problem 1
Relate the RNP **of the range space** and the denseness of norm attaining operators.

- The usual norm attainment notion does not work…
- Introduce a new notion?

Problem 2
Find **isometric** characterizations of Lindenstrauss property A (in the separable case).

- Improve the necessary conditions.
- Check whether they are sufficient somewhere.

Problem 3
Does every **finite-dimensional** Banach space satisfy Lindenstrauss property B?
Equivalently, are finite-rank operators always approachable by norm attaining ones?

- We do not even know if there is X such that every non-zero element of $\mathrm{NA}(X,\ell_2)$ is of rank one…

★ "Other" open problems

Problem 4 (Ostrovskii)
Does there exist an infinite-dimensional X such that $\mathrm{NA}(X,X) = \mathcal{L}(X,X)$?

- Please, attend Dantas' talk!

Problem 5
Does every dual space satisfy Lindenstrauss property A?

- Separable dual spaces have the RNP.
- What's about ℓ_∞ or JT^*?

Problem 6
Find necessary conditions for Lindenstrauss property B.

- Lindenstrauss' ones are not usable.
- They should be on smoothness properties...

Problem 7 (new "classical" spaces)
For which M_1, M_2, $\mathrm{NA}(\mathcal{F}(M_1), \mathcal{F}(M_2))$ is dense in $\mathcal{L}(\mathcal{F}(M_1), \mathcal{F}(M_2))$?

3 The RNP and range spaces: quasi norm-attainment

3.1 Quasi norm attainment

★ A weaker notion of norm attainment

Quasi norm attaining bounded linear operator (Godefroy, 2015; CCJM, 2022)
$T \in \mathcal{L}(X,Y)$ *quasi attains its norm* ($T \in \mathrm{QNA}(X,Y)$) if there exists $(x_n) \subset B_X$ such that $Tx_n \longrightarrow y \in \|T\|S_Y$.

- We say T quasi attains its norm *towards* y.
- Equivalently, $\overline{T(B_X)} \cap \|T\|S_Y \neq \emptyset$ $\;(T \in \mathrm{NA}(X,Y) \iff T(B_X) \cap \|T\|S_Y \neq \emptyset)$

First remarks

$$\mathrm{NA}(X,Y) \subset \mathrm{QNA}(X,Y) \qquad \mathcal{K}(X,Y) \subset \mathrm{QNA}(X,Y).$$

First positive examples
All pairs (X,Y) for which $\mathrm{NA}(X,Y)$ is dense or $\mathcal{L}(X,Y) = \mathcal{K}(X,Y)$.

Negative example 1 (Godefroy, 2015)
Y renorming of c_0 with the Kadets–Klee property $\implies \overline{\mathrm{QNA}(c_0, Y)} \neq \mathcal{L}(c_0, Y)$.

★ Negative results

Remarks

- $T \in \mathrm{QNA}(X,Y)$, $T(B_X)$ closed (for instance, T monomorphism) $\Longrightarrow\; T \in \mathrm{NA}(X,Y)$.
- Hence, if there is $T \in \mathcal{L}(X,Y)$ monomorphism, $T \notin \overline{\mathrm{NA}(X,Y)}$ $\Longrightarrow\; \overline{\mathrm{QNA}(X,Y)} \neq \mathcal{L}(X,Y)$.

Example 2 (improving a result by Johnson–Wolfe)
There exists S such that $\mathrm{QNA}(L_1[0,1], C(S))$ is not dense in $\mathcal{L}(L_1[0,1], C(S))$.

Example 3 (improving a result by Bourgain–Huff)
X failing RNP $\Longrightarrow$ there exist X_1, X_2 isomorphic to X such that $\mathrm{Id} \notin \overline{\mathrm{NA}(X_1,X_2)}$, hence $\mathrm{QNA}(X_1,X_2)$ is not dense in $\mathcal{L}(X_1,X_2)$.

3.2 The relation with the RNP of the range space

★ RNP of the range space

Theorem
$T \in \mathcal{L}(X,Y)$, $\overline{T(B_X)}$ RNP set, $\varepsilon > 0$. There exists $S \in \mathrm{QNA}(X,Y)$ such that:

- $\|T - S\| < \varepsilon$, $T - S$ is of rank-one
- moreover, there is $z_0 \in \overline{S(B_X)} \cap \|S\| S_Y$ such that whenever $(x_n) \subseteq B_X$ satisfies that $\|Sx_n\| \longrightarrow \|S\|$, we may find a sequence $(\theta_n) \subseteq \mathbb{T}$ such that $S(\theta_n x_n) \longrightarrow z_0$; in particular, there is $\theta_0 \in \mathbb{T}$ and a subsequence $(x_{\sigma(n)})$ of (x_n) such that $Sx_{\sigma(n)} \longrightarrow \theta_0 z_0$.

