Vector nonlocal calculus based on nonlocal gradients over bounded domains: Helmholtz decomposition and localization to the classical case

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Based on joint works with
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Abstract

One of the motivations of nonlocal models is that they may able to provide a framework for more general (or singular) phenomena, since they may not need to use classical derivatives, whereas the Helmholtz decomposition is a relevant result in mathematics and fluid mechanics which states that any (sufficiently smooth) vector field can be written as the sum of curl-free vector field plus a divergence-free one. Thus, the scope of this talk aims to provide a Helmholtz decomposition based on a strongly singular nonlocal gradient. In doing so we continue further with the nonlocal calculus developed for nonlocal gradients defined as

$$D_\delta^s u(x) = c_{n,s} \int_{B(x,\delta)} \frac{u(x)-u(y)}{|x-y|} \frac{x-y}{|x-y|} \frac{w_\delta(x-y)}{|x-y|^{n-1+s}}\, dy, \quad x \in \Omega$$

for $u : \Omega \cup B(0,\delta) \to \mathbb{R}$. This operator keeps a degree of fractional differentiability while providing a framework over bounded domains. Following some previous results regarding nonlocal versions of the Fundamental Theorem of Calculus, Poincaré Sobolev inequalities, Piola identity or compact embedding concerning the operator $D_\delta^s u$, we develop new tools such as nonlocal Divergence and Stokes theorems, three nonlocal Green identities as well as the study of the fundamental solution of the Laplacian given by $\Delta_\delta^s u := \operatorname{div}_\delta^s D_\delta^s u$, for a properly defined nonlocal divergence. Most of them are employed in the path leading to the nonlocal Helmholtz decomposition.

There have been previous results on Helmholtz decompositions concerning two-points nonlocal gradients(without integration), one-point gradient with integrable kernels or the Riesz fractional one (defined over the whole space).

This talk is complemented with localization results showing that the classical models are recovered when the nonlocality vanishes.