Tool: Bourgain–Stegall non-linear optimization principle
Suppose D is a bounded RNP set of a Banach space Y and $\phi\colon D \longrightarrow \mathbb{R}$ is upper semicontinuous and bounded above. Then, the set

$$\{y^* \in Y^* \colon \phi + \operatorname{Re} y^* \text{ strongly exposes } D\}$$

is a dense G_δ subset of Y^*.
★ $D := \overline{T(B_X)}$ and $\phi(y) = \|y\|$.

★ Consequences I

Corollary
If X or Y has the RNP, then $\mathrm{QNA}(X,Y)$ is dense in $\mathcal{L}(X,Y)$.

- The case of X having RNP needs Bourgain's result.
- The case of Y having RNP is the new one and it is false for NA.

Corollary
$\mathrm{QNA}(X,Y) \cap \mathcal{W}(X,Y)$ is always dense in $\mathcal{W}(X,Y)$.

Examples 1. There are many examples of pair of spaces (X,Y) for which $\mathrm{QNA}(X,Y)$ is dense while $\mathrm{NA}(X,Y)$ is not. For instance:

- There exists G such that $\mathrm{NA}(G, \ell_p)$ is not dense for $1 < p < \infty$, while $\mathrm{QNA}(X, \ell_p)$ is dense for every X.

★ Consequences II. Characterizing the RNP

Corollary

Z Banach space. TFAE:

(a) Z has the RNP.

(b) $\mathrm{QNA}(Z', Y)$ is dense in $\mathcal{L}(Z', Y)$ for every Banach space Y and every equivalent renorming Z' of Z.

(c) $\mathrm{QNA}(X, Z')$ is dense in $\mathcal{L}(X, Z')$ for every Banach space X and every equivalent renorming Z' of Z.

Remarks

- (a) $\iff$ (b) is true for NA,
- (c) $\implies$ (a) is true for NA,
- but (a) $\implies$ (c) is FALSE for NA, as shown by Gowers.

★ Characterizing properties using QNA

The idea is to discuss when each of the inclusions in the chain

$$\mathrm{NA}(X, Y) \subseteq \mathrm{QNA}(X, Y) \subseteq \mathcal{L}(X, Y)$$

is an equality.

$\mathrm{NA}(X, Y) = \mathrm{QNA}(X, Y)$

For a Banach space X, TFAE:

- X is reflexive,
- $\mathrm{NA}(X, Y) = \mathrm{QNA}(X, Y)$ for every Y,
- exists $Y \neq \{0\}$ such that $\mathrm{NA}(X, Y) = \mathrm{QNA}(X, Y)$.

$\mathrm{QNA}(X, Y) = \mathcal{L}(X, Y)$

- $\dim(X) = \infty \implies \mathcal{L}(X, c_0) \setminus \mathrm{QNA}(X, c_0) \neq \emptyset$.
- $\dim(Y) = \infty \implies \mathcal{L}(\ell_1, Y) \setminus \mathrm{QNA}(\ell_1, Y) \neq \emptyset$.

  ★ Read as follows: there is $K \subseteq B_Y$ **absolutely convex closed** with $\sup_{k\in K} \|k\| = 1$ but $K \cap S_Y = \emptyset$ (M.–Rao, 2010; Veselý, 2009).

3.3 Weak-star quasi norm attainment

★ Weak-star quasi norm attainment

Weak-star quasi norm attaining bounded linear operator (CJKM, 2024)

$T \in \mathcal{L}(X, Y^*)$ *weak-star quasi attains its norm* ($T \in w^*\text{-}\mathrm{QNA}(X, Y^*)$) if

$$\overline{T(B_X)}^{w^*} \cap \|T\| S_{Y^*} \neq \emptyset$$

The notion depends on the predual

$$w^*\text{-}\mathrm{QNA}(c_0, c_0^*) \neq w^*\text{-}\mathrm{QNA}(c_0, c^*)$$

Main result

For **every** X and Y, the set $w^*\text{-}\mathrm{QNA}(X, Y^*)$ is (norm) dense in $\mathcal{L}(X, Y^*)$.

Example

There is Y such that $\mathrm{QNA}(c_0, Y^*)$ is **not** dense in $\mathcal{L}(c_0, Y^*)$.

3.4 Open problems

★ Some related open problems

Open problem (Ostrovskii, 2005)
Does there exist X infinite-dimensional with $\mathrm{NA}(X,X) = \mathcal{L}(X,X)$?

- The only possible candidates for X are separable reflexive spaces without one-complemented subspaces with the AP (please, attend Dantas' talk!).

Open problem 1
Does there exist X infinite-dimensional with $\mathrm{QNA}(X,X) = \mathcal{L}(X,X)$?

Some remarks

- If X is reflexive, the two problems are the same.
- As $\mathcal{K}(X,X) \subseteq \mathrm{QNA}(X,X)$, one may think in testing spaces with "very few operators" (i.e. $\mathcal{L}(X,X) = \{\lambda \mathrm{Id} + S \colon \lambda \in \mathbb{K},\ S \in \mathcal{K}(X,X)\}$).
- But if $T = \lambda \mathrm{Id} + S$ with $\lambda \neq 0$ and $S \in \mathcal{K}(X,X)$ belongs to $\mathrm{QNA}(X,X)$, then $T \in \mathrm{NA}(X,X)$.

Open problem 2
Suppose that X satisfies that $\mathrm{QNA}(X,Y)$ is dense for every Y (property *quasi* A), does X satisfy (Lindenstrauss) property A?
★ For property B the result is false!

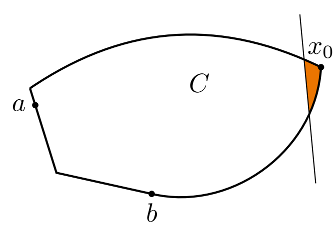
4 Property A and residuality of norm attaining things

4.1 A new necessary condition on property A

★ Definitions

Definition 1: strong exposition
$C \subset X$ bounded. $x_0 \in C$ is *strongly exposed* if there is $x^* \in X^*$ such that whenever $\{x_n\} \subset C$ satisfies $\mathrm{Re}\, x^*(x_n) \longrightarrow \sup \mathrm{Re}\, x^*(C)$, then $\{x_n\} \longrightarrow x_0$.
Equivalently, the slices of C defined by x^* contain x_0 and are arbitrarily small.

- In this case, we say that x^* *strongly exposes* C (at x_0).
- str-exp(C) set of strongly exposed points of C.
- SE(C) functionals which strongly expose C at some (strongly exposed) point.
- SE(C) is a G_δ subset of X^*.



For the case $C = B_X$...

- If $\mathrm{SE}(B_X)$ is dense, $\mathrm{NA}(X,\mathbb{K})$ is residual.
- Šmulyan's test:
$$x^* \in \mathrm{SE}(B_X) \iff \text{the norm of } X^* \text{ is Fréchet-differentiable at } x^*$$

★ Lindenstrauss's and Bourgain's necessary conditions vs a new one

Lindenstrauss, 1963

If X admits a LUR renorming and has property A
$\Longrightarrow$ B_X is the closed convex hull of $\text{str-exp}(B_X)$.

Bourgain, 1977

$C \subseteq X$ *separable bounded closed convex* such that for every Y the set

$$\Big\{T \in \mathcal{L}(X,Y) \colon \exists \max_{x\in C} \|Tx\|\Big\}$$

is dense in $\mathcal{L}(X,Y)$ (C has the Bishop–Phelps property in Bourgain's terminology)
$\Longrightarrow$ C is dentable (i.e. C contains slices of arbitrarily small diameter).

Jung-M.-Rueda, 2023

X admitting a LUR renorming, $C \subseteq X$ bounded with the Bishop–Phelps property
$\Longrightarrow$ $\mathrm{SE}(C)$ dense in X^* (hence $C = \overline{\mathrm{conv}}\big(\text{str-exp}(C)\big)$).
★ In particular, X separable with property A
$\Longrightarrow$ $\mathrm{SE}(B_X)$ is dense in X^*, hence $\mathrm{NA}(X,\mathbb{K})$ is residual.

★ Sketch of the proof of a particular case

Our result (particular case)

X admitting a LUR renorming, X with property A $\Longrightarrow$ $\mathrm{SE}(B_X)$ is dense in X^*.

Lemma

$S\colon X \longrightarrow Y$ monomorphism, Y LUR, $x_0 \in S_X$ such that $\|S\| = \|Sx_0\|$.
Then, x_0 is strongly exposed by S^*y^* for every $y^* \in S_{Y^*}$ with $\operatorname{Re} y^*(Sx_0) = \|S\|$.

- Consider a LUR norm $|\!|\!|\cdot|\!|\!|$ on X and let $Y = (X, |\!|\!|\cdot|\!|\!|) \oplus_2 \mathbb{K}$ which is LUR.
- For $x^* \in S_{X^*}$, define $T_n \in \mathcal{L}(X,Y)$ by $T_n(x) = (n^{-1}x, x^*(x))$, which are monomorphisms, and $S \in \mathcal{L}(X,Y)$ by $S(x) = (0, x^*(x))$. Observe $\{T_n\} \longrightarrow S$.
- We may find **monomorphisms** $S_n \in \mathrm{NA}(X,Y)$, $\|S_n\| = 1$, such that $\{S_n\} \longrightarrow S$.
- By the lemma, there are $y_n^* = (x_n^*, \lambda_n) \in Y^* = X^* \oplus_2 \mathbb{K}$ such that $S_n^* y_n^* \in \mathrm{SE}(B_X)$ and $\|S_n^* y_n^*\| = \|S_n\| = 1$.
- Suppose $\lambda_n \longrightarrow \lambda_0$ and observe
$$\|\lambda_0 x^* - S_n^* y_n^*\| = \|\lambda_0 x^* - (\lambda_n x^* - S^* y_n^*) - S_n^* y_n^*\| \leqslant |\lambda_0 - \lambda_n| + \|S^* - S_n^*\| \longrightarrow 0.$$
- As $\lambda_0 \neq 0$, $x^* = \lambda_0^{-1}(\lambda_0 x^*) \in \overline{\lambda_0^{-1}\,\mathrm{SE}(B_X)} = \overline{\mathrm{SE}(B_X)}$.

★ An interesting example

Example

The Lipschitz-free space on the Euclidean unit circle, $\mathcal{F}(\mathbb{T})$, satisfies:

- $\mathrm{SE}\big(B_{\mathcal{F}(\mathbb{T})}\big)$ is not dense in $\mathcal{F}(\mathbb{T})^* \equiv \mathrm{Lip}_0(\mathbb{T},\mathbb{R})$ (C-GL-M-RZ, 2021),
- hence, **by the new result**, $\mathcal{F}(\mathbb{T})$ fails Lindenstrauss property A.
- On the other hand, $B_{\mathcal{F}(\mathbb{T})} = \overline{\mathrm{conv}}\big(\text{str-exp}(B_{\mathcal{F}(\mathbb{T})})\big)$ (C-GL-M-RZ, 2021) (so it satisfies Lindenstrauss necessary condition for property A).

4.2 Sufficiency of the necessary condition?

★ The RNP and absolutely strongly exposing operators

Definition (Bourgain, 1977)

$T \in \mathcal{L}(X,Y)$ is *absolutely strongly exposing* ($T \in \mathrm{ASE}(X,Y)$) iff there exists $x_0 \in S_X$ such that whenever $\{x_n\} \subset B_X$ satisfies $\|T(x_n)\| \longrightarrow \|T\|$ then $\exists\{\theta_n\} \subset \mathbb{T}$ for which $\{\theta_n x_n\} \longrightarrow x_0$.

★ $\mathrm{ASE}(X,Y)$ is a G_δ-set. Therefore, if $\mathrm{ASE}(X,Y)$ is dense, $\mathrm{NA}(X,Y)$ is residual.

Bourgain, 1977

X RNP, Y arbitrary $\implies$ $\mathrm{ASE}(X,Y)$ is dense.

Much more: the non-linear Bourgain-Stegall variational principle (Stegall, 1978)

$C \subset X$ bounded RNP set, $\varphi\colon C \longrightarrow \mathbb{R}$ bounded upper semicontinuous.
Then, the set

$$\big\{x^* \in X^* \colon \varphi + \mathrm{Re}\, x^* \text{ strongly exposed } C\big\}$$

is residual in X^*.

★ Some comments and questions (I)

Observation (Chiclana–GarcíaLirola–M.–RuedaZoca, 2021)

ALL known sufficient conditions for property A actually imply that absolutely strongly exposing operators are dense:

- RNP,
- properties α and quasi-α,
- $B_X = \overline{\mathrm{conv}}(A)$, A uniformly strongly exposed.

This is a consequence of the following lemma...

If $T \in \mathcal{L}(X,Y)$ attains its norm at an element of str-exp(B_X), then $T \in \overline{\mathrm{ASE}(X,Y)}$.

Open problem 1 (still open)

Does the property A of X imply that $\mathrm{ASE}(X,Y)$ is dense for every Y**?**

★ Some comments and questions (II)

Observation

If $\mathrm{ASE}(X,Y)$ is dense for some Y $\implies$ $\mathrm{SE}(B_X)$ is dense.

Open problem 2 (still open)

Does the denseness of $\mathrm{SE}(B_X)$ imply that $\mathrm{ASE}(X,Y)$ is dense for every Y**?**

★ Some comments and questions (III)

Less ambitious question
If $\mathrm{SE}(B_X)$ is dense, for which Ys is $\mathrm{ASE}(X,Y)$ dense?

Examples of when $\mathrm{SE}(B_X)$ is dense

- If X has RNP,
- If X has property A and admits a LUR renorming,
- If X is LUR (a property which is not known to imply property A),
- If $\text{str-exp}(B_X) = S_X$ (a property which is not known to imply property A),
- $X = JT^*$ (the dual of the James-tree space, not known if it has property A).

The objective
To find spaces Y which are not known to have property B such that $\mathrm{ASE}(X,Y)$ is dense whenever $\mathrm{SE}(B_X)$ is dense.

★ A family of new examples. The general result

Theorem
X, Y Banach spaces, $\mathcal{I}(X,Y) \leqslant \mathcal{L}(X,Y)$ containing rank-one operators. Suppose:

- $\mathrm{SE}(B_X)$ is dense,
- there is $\{y_n^*\} \subset S_{Y^*}$ such that the set $\mathcal{A} = \{T \in \mathcal{I}(X,Y) \colon \|T\| = \|T^* y_n^*\| \text{ for some } n \in \mathbb{N}\}$ is residual in $\mathcal{I}(X,Y)$.

Then, $\mathrm{ASE}(X,Y) \cap \mathcal{I}(X,Y)$ is dense in $\mathcal{I}(X,Y)$.

Idea of the proof:

- The set $\mathcal{B} = \{T \in \mathcal{I}(X,Y) \colon T^* y_n^* \in \mathrm{SE}(B_X)\ \forall n \in \mathbb{N}\}$ is residual.

Lemma
$T \in \mathcal{L}(X,Y)$, $y^* \in S_{Y^*}$ with $T^*y^* \in \mathrm{SE}(B_X)$, $\|T^*y^*\| = \|T\|$, then there is $x_0 \in \text{str-exp}(B_X)$ such that $|[T^*y^*](x_0)| = \|Tx_0\| = \|T\|$, so $T \in \mathrm{ASE}(X,Y)$.

- $\mathcal{A} \cap \mathcal{B}$ is residual and contained in $\overline{\mathrm{ASE}(X,Y) \cap \mathcal{I}(X,Y)}$.

★ Consequences I

Consequence 1
$\mathrm{SE}(B_X)$ dense, Y^* RNP with $\text{str-exp}(B_{Y^*})$ countable up to rotations. Then:

$$\mathrm{ASE}(X,Y) \text{ dense in } \mathcal{L}(X,Y), \quad \mathrm{ASE}(X,Y) \cap \mathcal{K}(X,Y) \text{ dense in } \mathcal{K}(X,Y).$$

This result applies to...

- Y being a predual of ℓ_1,
- Y being finite-dimensional such that $\mathrm{ext}\,(B_{Y^*})$ is countable (up to rotation),
- $Y = \mathrm{lip}_0(M)$ when M is a countable compact metric space.

Consequence 2
$\text{SE}(B_X)$ dense, Y RNP with $\text{str-exp}(B_Y)$ countable up to rotations. Then:

$$\text{ASE}(X, Y^*) \text{ dense in } \mathcal{L}(X, Y^*), \quad \text{ASE}(X, Y^*) \cap \mathcal{K}(X, Y^*) \text{ dense in } \mathcal{K}(X, Y^*).$$

This result applies to...

- $Y = \mathcal{F}(M)$ (so $Y^* = \text{Lip}_0(M)$) when M is a countable proper metric space.

★ Consequences II

Consequence 3
$\text{SE}(B_X)$ dense, Y such that every separable subspace of Y admits a countable James boundary. Then:

$$\text{ASE}(X, Y) \cap \mathcal{K}(X, Y) \text{ dense in } \mathcal{K}(X, Y).$$

This result applies to...

- Y polyhedral (real) Banach space,
- Y closed subspace of (the real or complex space) $C(K)$ where K is a Hausdorff scattered compact space.

(no separability condition is needed!)

★ A second family of new examples. The general result

Theorem
X, Y Banach spaces, $\mathcal{I}(X, Y^*) \leqslant \mathcal{L}(X, Y^*)$ containing rank-one operators. Suppose:

- $\text{SE}(B_X)$ is dense,
- Y has the RNP and $\text{str-exp}(B_Y)$ is discrete up to rotations
(i.e. for every sequence $\{y_n\}$ of elements of $\text{str-exp}(B_Y)$ converging to an element $y_0 \in \text{str-exp}(B_Y)$, there is a sequence $\{\theta_n\} \subset \mathbb{T}$ such that $y_n = \theta_n y_0$ for large n).

Then, $\text{ASE}(X, Y^*) \cap \mathcal{I}(X, Y^*)$ is dense in $\mathcal{I}(X, Y^*)$.

Idea of the proof:

- We use Stegall variational principle in $\mathcal{L}(Y, X^*) \equiv \mathcal{L}(X, Y^*)$.
- We use Bourgain's ideas, the discreteness hypothesis, and the residuality of $\text{SE}(B_X)$, to get operators $T \colon Y \longrightarrow X^*$ and norm-one elements y such that $\|Ty\| = \|T\|$ and $Ty \in \text{SE}(B_X)$.
- The (pre)adjoints of these operators attains their norms at strongly exposed points of B_X. Hence, they belong to $\overline{\text{ASE}(X, Y^*)}$.

★ A second family of new examples. Consequence

Consequence 4
$\text{SE}(B_X)$ dense, Y RNP with $\text{str-exp}(B_Y)$ discrete up to rotations. Then:

$$\text{ASE}(X, Y^*) \text{ dense in } \mathcal{L}(X, Y^*), \quad \text{ASE}(X, Y^*) \cap \mathcal{K}(X, Y^*) \text{ dense in } \mathcal{K}(X, Y^*).$$

This result applies to...

- $Y = \mathcal{F}(M)$ (hence $Y^* = \text{Lip}_0(M)$) when M is a discrete metric space.

4.3 Open problems

★ Some related open problems

Open problem 1
Does Lindenstrauss property A **always** imply that $\mathrm{SE}(B_X)$ is dense in X^*?

- If YES, then ℓ_∞ would be a dual space failing property A.
- If NO, then for "big" spaces, property A behaves different than for separable ones. . .

Open problem 2
Does the denseness of $\mathrm{SE}(B_X)$ imply that $\mathrm{ASE}(X, Y)$ is dense for every Y?
Or, at least, that X has property A?

Open problem 3
Find conditions on Y to get that $\mathrm{ASE}(X, Y)$ is dense whenever $\mathrm{SE}(B_X)$ is:

- Y finite-dimensional?
- Y Asplund?
- $Y = Z^*$ with Z RNP?

5 Finite-rank operators

★ Preliminaries

Recall (M. 2014)
There are **compact** operators which cannot be approximated by norm attaining ones.

Open problem
Can finite-rank operators be always approached by norm attaining (finite-rank) ones?

Remark
If we look for properties allowing to approximate compact operators by norm attaining finite-rank ones, we need some kind of approximation property.
★ This is the "classical" approach from the 1970's

New ideas
We prefer to separate the problem of approximate compact operators (which cannot be always done) from the problem of approximate finite-rank operators (which is open).

5.1 "Classical" sufficient conditions to get density

★ Denseness of norm attaining finite-rank operators: first results

Notation
X, Y Banach spaces,

$$\mathrm{FRNA}(X, Y) := \mathcal{FR}(X, Y) \cap \mathrm{NA}(X, Y)$$

set of finite-rank norm attaining operators.

Property A
If X has property A, then $\mathcal{FR}(X, Y) \subset \overline{\mathrm{FRNA}(X, Y)}$:

- X RNP.

- Other geometrical properties (property α, quasi-α,...)
- Every separable Banach space can be renormed to have this property.

Open problem
Does property B of Y imply that $\mathcal{FR}(X,Y) \subset \overline{\mathrm{FRNA}(X,Y)}$ for every X?

★ Denseness of norm attaining finite-rank operators: approximation properties

For domain spaces (Johnson-Wolfe, 1979)
X Banach space. Suppose that there is a net (P_α) of finite-rank norm-one projections on X such that $(P_\alpha^* x^*) \longrightarrow x^*$ in norm $\forall x^* \in X^*$.
Then, $\mathcal{K}(X,Y) = \overline{\mathrm{FRNA}(X,Y)}$ for every Y.

Idea of the proof

- $(TP_\alpha) \longrightarrow T$ for every $T \in \mathcal{K}(X,Y)$
- $TP_\alpha \in \mathrm{FRNA}(X,Y)$ as $P_\alpha(X)$ is finite-dimensional and $B_{P_\alpha(X)} = P_\alpha(B_X)$.

It applies to...

- $X = C(K)$, $X = C_0(L)$, and $X = L_1(\mu)$.
- $X^* \equiv \ell_1$ and X subspace of c_0 with *monotone* Schauder basis (M., 2014).

For range spaces (Johnson–Wolfe, 1979)
Suppose that every finite-dimensional subspace of Y is contained in a polyhedral finite-dimensional subspace of Y. Then, $\mathcal{RF}(X,Y) \subset \overline{\mathrm{FRNA}(X,Y)}$ for every X.

Idea of the proof

- $T \in \mathcal{FR}(X,Y)$ can be view as $T \in \mathcal{L}(X,F)$ with F polyhedral

It applies to...
$Y^* \equiv L_1(\mu)$, $Y = L_1(\mu)$ (only real case), Y polyhedral with the AP.

5.2 New sufficient conditions to get density

★ A new condition: lineability of norm attaining functionals

Theorem (KLMW, 2020)
X Banach space. Suppose that given $x_1^*,\ldots,x_n^* \in X^*$ and $\varepsilon > 0$, there are $z_1^*,\ldots,z_n^* \in X^*$ such that $\|x_i^* - z_i^*\| < \varepsilon$ and

$$\mathrm{span}\{z_1^*,\ldots,z_n^*\} \subset \mathrm{NA}(X,\mathbb{K}).$$

Then, $\mathcal{FR}(X,Y) \subset \overline{\mathrm{FRNA}(X,Y)}$ for every Y.

Idea of the proof

- Write $T \in \mathcal{FR}(X,Y)$ as a finite-sum $T = \sum x_i^* \otimes y_i$.
- Take $\{z_i^*\}$ of the hypothesis and consider $S = \sum z_i^* \otimes y_i$.
- Then, $T \sim S$ and $S^*(Y^*) = [\ker S]^\perp \subset \mathrm{span}\{z_i^*\} \subset \mathrm{NA}(X,\mathbb{K})$.

- $S \in \mathrm{NA}(X, Y)$ by the following lemma...

Lemma
$T\colon X \longrightarrow Y$, exists $y^* \in S_{Y^*}$ such that $\|T^*y^*\| = \|T\|$ and $T^*y^* \in \mathrm{NA}(X, \mathbb{K})$,
then $T \in \mathrm{NA}(X, Y)$.

It applies to...

- All cases covered by Johnson–Wolfe result ($X = C_0(L)$, $X = L_1(\mu)$, $X^* \equiv \ell_1 \ldots$)
- When $\mathrm{NA}(X, \mathbb{K})$ is a linear subspace:
  - $X = \mathcal{K}(\ell_2, \ell_2)$;
  - X being a c_0-sum of reflexive spaces;
  - X being a proximinal finite-codimensional subspace of c_0 or of $\mathcal{K}(\ell_2, \ell_2)$.

5.3 Existence of rank-two norm attaining operators

★ Getting finite-rank norm attaining operators

Open problem
X, Y Banach spaces, $\dim(Y) \geqslant 2$, does there exist $T \in \mathrm{FRNA}(X, Y)$ with rank-two?

Notation
X, Y Banach spaces, $\mathrm{NA}^{(2)}(X, Y) := \big\{T \in \mathrm{NA}(X, Y)\colon T \text{ of rank-two}\big\}$.

Remark
If Y is not strictly convex $\mathrm{NA}^{(2)}(X, Y) \neq \emptyset$ $\forall X$ (put a copy of $B_{\ell_\infty^2}$ into B_Y).

Remark
$\mathrm{NA}^{(2)}(X, \ell_2) \neq \emptyset \implies \mathrm{NA}^{(2)}(X, Y) \neq \emptyset$ when $\dim(Y) \geqslant 2$ (using transitivity of ℓ_2^2).

Open problem
Does there exists X (with $\dim(X) \geqslant 2$) such that $\mathrm{NA}^{(2)}(X, \ell_2) = \emptyset$?
Observe that in this case, $\mathrm{NA}(X, \ell_2^2)$ would be not dense!!

★ Sufficient conditions to get $\mathrm{NA}^{(2)}(X, \ell_2) \neq \emptyset$

Condition 1 (folklore)
If there is a norm-one rank-two projection $P\colon X \longrightarrow X$, then $\mathrm{NA}^{(2)}(X, \ell_2) \neq \emptyset$.
★ Take $S \in \mathcal{L}(P(X), \ell_2) = \mathrm{NA}(P(X), \ell_2)$ of rank-two and use that $P(B_X) = B_{P(X)}$
to get that $SP \in \mathrm{NA}^{(2)}(X, \ell_2)$.

But... (Bosznay–Garay, 1986)
There are Banach spaces for which every norm-one projections is either of rank one or the identity.

Condition 2 (folklore)
If X contains a proximinal subspace Z of codimension two, then $\mathrm{NA}^{(2)}(X, \ell_2) \neq \emptyset$.
★ Take $S \in \mathcal{L}(X/Z, \ell_2)$ of rank-two and use that $q_Z(B_X) = B_{X/Z}$ by proximinality
to get that $Sq_Z \in \mathrm{NA}^{(2)}(X, \ell_2)$.

But... (Read, 2018, solving an open question by Singer of the 1970's)
There are Banach spaces with no proximinal subspaces of codimension two

★ Sufficient conditions to get $\mathrm{NA}^{(2)}(X, \ell_2) \neq \emptyset$ II

Condition 3 (folklore)

If $\mathrm{NA}(X, \mathbb{K})$ contains a two-dimensional subspace W, then $\mathrm{NA}^{(2)}(X, \ell_2) \neq \emptyset$.
★ Take $\widetilde{S} \in \mathcal{L}(X/W_\perp, \ell_2)$ of rank-two and write $S = \widetilde{S} q_{W_\perp}$; then S is rank-two and $S^* \in \mathrm{NA}(\ell_2, X^*)$ with $S^*(\ell_2) \subset W = (W_\perp)^\perp \subset \mathrm{NA}(X, \mathbb{K})$.

Lemma

$T\colon X \longrightarrow Y$, exists $y^* \in S_{Y^*}$ such that $\|T^* y^*\| = \|T\|$ and $T^* y^* \in \mathrm{NA}(X, \mathbb{K})$, then $T \in \mathrm{NA}(X, Y)$.

But… (Rmoutil, 2017; solving an open question by Godefroy from 2000)

The Read's space $\mathcal{R}$ satisfies that $\mathrm{NA}(\mathcal{R}, \mathbb{R})$ contains no two-dimensional subspace.

What can we do next??

★ Sufficient conditions to get $\mathrm{NA}^{(2)}(X, \ell_2) \neq \emptyset$ III

Remark

Read's space $\mathcal{R}$ (and other constructions with the same property) are not smooth.

★ Therefore, taking $f_1, f_2 \in S_{X^*}$ linearly independent and $x_0 \in S_X$ such that $f_1(x_0) = f_2(x_0) = 1$, we get that the operator $S(x) = (f_1(x), f_2(x))$ from X to ℓ_2^2
is onto and norm attaining.

Debs–Godefroy–SaintRaymond, 1995

Given X separable, there exists Z smooth renorming of X such that $\mathrm{NA}(X, \mathbb{K}) = \mathrm{NA}(Z, \mathbb{K})$.

Smooth *a posteriori* Read spaces

There are *smooth* spaces X for which $\mathrm{NA}(X, \mathbb{K})$ has not two-dimensional subspaces.

However…

For the smooth Read spaces above, $\mathrm{NA}(X, \mathbb{K})$ contains non-trivial cones.

★ Sufficient conditions to get $\mathbf{NA}^{(2)}(X, \ell_2) \neq \emptyset$ IV

Kadets–López–M.–Werner, 2000

X Banach space, if $\mathrm{NA}(X, \mathbb{K})$ containing non-trivial cones, then $\mathrm{NA}^{(2)}(X, \ell_2) \neq \emptyset$.

A simpler proof using ideas from Cabello brothers

1. (Cabello–Cabello) For any two-dimensional space Z, the elements in S_Z at which a non-degenerate ellipsoid contained in B_Z touch S_Z are dense in S_Z.
2. As $\mathrm{NA}(X, \mathbb{K})$ contains non-trivial cones, there is a two-dimensional subspace Z for which $\mathrm{NA}(X, \mathbb{K}) \cap S_Z$ has non-empty interior.
3. Hence, there is a non-degenerate ellipsoid contained in B_Z touching S_Z at an element of $\mathrm{NA}(X, \mathbb{K}) \cap S_Z$.
4. This ellipsoid define an injective $S \in \mathcal{L}(\ell_2^2, X^*)$ satisfying that $\|S\| = 1 = \|Sw_0\|$ with $Sw_0 \in \mathrm{NA}(X, \mathbb{K})$.
5. Hence, $T := S^*|_X \in \mathcal{L}(X, \ell_2^2)$ is onto and $T^* = S$, so $T \in \mathrm{NA}^{(2)}(X, \ell_2^2)$.

Open problem??

To get a possible counterexample, we have to eliminate cones from $\mathrm{NA}(X, \mathbb{K})$.

★ AI ultimate example

Example

There is a Banach space $\mathfrak{X}$ such that $\mathrm{NA}(\mathfrak{X}, \mathbb{R})$ does not contain non-trivial cones. Moreover:

- For every $Z \leqslant \mathfrak{X}^*$ of dimension two, $\mathrm{NA}(\mathfrak{X}, \mathbb{R}) \cap S_Z$ contains, at most, four points.
- Given two linearly independent elements $f_1, f_2 \in \mathrm{NA}(\mathfrak{X}, \mathbb{R})$, no other element in the segment between f_1 and f_2 belongs to $\mathrm{NA}(\mathfrak{X}, \mathbb{R})$.

M. Martín.
A Banach space whose set of norm attaining functionals is algebraically trivial
Preprint (2024), https://arxiv.org/abs/2406.07273

The proof is similar to the construction of Read norms given in

V. Kadets, G. López, M. Martín, and D. Werner
Equivalent norms with an extremely nonlineable set of norm attaining functionals
J. Inst. Math. Jussieu (2020)

applied to a smooth renorming X of c_0 with $\mathrm{NA}(X, \mathbb{R}) = c_{00} \leqslant X^*$, using an ℓ_1-sum of smooth renorming of ℓ_1 whose duality is "asymptotically" the one of (ℓ_1, ℓ_∞).

5.4 Open problems

★ Open problems

Problem 1

Does every finite-dimensional Banach space satisfy Lindenstrauss property B?
Equivalently, are finite-rank operators always approachable by norm attaining ones?

Problem 2 (The most irritating one according to Johnson–Wolfe)

Is $\mathrm{NA}(X, \ell_2^2)$ dense in $\mathcal{L}(X, \ell_2^2)$ for every X?

Problem 3

We do not even know if there is X such that $\mathrm{NA}^{(2)}(X, \ell_2) = \emptyset$.

Problem 4

Is $\mathrm{NA}^{(2)}(\mathfrak{X}, \ell_2)$ empty? If not, is $\mathrm{FRNA}(\mathfrak{X}, \ell_2)$ dense in $\mathcal{K}(\mathfrak{X}, \ell_2)$?

★ Open problems II

Problem 5

Find conditions on X to assure that $\mathcal{FR}(X, \ell_2) \subset \mathrm{FRNA}(X, \ell_2)$ or, at least,
that $\mathrm{NA}^{(2)}(X, \ell_2) \neq \emptyset$.

Idea

- A positive answer to the second question implies that there is a two-dimensional subspace Z of X^* whose intersection with $\mathrm{NA}(X, \mathbb{R})$ contains an "inner point" (that is, an element in S_Z at which a non-degenerate ellipsoid contained in B_Z touch S_Z).
- Everything would be easier if $\mathrm{NA}(X, \mathbb{R})$ and the set of inner points were residual, but:
  - $\mathrm{NA}(X, \mathbb{R})$ is residual in many cases (X LUR...)
  - The set of inner points can be not residual (Cabello, private communication)...